Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 3

Explicit complete curves in the moduli space of curves of genus three

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Consider M_g, the moduli space of smooth curves of genus g over the field of complex numbers $\mathbf{C}$. For $g \geq 2$, M_g is a quasi-projective variety of dimension $3g-3$. Note that M_g is not complete as smooth curves can degenerate to singular ones. In fact, M_2 is affine. However, M_g contains complete curves if $g \geq 3$. This follows from the existence of a projective compactification $\widetilde{M}_g$ of M_g in which the boundary $\widetilde{M}_g \setminus M_g$ has codimension ≥ 2 (take the closure of the image of M_g in the *Satake* compactification of $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties). The complete curves are obtained by cutting $\widetilde{M}_g$ with sufficiently many hypersurfaces in general position. An upper bound for the dimension of a complete subvariety of M_g is $g-2$ if $g \geq 2$ (see [14]). So these complete curves achieve this bound if $g = 3$.

Harris (see [25]) notes that these curves are not very explicit, although constructions of explicit complete families of smooth curves are known. In [17] an explicit one-dimensional family is given for every genus $g \geq 4$, but for $g = 3$ a less explicit family is exhibited. The aim of this note is to produce explicit examples of complete families of smooth curves of genus 3 having a moduli theoretic interpretation.

In short the construction is the following. Fix a smooth base curve C_3 of genus 3 and fix a complete curve $F \subset C_3 \times C_3 \setminus \Delta$. Construct a complete family of smooth double covers $(C_f \overset{\pi_f}{\rightarrow} C_3)_{f \in F}$, where π_f is branched over the two points determined by $f \in F$ (in fact such a family may exist only over a finite cover of F due to monodromy obstructions). To the covers $C_f \overset{\pi_f}{\rightarrow} C_3$ we can associate their *Prym* varieties. We obtain a complete family of 3-dimensional principally polarized abelian varieties $\mathrm{Prym}(C_f/C_3)_{f \in F}$, which turn out to be Jacobians of smooth curves if C_3 is not hyperelliptic. So we get a complete family of Jacobians of smooth genus 3 curves.

In the first section we present a specific construction of complete families of 3-dimensional principally polarized abelian varieties, depending on five parameters. The base is 1-dimensional and the fibers are Prym varieties of branched double covers of curves. We will see in section 2 that these fibers are actually Jacobians of smooth curves. The corresponding families of curves are then constructed via the so-called "trigonal construction". Section 3 contains a calculation of invariants of these families, in particular the number of hyperelliptic fibers. Finally, we show that these families are general in the following sense: the generic smooth curve of genus 3 occurs as a fiber of one of the families.

3.1 The construction

Fix an elliptic curve $E = (E, 0)$. Let $\phi : C_3 \to E$ be a double cover ramified at 4 points of E, so that the genus C_3 is 3. We want C_3 to be non-hyperelliptic. This turns out to be an open condition on the four branch points: let B the branch divisor, let $\tilde{B}$ the ramification divisor of ϕ, and let L denote the unique line bundle on E satisfying $\phi^* L \cong \mathcal{O}_{C_3}(\tilde{B})$. Then C_3 is hyperelliptic if and only if $B = B_1 + B_2$ for some $B_1, B_2 \in |L|$ (if C_3 is hyperelliptic, then σ acts freely on $\mathrm{Supp}(\tilde{B})$ and $L = \phi_*(x + \sigma x)$ for an $x \in \mathrm{Supp}(\tilde{B})$).

Fix a point $t \neq 0$ in E. Set $\Delta_t = \{ (x, x+t) \mid x \in E \} \subset E \times E$. Let $F = F(t) \subset C_3 \times C_3$ be the inverse image of Δ_t under the natural map $\phi \times \phi : C_3 \times C_3 \to E \times E$. Then in $C_3 \times C_3$ we have that $F \cap \Delta = \emptyset$, thus F parametrizes pairs of *distinct* points on C_3. If we suppose—as we shall do in the following—that $t \notin \{ \phi(p) - \phi(q) \mid p, q \text{ ramification points of } \phi \}$, then one easily verifies that F is smooth of genus 9.

Denote by $\pi_1, \pi_2 : F \to C_3$ the maps induced by the projections of $C_3 \times C_3$ onto the first, respectively the second coordinate, and denote by $\Gamma_{\pi_i} \subset F \times C_3$ the graph of π_i, for $i = 1, 2$. We want to have a double cover of $F \times C_3$ ramified precisely over $\Gamma_{\pi_1} + \Gamma_{\pi_2}$. Such a cover may not exist due to monodromy obstructions. To overcome these, we consider the natural map $F \to \mathrm{Pic}^2(C_3)$, $x \mapsto [\pi_1(x) + \pi_2(x)]$, and the squaring map $sq : \mathrm{Pic}^1(C_3) \to \mathrm{Pic}^2(C_3)$, $L \mapsto L \otimes L$. Define the curve B as the fiberd product $B = F \times_{\mathrm{Pic}^2(C_3)} \mathrm{Pic}^1(C_3)$.

$$\begin{array}{ccc} B & \to & \mathrm{Pic}^1(C_3) \\ f\downarrow & & \downarrow \mathrm{sq} \\ F & \to & \mathrm{Pic}^2(C_3) \end{array}$$

Denote the map $B \to F$ by f, and set $\Gamma = \Gamma_{\pi_1 \circ f} + \Gamma_{\pi_2 \circ f} \subset B \times C_3$. The pull-back to $B \times C_3$ of the universal line bundle over $\mathrm{Pic}^1(C_3) \times C_3$ gives a line bundle $\mathcal{L}$ on $B \times C_3$ satisfying $\mathcal{L}^2 \cong \mathcal{O}_{B \times C_3}(\Gamma)$.

