Complete subvarieties of moduli spaces of algebraic curves

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Publication date
2005

Citation for published version (APA):
CHAPTER 3

Explicit complete curves in the moduli space of curves of genus three

This chapter appeared as Geometriae Dedicata 56: 185–196 (1995) and appears here with minor modifications.

Consider $M_g$, the moduli space of smooth curves of genus $g$ over the field of complex numbers $\mathbb{C}$. For $g \geq 2$, $M_g$ is a quasi-projective variety of dimension $3g - 3$. Note that $M_g$ is not complete as smooth curves can degenerate to singular ones. In fact, $M_2$ is affine. However, $M_g$ contains complete curves if $g \geq 3$. This follows from the existence of a projective compactification $\overline{M_g}$ of $M_g$ in which the boundary $\overline{M_g} \setminus M_g$ has codimension $\geq 2$ (take the closure of the image of $M_g$ in the Satake compactification of $A_g$, the moduli space of principally polarized abelian varieties).

The complete curves are obtained by cutting $\overline{M_g}$ with sufficiently many hypersurfaces in general position. An upper bound for the dimension of a complete subvariety of $M_g$ is $g - 2$ if $g \geq 2$ (see [14]). So these complete curves achieve this bound if $g = 3$.

Harris (see [25]) notes that these curves are not very explicit, although constructions of explicit complete families of smooth curves are known. In [17] an explicit one-dimensional family is given for every genus $g \geq 4$, but for $g = 3$ a less explicit family is exhibited. The aim of this note is to produce explicit examples of complete families of smooth curves of genus 3 having a moduli theoretic interpretation.

In short the construction is the following. Fix a smooth base curve $C_3$ of genus 3 and fix a complete curve $F \subset C_3 \times C_3 \setminus \Delta$. Construct a complete family of smooth double covers $(C_f \to C_3)_{f \in F}$, where $\pi_f$ is branched over the two points determined by $f \in F$ (in fact such a family may exist only over a finite cover of $F$ due to monodromy obstructions). To the covers $C_f \to C_3$ we can associate their Prym varieties. We obtain a complete family of 3-dimensional principally polarized abelian varieties $\text{Prym}(C_f/C_3)_{f \in F}$, which turn out to be Jacobians of smooth curves if $C_3$ is not hyperelliptic. So we get a complete family of Jacobians of smooth genus 3 curves.

In the first section we present a specific construction of complete families of 3-dimensional principally polarized abelian varieties, depending on five parameters. The base is 1-dimensional and the fibers are Prym varieties of branched double covers of curves. We will see in section 2 that these fibers are actually Jacobians of smooth curves. The corresponding families of curves are then constructed via the so-called “trigonal construction”. Section 3 contains a calculation of invariants of these families, in particular the number of hyperelliptic fibers. Finally, we show that these families are general in the following sense: the generic smooth curve of genus 3 occurs as a fiber of one of the families.
3.1 The construction

Fix an elliptic curve $E = (E,0)$. Let $\phi : C_3 \to E$ be a double cover ramified at 4 points of $E$, so that the genus $C_3$ is 3. We want $C_3$ to be non-hyperelliptic. This turns out to be an open condition on the four branch points: let $B$ the branch divisor, let $\tilde{B}$ the ramification divisor of $\phi$, and let $L$ denote the unique line bundle on $E$ satisfying $\phi^*L \cong \mathcal{O}_{C_3}(\tilde{B})$. Then $C_3$ is hyperelliptic if and only if $B = B_1 + B_2$ for some $B_1, B_2 \in |L|$ (if $C_3$ is hyperelliptic, then $\sigma$ acts freely on $\text{Supp}(B)$ and $L = \phi_*(x + \sigma x)$ for an $x \in \text{Supp}(\tilde{B})$).

Fix a point $t \neq 0$ in $E$. Set $\Delta_t = \{(x, x + t) \mid x \in E\} \subset E \times E$. Let $F = F(t) \subset C_3 \times C_3$ be the inverse image of $\Delta_t$ under the natural map $\phi \times \phi : C_3 \times C_3 \to E \times E$. Then in $C_3 \times C_3$ we have that $F \cap \Delta = \emptyset$, thus $F$ parametrizes pairs of distinct points on $C_3$. If we suppose—as we shall do in the following—that $t \notin \{\phi(p) - \phi(q) \mid p, q$ ramification points of $\phi\}$, then one easily verifies that $F$ is smooth of genus 9.