Using this $\mathcal{L}$ we make a double cover $\mathcal{C} \to B \times C_3$, ramified precisely along Γ. This is a well-known construction: embed $B \times C_3$ in the total space of the line bundle corresponding to $\mathcal{O}_{B \times C_3}(\Gamma)$ with a section s of $\mathcal{O}_{B \times C_3}(\Gamma)$ which vanishes precisely once along Γ. Define $\mathcal{C}$ as the inverse image of $s(B \times C_3)$ under the natural squaring map on local sections $\mathcal{L} \stackrel{sq}{\to} \mathcal{L}^2 \cong \mathcal{O}_{B \times C_3}(\Gamma)$. Here $\mathcal{L}$ denotes the total space of the line bundle $\mathcal{L}$, by abuse of notation.

$$\begin{array}{ccccccc} \mathcal{C} & \to & \mathcal{L} & \to & \mathcal{L}^2 & \stackrel{\cong}{\longrightarrow} & \mathcal{O}_{B \times C_3}(\Gamma) \\ & & \downarrow & & \downarrow & & \downarrow\uparrow \\ & & B \times C_3 & = & B \times C_3 & = & B \times C_3 \end{array}$$

The composition with the projection $B \times C_3 \to B$ yields a complete family of curves $\pi_{\mathcal{C}} : \mathcal{C} \to B$. Let $\mathcal{C}_b$ denote the fiber of $\pi_{\mathcal{C}} : \mathcal{C} \to B$ over $b \in B$. Each $\mathcal{C}_b$ is smooth of genus 6 and maps $2:1$ to C_3 with two ramification points, via the composition $\mathcal{C} \to B \times C_3 \to C_3$. Note that the map $f : B \to F$ is étale of degree 64, so that B is a smooth curve, but possibly *reducible*. Thus $\pi_{\mathcal{C}} : \mathcal{C} \to B$ is a complete, possibly reducible family of smooth curves.

We can associate to every double covering $\mathcal{C}_b \to C_3$ its Prym variety (see [39]). This is a 3-dimensional, principally polarized abelian variety $\text{Prym}(\mathcal{C}_b/C_3) = (P, \Xi)$. P is defined as the kernel of the norm map $\text{Nm} : \text{Jac}(\mathcal{C}_b) \to \text{Jac}(C_3)$ (which is connected since $\mathcal{C}_b \to C_3$ is ramified). The Jacobian of C_3 injects in the Jacobian of $\mathcal{C}_b$, so up to isogeny $\text{Jac}(\mathcal{C}_b)$ splits as the product of $\text{Jac}(C_3)$ and another factor, which is P. The polarization on $\text{Jac}(\mathcal{C}_b)$ gives rise to a polarization of $P \times \text{Jac}(C_3)$: $P \times \text{Jac}(C_3) \to \text{Jac}(\mathcal{C}_b) \to \widehat{\text{Jac}(\mathcal{C}_b)} \to \widehat{P} \times \widehat{\text{Jac}(C_3)}$. This polarization splits and induces on P twice a principal polarization Ξ.

It is straightforward to globalize this construction. The family $\pi_{\mathcal{C}} : \mathcal{C} \to B$ has sections. Thus we have the associated family $\pi_{\mathcal{C}} : \mathcal{C} \to B$ of Jacobians $J\mathcal{C} \to B$. Consider the norm map $\text{Nm} : J\mathcal{C} \to B \times \text{Jac}(C_3)$, induced by the usual norm map $\text{Nm} : J\mathcal{C}_b \to b \times \text{Jac}(C_3)$. Define $\mathcal{P}$ as $\text{Nm}^{-1}(B \times 0) \subset J\mathcal{C}$. The projection $J\mathcal{C} \to B$ yields a family $\pi_{\mathcal{P}} : \mathcal{P} \to B$ of 3-dimensional principally polarized abelian varieties.

We need Theorem 2.1. The theorem says that the 1-dimensional family of Pryms $\mathcal{P} \to B$ is in fact a family of Jacobians of smooth curves. Note that here we use that C_3 is non-hyperelliptic. For readability the statement and proof of this theorem is postponed to the next section. Summarizing, we have the following result:

Corollary 3.1 *The above construction yields a complete, one-dimensional family $\pi_{\mathcal{C}} : \mathcal{P} \to B$ of 3-dimensional Jacobians of smooth curves.*

Note that the construction depends on five parameters: namely on E, on the double cover $\phi : C_3 \to E$ and on the point $t \in E$. Thus we obtain in fact a five-dimensional family of complete, one-dimensional families of 3-dimensional Jacobians.

3.2 The corresponding family of curves

We will exhibit the family of smooth curves corresponding to the family of Jacobians $\pi_{\mathcal{P}} : \mathcal{P} \to B$, using the so-called trigonal construction. We explain this first for a single Prym variety $\text{Prym}(\mathcal{C}_b/C_3)$. For the details and proofs we refer to [18]. The map $\mathcal{C}_b \to C_3$ has two branch points, which we denote by $p, p' \in C_3$. Consider C_3 canonically embedded in $|K_{C_3}|^* \cong \mathbf{P}^2$. The line through p and p' cuts out on C_3 two other points. Choose one of them and call it p''. Let $\mathbf{P}^1$ denote the linear system $|K_{C_3} - p''|$. Projecting from p'' determines a three-to-one map $C_3 \to \mathbf{P}^1 \cong \mathbf{P}^1$.

Assume that this map does not ramify at p or p'. Let $\mathcal{C}_b^{(3)}$ (respectively $C_3^{(3)}$) denote the threefold symmetric product of $\mathcal{C}_b$ (respectively C_3). The line $\mathbf{P}^1$ embeds naturally in $C_3^{(3)}$. Consider the inverse image of $\mathbf{P}^1 \subset C_3^{(3)}$ under the natural map $\mathcal{C}_b^{(3)} \to C_3^{(3)}$. This is a curve with two smooth, isomorphic components T_0, T_1. Their intersection consists of two distinct points q, q', in which they meet transversally. Both curves map with degree four to $\mathbf{P}^1$ and q, q' are simple ramification points on each curve mapping to the same point on $\mathbf{P}^1$. Apart from q and q' the ramification on the tetragonal curves T_0 and T_1 arises from the ramification of the g_3^1 on C_3. Namely if $C_3 \to \mathbf{P}^1$ is simply branched over $P \in \mathbf{P}^1$, then also the maps $T_i \to \mathbf{P}^1$ are simply branched over P. If $C_3 \to \mathbf{P}^1$ is completely branched over $P \in \mathbf{P}^1$ then the maps $T_i \to \mathbf{P}^1$ have a point of ramification index 3 over P. In particular the genera of the T_i are 3.

If on the other hand the triple cover $C_3 \to \mathbf{P}^1$ is ramified at p or p', then the situation is different. The points q and q' coincide, i.e., the inverse image still has two smooth isomorphic components T_0 and T_1 which meet in one point q, and the fourfold cover $T_i \to \mathbf{P}^1$ is totally ramified in q.

The curves T_0 and T_1 map naturally to $\mathrm{Pic}^3(\mathcal{C}_b)$, since they live in $\mathcal{C}_b^{(3)}$. The norm map $\mathrm{Nm} : \mathrm{Pic}^3(\mathcal{C}_b) \to \mathrm{Pic}^3(C_3)$ maps their images to the image of $\mathbf{P}^1$ in $\mathrm{Pic}^3(C_3)$, which is a point. The images of T_0 and T_1 in $\mathrm{Pic}^3(\mathcal{C}_b)$ lie thus in a suitable translate of $P = \ker(\mathrm{Nm} : \mathrm{Jac}(\mathcal{C}_b) \to \mathrm{Jac}(C_3))$. Moreover, the maps $T_i \to \mathrm{Pic}^3(\mathcal{C}_b)$, $i = 0, 1$, induce isomorphisms of principal polarized abelian varieties $\mathrm{Jac}(T_i) \to \mathrm{Prym}(\mathcal{C}_b/C_3)$. The trigonal construction goes back to Recillas and was generalized by Donagi.

To globalize this construction reconsider the curve $F \subset C_3 \times C_3$. Every point $x \in F = F(t)$ determines two distinct points $\pi_1(x), \pi_2(x) \in C_3$, and the line spanned by these points in the canonical embedding of C_3 we denote by $\ell(x)$. Let $K(x)$ denote the hyperplane section $C_3 \cdot \ell(x)$. Consider the curve

$$\tilde{F} = \tilde{F}(t) = \{ (x, c) \in F \times C_3 \mid c \in \mathrm{Supp}(K(x) - \pi_1(x) - \pi_2(x)) \}.$$

Then $\tilde{F}$ is smooth since F and C_3 are. Clearly $\tilde{F}$ maps with degree two onto F. This map is ramified in 64 points (see Lemma 3.3 for the computation), hence $\tilde{F}$ is irreducible of genus 49. A point (x, p'') of $\tilde{F}$ determines on C_3 a g^1_3, namely $|K_{C_3} - p''|$, plus two unordered, distinct points contained in a divisor of this g^1_3, namely $\pi_1(x), \pi_2(x)$. $\tilde{F}$ maps to $\mathrm{Pic}^2(C_3)$ via the composition $\tilde{F} \to F \to \mathrm{Pic}^2(C_3)$, and also $\mathrm{Pic}^1(C_3)$ does via the squaring map $\mathrm{Pic}^1(C_3) \to \mathrm{Pic}^2(C_3)$. Let $\tilde{B} = \tilde{F} \times_{\mathrm{Pic}^2(C_3)} \mathrm{Pic}^1(C_3)$. Note that $\tilde{B}$ maps two-to-one to B.

The pull-back of $\mathcal{C} \to B$ to $\tilde{B}$ yields a double cover $\tilde{\mathcal{C}} \to \tilde{B} \times C_3$. The composition $\tilde{\mathcal{C}} \to \tilde{B} \times C_3 \to \tilde{B}$ is a one-dimensional family of double covers. The extra structure is a three-to-one map $\tilde{B} \times C_3 \to S$, where S is a ruled surface $S \to \tilde{B}$ with fiber $S_b = |K_{C_3} - p''|$. Then S embeds in $\tilde{B} \times C_3^{(3)}$ and its inverse image along the natural map $\tilde{\mathcal{C}}^{(3)}_{\tilde{B}} \to (\tilde{B} \times C_3)^{(3)}_{\tilde{B}} = \tilde{B} \times C_3^{(3)}$ we denote by $\mathcal{T}'$. The (singular) surface $\mathcal{T}'$ has an involution which preserves the fibers of $\mathcal{T}' \to S$ setwise, and its quotient by this involution we denote by $\mathcal{T}$. The surface $\mathcal{T}$ is smooth. The natural map $\mathcal{T} \to S$ composed with the projection $S \to \tilde{B}$ yields a family $\mathcal{T} \to \tilde{B}$ of smooth tetragonal curves.

Corollary 3.2 *The above construction gives a complete, one-dimensional family $\mathcal{T} \to S \to \tilde{B}$ of smooth, tetragonal curves of genus 3.*

3.3 Pryms of ramified double coverings of genus 3 curves

This section is devoted to the proof of the theorem below. It shows that the Prym variety associated to a ramified double covering $\tilde{Z}/Z$ ramified in two points, where the genus of Z is 3, is a Jacobian of a *smooth* curve as long as the base curve is not hyperelliptic.

In the following let Z be a smooth genus 3 curve, and let $\pi_Z : \tilde{Z} \to Z$ be a double cover of smooth curves, with two branch points $p, p' \in Z$ and ramification points $\tilde{p}, \tilde{p}' \in \tilde{Z}$. Associated to this cover is a (unique) line bundle $L_{\tilde{Z}/Z}$ satisfying $(\pi_Z)^*(L_{\tilde{Z}/Z}) \cong \mathcal{O}_{\tilde{Z}}(\tilde{p} + \tilde{p}')$. The data $(Z, p + p', L_{\tilde{Z}/Z})$ determine the cover $\pi_Z : \tilde{Z} \to Z$ up to isomorphism. The Prym variety $P = \mathrm{Prym}(\tilde{Z}/Z)$ is a 3-dimensional abelian variety which comes with a natural principal polarization, denoted by Ξ.