Denote by $\pi_1, \pi_2 : F \to C_3$ the maps induced by the projections of $C_3 \times C_3$ onto the first, respectively the second coordinate, and denote by $\Gamma_{\pi_i} \subset F \times C_3$ the graph of $\pi_i$, for $i = 1, 2$. We want to have a double cover of $F \times C_3$ ramified precisely over $\Gamma_{\pi_1} + \Gamma_{\pi_2}$. Such a cover may not exist due to monodromy obstructions. To overcome these, we consider the natural map $F \to \text{Pic}^2(C_3), x \mapsto [\pi_1(x) + \pi_2(x)]$, and the squaring map $sq : \text{Pic}^1(C_3) \to \text{Pic}^2(C_3), L \mapsto L \otimes L$. Define the curve $B$ as the fibered product $B = F \times_{\text{Pic}^2(C_3)} \text{Pic}^1(C_3)$.

$$
\begin{array}{ccc}
B & \to & \text{Pic}^1(C_3) \\
\downarrow f & \downarrow \downarrow sq & \\
F & \to & \text{Pic}^2(C_3)
\end{array}
$$

Denote the map $B \to F$ by $f$, and set $\Gamma = \Gamma_{\pi_1} + \Gamma_{\pi_2} \subset B \times C_3$. The pull-back to $B \times C_3$ of the universal line bundle over $\text{Pic}^1(C_3) \times C_3$ gives a line bundle $\mathcal{L}$ on $B \times C_3$ satisfying $\mathcal{L}^2 \cong \mathcal{O}_{B \times C_3}(\Gamma)$.

Using this $\mathcal{L}$ we make a double cover $C \to B \times C_3$, ramified precisely along $\Gamma$. This is a well-known construction: embed $B \times C_3$ in the total space of the line bundle corresponding to $\mathcal{O}_{B \times C_3}(\Gamma)$ with a section $s$ of $\mathcal{O}_{B \times C_3}(\Gamma)$ which vanishes precisely once along $\Gamma$. Define $\mathcal{C}$ as the inverse image of $s(B \times C_3)$ under the natural squaring map on local sections $\mathcal{L} \overset{sq}{\to} \mathcal{L}^2 \cong \mathcal{O}_{B \times C_3}(\Gamma)$. Here $\mathcal{L}$ denotes the total space of the line bundle $\mathcal{L}$, by abuse of notation.

$$
\begin{array}{ccc}
\mathcal{C} & \to & \mathcal{L} \\
\downarrow & \downarrow & \cong \mathcal{O}_{B \times C_3}(\Gamma) \\
B \times C_3 & \to & B \times C_3
\end{array}
$$

The composition with the projection $B \times C_3 \to B$ yields a complete family of curves $\pi^*_C : \mathcal{C} \to B$. Let $C_b$ denote the fiber of $\pi^*_C : \mathcal{C} \to B$ over $b \in B$. Each $C_b$ is smooth of genus 6 and maps $2 : 1$ to $C_3$ with two ramification points, via the composition $\mathcal{C} \to B \times C_3 \to C_3$. Note that the map $f : B \to F$ is étale of degree 64, so that $B$ is a smooth curve, but possibly reducible. Thus $\pi^*_C : \mathcal{C} \to B$ is a complete, possibly reducible family of smooth curves.
We can associate to every double covering $C_b \to C_3$ its Prym variety (see [39]). This is a 3-dimensional, principally polarized abelian variety $\text{Prym}(C_b/C_3) = (P, \Xi)$. $P$ is defined as the kernel of the norm map $\text{Nm} : \text{Jac}(C_b) \to \text{Jac}(C_3)$ (which is connected since $C_b \to C_3$ is ramified). The Jacobian of $C_3$ injects in the Jacobian of $C_b$, so up to isogeny $\text{Jac}(C_b)$ splits as the product of $\text{Jac}(C_3)$ and another factor, which is $P$. The polarization on $\text{Jac}(C_b)$ gives rise to a polarization of $P \times \text{Jac}(C_3)$: $P \times \text{Jac}(C_3) \to \widehat{\text{Jac}(C_b)} \to \widehat{P \times \text{Jac}(C_3)}$. This polarization splits and induces on $P$ twice a principal polarization $\Xi$.