Theorem 3.3

1. *If Z is hyperelliptic, and if p, p' are interchanged by the hyperelliptic involution of Z, then (P, Ξ) is the Jacobian of a smooth irreducible hyperelliptic curve, or a product of two such.*
2. *If Z is hyperelliptic and p, p' are not interchanged by the hyperelliptic involution of Z, then (P, Ξ) is a Jacobian of a smooth irreducible hyperelliptic curve.*
3. *If Z is not hyperelliptic, then (P, Ξ) is a Jacobian of a smooth irreducible curve. This curve is hyperelliptic if and only if for some $p'' \in Z$, $p + p' + 2p''$ is a canonical divisor and $h^0(L^{-1}_{\tilde{Z}/Z}(p + p' + p'')) = 0$.*

PROOF. Let $W = Z/(p \sim p')$ be the curve obtained from Z by identifying p and p', and let $\tilde{W} = \tilde{Z}/(\tilde{p} \sim \tilde{p}')$. Denote by $f_N : Z \to W$ the normalisation mapping. Note that $p_a(W) = 4$, $p_a(\tilde{W}) = 7$.

The idea is to use the theory from [6]: the induced map $\pi_W : \tilde{W} \to W$ is an admissible covering in the sense of Beauville, and $\mathrm{Prym}(\tilde{Z}/Z) = \mathrm{Prym}(\tilde{W}/W)$. Here the right hand side is the Prym variety associated by Beauville to the covering $\tilde{W}/W$. It lives canonically in

$$J\tilde{W}^* = \{\text{line bundles } L \text{ on } \tilde{W} \mid 2 \deg L = \deg \omega_{\tilde{W}}\}$$

as $P = \mathrm{Prym}(\tilde{W}/W) = \{L \text{ on } \tilde{W} \mid \mathrm{Nm}L \cong \omega_W h^0(L) \text{ even}\}$, and the polarization Ξ then is $\Xi = \{L \in P \mid h^0(L) \geq 2\}$.

To prove the theorem, we will show that, unless we are in case (1), $\dim \mathrm{Sing}\Xi \leq 0$. Since $\dim P = 3$, this shows that in the cases (2) and (3) (P, Ξ) is a Jacobian. If $\mathrm{Sing}\Xi$ is a point $L \in \Xi$—in which case (P, Ξ) is a hyperelliptic Jacobian—we show how L arises from the geometry of Z.

Note that the proof is an adaptation of the proof of Theorem 4.10 in [6]. Also note that case (1) is already treated by Beauville (see [6], p. 171).

Start by assuming that $\mathrm{Sing}\Xi$ is non-empty, say $L \in \mathrm{Sing}(\Xi)$. We will show that in the cases (2) and (3) $L \in \Xi$ is a uniquely determined point. By [6], Lemma 4.1, either $h^0(L)$ is ≥ 4 and even or $h^0(L) = 2$, and $\imath^* s \otimes t = s \otimes \imath^* t$ for a base $\{s, t\}$ of $H^0(\tilde{W}, L)$, where $\imath : \tilde{W} \to \tilde{W}$ is the involution interchanging the sheets of $\pi_W : \tilde{W} \to W$.

First suppose that there is an $L \in \mathrm{Sing}(\Xi)$, with $h^0(L) \geq 4$ and $h^0(L)$ even. Let L_1 be the pull-back of L to $\tilde{Z}$. Then $\deg(L_1) = 6$, and $h^0(L_1) \geq 4$. From Clifford's theorem it follows that $\tilde{Z}$ is hyperelliptic. The hyperelliptic involution σ of $\tilde{Z}$ commutes with $\imath$. Moreover, σ interchanges the fixed points of $\imath$, since all sections of L_1 come from sections of L. Hence σ descends and Z is hyperelliptic . If we denote the hyperelliptic involution of Z also with σ, we see that σ interchanges p, p'. So we are in case (1). This is the case in which W has a so-called non-singular g^1_2. Non-singular means that the g^1_2 contains a divisor with non-singular support. It follows that (P, Ξ) is a product of hyperelliptic Jacobians (see [6], p. 171).

Secondly, suppose that there is an $L \in \mathrm{Sing}(\Xi)$ with $h^0(L) = 2$, with a base $\{s, t\}$ for $h^0(L)$ satisfying $s \otimes \imath^* t = \imath^* s \otimes t$.

We first assume that s, t have the property that either s or t is not zero at the singular point. By [6], lemma 4.4, L is of the form $(\pi_W)^* M(E)$. Here M is a line bundle on W, $h^0(M) \geq 2$, $\deg((\pi_W)^* M(E)) = \deg(L) = \deg(\omega_W) = 6$. M is non-singular, and E is an effective divisor on $\tilde{W}$ with non-singular support such that $(\pi_W)_* E \in |\omega_W \otimes M^{-2}|$. Let M_1 be the pull-back of M to Z. Then M_1 gives an g^1_2 or a g^1_3 on Z.

If Z is hyperelliptic, a g^1_3 on Z is the unique g^1_2 plus a point. We see that also in this case the two branch points p, p' of $\tilde{Z} \to Z$ are interchanged by the hyperelliptic involution, and we are in case (1) again.

If Z is not hyperelliptic, then $|M_1|$ is a g^1_3, $E = 0$, and the branch divisor $p + p'$ is contained in an element of the system $|M_1|$, say $p + p' + p'' \in |M_1|$. The relation $\mathrm{Nm}_W((\pi_W)^* M) = \omega_W$ gives by pull-back $\mathrm{Nm}_Z((\pi_Z)^* M_1) = \omega_Z(p + p')$. It follows that $p + p' + 2p''$ is a canonical divisor, and $|M_1|$ is the pencil $|K_Z - p''|$. So we are in case (3) and $L = (\pi_W)^* M$.

Finally assume that the two sections s, t of L vanish simultaneously at the singular point. Let L_1 be the pull-back to $\tilde{Z}$ of L, and set $L_2 = L_1(-\tilde{p} - \tilde{p}')$. Then $\mathrm{Nm}(L_2) = \omega_Z$, and L_1 has sections induced by s, t, which we will also denote by s, t, and which still satisfy $s \otimes \imath^* t = \imath^* s \otimes t$. Then [6], lemma 4.4 applies, giving that $L_2 = (\pi_Z)^*(M)$, where M is a line bundle on Z, $h^0(M) \geq 2$. It follows that Z is hyperelliptic and that M is the line bundle associated with the unique g^1_2 on Z. So we are in case (2) and L is an extension of ${\pi_Z}^*(M)(\tilde{p} + \tilde{p}')$ to $\tilde{W}$.

This finishes the first part of the proof. We will show in the second part that the above procedure can be reversed to obtain precisely one singularity of Ξ if we are in case (2). If p and p' satisfy the condition of case (3) then we also obtain precisely one singularity of Ξ.

First assume that we are in case (2), i.e., Z is hyperelliptic and that p, p' are not interchanged by the hyperelliptic involution. The g^1_2 on Z pulls back to a line bundle L_2 on $\tilde{Z}$. All extensions of the line bundle $L_1 = L_2(\tilde{p} + \tilde{p}')$ to a line bundle L on $\tilde{W}$ admit sections s, t satisfying $s \otimes \imath^* t = \imath^* s \otimes t$. There are precisely two extensions L such that $\mathrm{Nm}_W L = \omega_W$, and only one of these is even (see [6], Prop. 3.5). This takes care of the second case of the theorem, since we obtain indeed precisely one singularity L of Ξ, showing that (P, Ξ) is the Jacobian of a smooth, hyperelliptic curve.

Secondly, suppose that we are in case (3), and that we have a line bundle M_1 on Z, with $h^0_Z(M_1) = 2$ and $M_1 \cong \mathcal{O}_Z(p + p' + p'') \cong \mathcal{O}_Z(K_Z - p'')$. This M_1 has a unique extension M to W such that $h^0_W(M) = 2$. Then $\mathrm{Nm}_W((\pi_W)^*(M)) = \omega_W$.

For $\mathrm{Nm}_W((\pi_W)^*(M)) = M^2$ is an extension of $M_1^2 = \omega_Z(p+p')$ to W having a section non-zero at $p = p'$, and ω_W is the unique such extension.

The pullback $(\pi_W)^*(M)$ will yield an element of $\mathrm{Sing}(\Xi)$ if $h^0_{\tilde{W}}((\pi_W)^*M) = 2$. We have the decomposition

$$H^0_{\tilde{Z}}((\pi_Z)^*M_1) \cong H^0_Z(M_1) \oplus H^0_Z(M_1 \otimes L^{-1}_{\tilde{Z}/Z})$$

which is the decomposition into even and odd parts with respect to the action of the involution $\imath$ on $H^0_{\tilde{Z}}((\pi_Z)^*M_1)$. We already have seen that all sections of $H^0_{\tilde{Z}}((\pi_Z)^*M_1)^+ \cong H^0_Z(M_1)$ extend to $\tilde{W}$. Also all sections of $H^0_{\tilde{Z}}((\pi_Z)^*M_1)^- \cong H^0_Z(M_1 \otimes L^{-1}_{\tilde{Z}/Z})$ extend to $\tilde{W}$, since such a section is zero at $\tilde{p}$ and $\tilde{p}'$. It follows that $h^0_{\tilde{W}}((\pi_W)^*M) = h^0_{\tilde{Z}}((\pi_Z)^*M_1)$, and that $h^0_{\tilde{W}}((\pi_W)^*M) = 2$ if and only if $h^0_Z(M_1 \otimes L^{-1}_{\tilde{Z}/Z}) = 0$. So we obtain precisely one singularity $(\pi_W)^*M$ of Ξ if and only if $h^0(M_1 \otimes L^{-1}_{\tilde{Z}/Z}) = 0$. Note that $M_1 \otimes L^{-1}_{\tilde{Z}/Z}$ is a theta characteristic: $M_1^2 \otimes L^{-2}_{\tilde{Z}/Z} \cong \mathcal{O}_Z(p+p'+2p'') \cong \omega_Z$ by assumption. □

3.4 The degree of λ and the number of hyperelliptic fibers

Consider the family $\mathcal{P} \to B$ constructed in section 1. Our aim in this section is to compute the degree of $\lambda = c_1((\pi_{\mathcal{P}})_*(\Omega^1_{\mathcal{P}/B}))$ and the number of hyperelliptic fibers of the family $\pi_{\mathcal{P}} : \mathcal{P} \to B$.

Proposition 3.4 *The degree of $\lambda_{\mathcal{P}/B}$ equals 2^7.*

PROOF. Denote by p_B the projection $B \times C_3 \to C_3$, by p_F the projection $F \times C_3 \to C_3$, and let $p_{\mathcal{C}}$ be the composition $\mathcal{C} \to B \times C_3 \xrightarrow{p_B} C_3$. The branch locus of $\mathcal{C} \to B \times C_3$ is $\Gamma = \Gamma_{\pi_1 \circ f} + \Gamma_{\pi_2 \circ f}$. Here $\Gamma_{\pi_i \circ f}$ denotes the graph of $\pi_i \circ f$ in $B \times C_3$, $i = 1, 2$. We denote the ramification locus of $\mathcal{C} \to B \times C_3$ by $\tilde{\Gamma} = \tilde{\Gamma}_{\pi_1 \circ f} + \tilde{\Gamma}_{\pi_2 \circ f} \subset \mathcal{C}$. Let ω be a holomorphic 1-form on C_3. Let $\Omega = p_B^*\omega$ (respectively $\tilde{\Omega} = p_{\mathcal{C}}^*\omega$) be its pull-back to $B \times C_3$ (respectively $\mathcal{C}$), and consider it as a section of the line bundle $\omega_{B \times C_3/B}$ (respectively $\omega_{\mathcal{C}/B}$).

We compute the intersection number $(\tilde{\Omega})^2$, where by $(\tilde{\Omega})$ we denote the divisor of $\tilde{\Omega}$. Clearly, $(\tilde{\Omega}) = p_{\mathcal{C}}^*(\omega) + \tilde{\Gamma}$. So

$$\begin{aligned}(\tilde{\Omega})^2 &= 2\tilde{\Gamma} \cdot p_{\mathcal{C}}^*(\omega) + \tilde{\Gamma}^2 \\ &= 2\Gamma \cdot p_B^*(\omega) + \tfrac{1}{2}\Gamma^2.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\Gamma \cdot p_B^*(\omega) &= \deg(f)(\Gamma_{\pi_1} + \Gamma_{\pi_2}) \cdot p_F^*(\omega) \\ &= 2\deg(f)\Gamma_{\pi_1} \cdot p_F^*(\omega) \\ &= 2\deg(f)\deg(\pi_1)\deg(\omega) \\ &= 16\deg(f),\end{aligned}$$

and

$$\begin{aligned}\Gamma^2 &= \deg(f)(\Gamma_{\pi_1} + \Gamma_{\pi_2})^2 \\ &= \deg(f)(2\Gamma_{\pi_1}^2 + 2\Gamma_{\pi_1} \cdot \Gamma_{\pi_2}) \\ &= \deg(f)(2\Gamma_{\pi_1}^2) \\ &= 2\deg(f)((\pi_1 \times \mathrm{id})^*\Delta_{C_3 \times C_3})^2 \\ &= -16\deg(f).\end{aligned}$$

Hence

$$\begin{aligned}(\tilde{\Omega})^2 &= 32\deg(f) - 8\deg(f) \\ &= 24\deg(f).\end{aligned}$$

Thus for the family $\pi_{\mathcal{C}} : \mathcal{C} \to B$ we get $\deg((\pi_{\mathcal{C}})_*(c_1(\omega_{\mathcal{C}/B})^2)) = 24\deg(f)$. The degree of $\lambda_{\mathcal{C}/B}$ on B follows from the relation $12\lambda = (\pi_{\mathcal{C}})_*(c_1(\omega_{\mathcal{C}/B})^2)$, which is a consequence of the Grothendieck-Riemann-Roch formula. Hence

$$\deg(\lambda_{\mathcal{C}/B}) = 2\deg(f) = 2^7.$$

We show that this is equal to the degree of the class $\lambda_{\mathcal{P}/B}$ corresponding to the family of Pryms $\pi_{\mathcal{P}} : \mathcal{P} \to B$. By definition, $\lambda_{\mathcal{P}/B}$ is the first Chern class of the vectorbundle $(\pi_{\mathcal{P}})_*(\Omega^1_{\mathcal{P}/B})$ on B. We have the splitting $\Omega^1_{J\mathcal{C}/B} = \Omega^1_{\mathcal{P}/B} \oplus \Omega^1_{\mathrm{Jac}(C_3)/B}$, so that

$$(\pi_J)_*(\Omega^1_{J\mathcal{C}/B}) = (\pi_{\mathcal{P}})_*(\Omega^1_{\mathcal{P}/B}) \oplus H^0(C_3, \omega_{C_3}).$$

On the other hand we have $(\pi_J)_*(\Omega^1_{J\mathcal{C}/B}) \cong (\pi_{\mathcal{C}})_*(\omega_{\mathcal{C}/B})$. We conclude that $\lambda_{\mathcal{P}/B} = \lambda_{\mathcal{C}/B}$. In particular $\deg(\lambda_{\mathcal{P}/B}) = 2^7$. □

Proposition 3.5 *The number of hyperelliptic fibers of $\pi_{\mathcal{P}} : \mathcal{P} \to B$ is $2^8 \cdot 9$.*

PROOF. Let $\mathcal{L}_b = \mathcal{L}|_{b \times C_3}$. Here $\mathcal{L}$ is the line bundle on $B \times C_3$ used to construct the cover $\mathcal{C} \to B \times C_3$. The cover $\mathcal{C}_b/C_3$ is determined by the data $(C_3, f(\pi_1(b)) + f(\pi_2(b)), \mathcal{L}_b)$. By Theorem 2.1 $\mathrm{Prym}(\mathcal{C}_b/C_3)$ is hyperelliptic if and only if there is a $p \in C_3$ for which the divisor $D = f(\pi_1(b)) + f(\pi_2(b)) + 2p$ is canonical and the theta characteristic $\mathcal{L}_b^{-1}(D-p)$ is even.

The first condition depends only on the image $f(b) \in F$. Let

$$T = \{\, x \in F \mid \mathcal{O}_{C_3}(\pi_1(x) + \pi_2(x) + 2p) \cong \omega_{C_3} \text{ for some } p \in C_3 \,\}.$$

For $x \in T$ let D be the unique canonical divisor $\pi_1(b) + \pi_2(b) + 2p$. The set $\{\, \mathcal{L}_b \mid y \in f^{-1}(x) \,\}$ consists of all $2^{2g} = 64$ distinct roots of $\pi_1(x) + \pi_2(x)$ in $\mathrm{Pic}(C_3)$. It follows that the set $\{\, \mathcal{L}_b^{-1}(D-p) \mid b \in f^{-1}(x) \,\}$ consists of 64 distinct theta characteristics, of which 36 are even and 28 are odd. Hence $h = 36 \cdot \#T$. The proposition follows from the following lemma. □

Lemma 3.6 *The cardinality of T equals 64.*

PROOF. Consider the curve $\tilde{F}=\tilde{F}(t) = \{ (x,p) \mid p \in \mathrm{Supp}(K(x) - \pi_1(x) - \pi_2(x)) \}$ in $F \times C_3$. $\tilde{F}$ is smooth, since F and C_3 are. Clearly T is the branch locus of the projection $\tilde{F} \to F$. We compute $\#T$ by computing the genus of $\tilde{F}$.

Let d be the degree of the projection of F onto F and d' be the degree of the projection of $\tilde{F}$ onto C_3. By definition $d = 2$.

We compute d' first by computing this degree not for $\tilde{F}(t)$ but for $\tilde{F}(0)$ instead. Note that $\tilde{F}(0) \subset C_3 \times C_3$ is reducible: $\tilde{F}(0) = \Delta + \Delta'$. Here $\Delta' \subset C_3 \times C_3$ denotes the graph of i_ϕ, the involution interchanging the sheets of $\phi : C_3 \to E$. Then $\tilde{F}(0)$ consists of the disjoint union of the tangential correspondence

$$\{ (p,q) \mid q \text{ on tangent line to } q \} \subset C_3 \times C_3$$

and the correspondence

$$\{ (p,q) \mid p + i_\phi(p) + q + i_\phi(q) \in |K_{C_3}| \} \subset C_3 \times C_3.$$

It follows that $d' = 10 + 2 = 12$.

Denote by E a 'horizontal' fiber $F \times \{c\}$ and by F a 'vertical' fiber $\{y\} \times C_3$ in $F \times C_3$. Let Γ_{π_i} be the graph of the maps $\pi_i : F \to C_3$, $i = 1, 2$. Then we have (see [24], p. 285):

$$\tilde{F} \sim (d+2)E + (d'+4)F - \Gamma_{\pi_1} - \Gamma_{\pi_2} = 4E + 16F - \Gamma_{\pi_1} - \Gamma_{\pi_2}.$$

Here $\sim$ denotes linear equivalence. Furthermore, $\Gamma_{\pi_1}^2 = \Gamma_{\pi_2}^2 = 2 \cdot \Delta_{C_3}^2 = -8$ and $\Gamma_{\pi_1} \cdot \Gamma_{\pi_2} = 0$ by construction, and $K_{F \times C_3} \sim 4E + 16F$. From the adjunction formula it follows that $2p_a(\tilde{F}) - 2 = 96$, so in particular $\tilde{F}$ is irreducible. Applying the Riemann-Hurwitz formula to the projection $\tilde{F} \to F$ yields the number of points of T. It equals $\deg(K_{\tilde{F}}) - d \cdot \deg(K_{C_3}) = 96 - 2 \cdot 16 = 64$. □

These results enable us to determine the intersection of the families $\mathcal{T} \to \tilde{B}$ with the hyperelliptic locus. In the Chow group $A^1(M_3)$ (with $\mathbf{Q}$-coefficients) denote by $[\mathcal{H}3]_Q$ the Q-class of the hyperelliptic locus, and by λ the class $c_1(\pi_*(\omega_{C_3/M_3}))$ (see [41]). In $A^1(M_3)$ we have that $[\mathcal{H}3]_Q = 9\lambda$ (see [25]). Let $[\tilde{B}]_Q$ be the Q-class of the image of $\tilde{B}$ in M_3. Then

$$\deg([\tilde{B}]_Q{\cdot}[\lambda]_Q) = \deg(\lambda|_{\tilde{B}}) = 2\deg(\lambda|_B) = 2^8.$$

The relation $[\mathcal{H}3]_Q = 9\lambda_Q$ implies that $\deg([\tilde{B}]_Q \cdot [\mathcal{H}3]_Q) = 9 \cdot 2^8$. We have $9 \cdot 2^8$ hyperelliptic fibers on $\tilde{B}$ by Corollary 3.3. If we let $b \in \tilde{B}$ be a point such that the fiber $\mathcal{T}_b$ is hyperelliptic, it follows that the intersection multiplicity $i([\tilde{B}]_Q, [\mathcal{H}3]_Q; b) = 1$.

This multiplicity is computed on the base of the versal deformation space of $\mathcal{T}_b$. The tangent space to the versal deformation space in the point corresponding to $\mathcal{T}_b$ is $H^1(\mathcal{T}_b, \Theta_b)$, where Θ_b is the tangent bundle of $\mathcal{T}_b$. The tangent direction of $\tilde{B}$ in $H^1(\mathcal{T}_b, \Theta_b)$ is the image of the Kodaira-Spencer map $T_{\tilde{B},b} \to H^1(\mathcal{T}_b, \Theta_b)$, and the tangent space to the hyperelliptic locus is the subspace of $H^1(\mathcal{T}_b, \Theta_b)$ invariant under the hyperelliptic involution on $H^1(\mathcal{T}_b, \Theta_b)$. So we conclude that the image of the Kodaira-Spencer map in $H^1(\mathcal{T}_b, \Theta_b)$ is not contained in the invariant subspace of the hyperelliptic involution acting on $H^1(\mathcal{T}_b, \Theta_b)$. However, we don't know how to prove this directly.

3.5 The surjectivity of the Prym map

Let $\mathcal{R}_{3,2}$ be the moduli space of tuples $(C, p+p', L)$, where C is a smooth curve of genus 3, $p+p'$ is an effective divisor of degree two on C such that $p \neq p'$, and L is a line bundle on C such that $L^2 \cong \mathcal{O}_C(p+p')$. Such a tuple $(C, p+p', L)$ determines a double cover of smooth curves $\tilde{C} \to C$ to which we can associate its Prym variety $\mathrm{Prym}(C, p+p', L)$. This operation yields a morphism $\mathcal{P}r : \mathcal{R}_{3,2} \to \mathcal{A}_3$, where $\mathcal{A}_3$ is the moduli space of principally polarized abelian varieties of dimension 3. We call this the *Prym* map.

Let $\mathcal{R}_{3,2}^{ell} \subset \mathcal{R}_{3,2}$ be the locus of tuples $(C, p+p', L)$ with the property that C admits a map of degree 2 to an elliptic curve. We will denote the restriction of the Prym map to $\mathcal{R}_{3,2}^{ell}$ also by $\mathcal{P}r$. The aim of this section is to show that $\mathcal{P}r : \mathcal{R}_{3,2}^{ell} \to \mathcal{A}_3$ is dominant. Note that the dimension of $\mathcal{R}_{3,2}$ is 8 and that both $\mathcal{R}_{3,2}^{ell}$ and $\mathcal{A}_3$ are of dimension 6. Hence if the morphism $\mathcal{P}r : \mathcal{R}_{3,2}^{ell} \to \mathcal{A}_3$ is dominant, it is also generically finite.

We compute the codifferential of $\mathcal{P}r$. After taking appropiate finite level structures, we may assume that $\mathcal{R}_{3,2}^{ell}$ and $\mathcal{A}_3$ are smooth. Since $\mathcal{R}_{3,2}$ is an unramified cover of $M_{3,2}$, we identify the cotangent spaces at the points $(C, p+p', L) \in \mathcal{R}_{3,2}$ and $(C, p+p') \in M_{3,2}$. The identification of the cotangent space $T^*_{M_3,(C,p+p')}$ with $H^0(C, \omega_C^2(p+p'))$ thus gives us $T^*_{\mathcal{R}_{3,2},(C,p+p',L)} \cong H^0(C, \omega_C^2(p+p'))$.

Let (A, Θ) be a point of $\mathcal{A}_3$. We identify its cotangent space $T^*_{\mathcal{A}_3,(A,\Theta)}$ with $S^2T^*_{A,0}$, the symmetric square of the cotangent space to A at the origin, by standard identifications. In our case $(A, \Theta) = \mathrm{Prym}(C, p+p', L)$. From the definition of $\mathrm{Prym}(\tilde{C}/C)$ as the odd part of $\mathrm{Jac}(\tilde{C})$ (see [39]) and the splitting $H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes L)$ it follows that $T^*_{\mathrm{Prym}(C,p+p',L),0} \cong H^0(C, \omega_C \otimes L)$.

Using these identifications, the codifferential

$$\mathcal{P}r^* : S^2H^0(C, \omega_C \otimes L) \to H^0(C, \omega_C^2(p+p'))$$

is the composition of two natural maps: the cup-product $S^2H^0(C, \omega_C \otimes L) \to H^0(C, \omega_C^2 \otimes L^2)$ followed by the identification $H^0(C, \omega_C^2 \otimes L^2) \cong H^0(C, \omega_C^2(p+p'))$ induced by the isomorphism $L^2 \cong \mathcal{O}_C(p+p')$.

The codifferential is computed in [6] for the Prym map $\overline{\mathcal{R}}_{g+1} \to \mathcal{A}_g$, where $\overline{\mathcal{R}}_{g+1}$ is the moduli space of "admissible" double covers. We obtain the above result by embedding $\mathcal{R}_{3,2}$ into $\overline{\mathcal{R}}_4$ via $(C, p+p', L) \mapsto (C/(p \sim p'), L)$.

Suppose that C admits an elliptic involution, i.e., an automorphism of order two such that the quotient is an elliptic curve. Then $H^0(C, \omega_C^2)$ decomposes into (± 1)-eigenspaces $H^0(C, \omega_C^2)^{\pm}$. Embed the space $H^0(C, \omega_C^2)$ into $H^0(C, \omega_C^2(p+p'))$. Then the cotangent space $T^*(\mathcal{R}_{3,2}^{ell})_{(C,p+p',L)}$ to $\mathcal{R}_{3,2}^{ell}$ at $(C, p+p', L)$ can be identified with the quotient $H^0(C, \omega_C^2(p+p'))/H^0(C, \omega_C^2)^-$. The codifferential of $\mathcal{R}_{3,2}^{ell} \to \mathcal{A}_3$ in the point $(C, p+p', L)$ is the composition

$$S^2(H^0(C, \omega_C \otimes L)) \to H^0(C, \omega_C^2(p+p')) \to H^0(C, \omega_C^2(p+p'))/H^0(C, \omega_C^2)^- \quad (*)$$

We compute this codifferential in one specific point of $\mathcal{R}_{3,2}^{ell}$. Let C_{48} be the plane quartic curve given by the equation $X^4 + Y(Y^3 + Z^3) = 0$. It has an automorphism group of order 48 and admits the elliptic involution $X \mapsto -X, Y \mapsto Y, Z \mapsto Z$. This curve has 4 hyperflexes and 16 ordinary flexes. Let $q = (0, 1, -1)$, $p = (0, 0, 1)$ and $p' = p'' = (1, \eta, 0)$, with η a fourth root of -1. The points p and q are hyperflexes and $p' = p''$ is a ordinary flex. If we let L be the line bundle $\mathcal{O}_{C_{48}}(2q - p'')$, the tuple $(C_{48}, p + p', L)$ is a point of $\mathcal{R}^{ell}_{3,2}$.

Lemma 3.7 *The codifferential of the Prym map $\mathcal{P}r : \mathcal{R}^{ell}_{3,2} \to \mathcal{A}_3$ in the point $(C_{48}, p + p', L)$ has maximal rank.*

PROOF. The canonical embedding of $C_{48} \subset \mathbf{P}^2$ and the isomorphism $H^0(C_{48}, \omega \otimes L) \cong H^0(C_{48}, \omega^2(-2q - p''))$ enable us to write down an explicit base for the space $H^0(C_{48}, \omega \otimes L)$. This gives six generators for the image of $S^2 H^0(C_{48}, \omega \otimes L)$ in $H^0(C_{48}, \omega^2(p + p'))$. Embedding $H^0(C_{48}, \omega^2(p + p')) \cong H^0(C_{48}, \omega^3(-2p''))$ in $H^0(C_{48}, \omega^3) \cong H^0(\mathbf{P}^2, O(3))$ we get six homogeneous forms of degree 3.

Also we embed $H^0(C_{48}, \omega^2)^-$ in $H^0(\mathbf{P}^2, O(3))$ and write down a base. In fact, if $L_{p''}$ is the equation of the flex line through p'' then the two forms $XYL_{p''}$ and $XZL_{p''}$) form a base. A straightforward calculation shows that these eight forms are linearly independent in $H^0(\mathbf{P}^2, \mathcal{O}(3))$. □

Corollary 3.8 *The Prym map $\mathcal{P}r : \mathcal{R}^{ell}_{3,2} \to \mathcal{A}_3$ is dominant and generically finite.*

Corollary 3.9 *The generic curve of genus three occurs as a fiber of a family $\mathcal{T} \to \tilde{B}$ appearing in Corollary 1.2.*