It is straightforward to globalize this construction. The family $\pi_C : \mathcal{C} \to B$ has sections. Thus we have the associated family $\pi_C : \mathcal{J} \to B$ of Jacobians $\mathcal{J} \to B$. Consider the norm map $\text{Nm} : \mathcal{J} \to B \times \text{Jac}(C_3)$, induced by the usual norm map $\text{Nm} : \mathcal{J} \to B \times \text{Jac}(C_3)$. Define $\mathcal{P}$ as $\text{Nm}^{-1}(B \times \{0\}) \subset \mathcal{J}$. The projection $\mathcal{J} \to B$ yields a family $\pi_{\mathcal{P}} : \mathcal{P} \to B$ of 3-dimensional principally polarized abelian varieties.

We need Theorem 2.1. The theorem says that the 1-dimensional family of Pryms $\mathcal{P} \to B$ is in fact a family of Jacobians of smooth curves. Note that here we use that $C_3$ is non-hyperelliptic. For readability the statement and proof of this theorem is postponed to the next section. Summarizing, we have the following result:

**Corollary 3.1** The above construction yields a complete, one-dimensional family $\pi_C : \mathcal{P} \to B$ of 3-dimensional Jacobians of smooth curves.

Note that the construction depends on five parameters: namely on $E$, on the double cover $C_3 \to E$ and on the point $t \in E$. Thus we obtain in fact a five-dimensional family of complete, one-dimensional families of 3-dimensional Jacobians.

### 3.2 The corresponding family of curves

We will exhibit the family of smooth curves corresponding to the family of Jacobians $\pi_{\mathcal{P}} : \mathcal{P} \to B$, using the so-called trigonal construction. We explain this first for a single Prym variety $\text{Prym}(C_b/C_3)$. For the details and proofs we refer to [18]. The map $C_b \to C_3$ has two branch points, which we denote by $p, p' \in C_3$. Consider $C_3$ canonically embedded in $|K_{C_3}|^\ast \cong \mathbb{P}^2$. The line through $p$ and $p'$ cuts out on $C_3$ two other points. Choose one of them and call it $p''$. Let $\mathbb{P}^1$ denote the linear system $|K_{C_3} - p''|$. Projecting from $p''$ determines a three-to-one map $C_3 \to \mathbb{P}^1 \cong \mathbb{P}^1$.

Assume that this map does not ramify at $p$ or $p'$. Let $C_b^{(3)}$ (respectively $C_3^{(3)}$) denote the threefold symmetric product of $C_b$ (respectively $C_3$). The line $\mathbb{P}^1$ embeds naturally in $C_b^{(3)}$ (respectively $C_3^{(3)}$). Consider the inverse image of $\mathbb{P}^1 \subset C_b^{(3)}$ under the natural map $C_b^{(3)} \to C_3^{(3)}$. This is a curve with two smooth, isomorphic components $T_0, T_1$. Their intersection consists of two distinct points $q, q'$, in which they meet transversally. Both curves map with degree four to $\mathbb{P}^1$ and $q, q'$ are simple ramification points on each curve mapping to the same point on $\mathbb{P}^1$. Apart from $q$ and $q'$ the ramification on the tetragonal curves $T_0$ and $T_1$ arises from the ramification of the $g_9^1$ on $C_3$. Namely if $C_3 \to \mathbb{P}^1$ is simply branched over $P \in \mathbb{P}^1$, then also the maps $T_i \to \mathbb{P}^1$ are simply branched over $P$. If $C_3 \to \mathbb{P}^1$ is completely branched over $P \in \mathbb{P}^1$ then the maps $T_i \to \mathbb{P}^1$ have a point of ramification index 3 over $P$. In particular the genera of the $T_i$ are 3.
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If on the other hand the triple cover $C_3 \to \mathbf{P}^1$ is ramified at $p$ or $p'$, then the situation is different. The points $q$ and $q'$ coincide, i.e., the inverse image still has two smooth isomorphic components $T_0$ and $T_1$ which meet in one point $q$, and the fourfold cover $T_i \to \mathbf{P}^1$ is totally ramified in $q$.

The curves $T_0$ and $T_1$ map naturally to $\text{Pic}^3(C_b)$, since they live in $\mathcal{C}_b^{(3)}$. The norm map $\text{Nm} : \text{Pic}^3(C_b) \to \text{Pic}^3(C_3)$ maps their images to the image of $\mathbf{P}^1$ in $\text{Pic}^3(C_3)$, which is a point. The images of $T_0$ and $T_1$ in $\text{Pic}^3(C_b)$ lie thus in a suitable translate of $P = \ker(\text{Nm} : \text{Jac}(C_b) \to \text{Jac}(C_3))$. Moreover, the maps $T_i \to \text{Pic}^3(C_b)$, $i = 0, 1$, induce isomorphisms of principal polarized abelian varieties $\text{Jac}(T_i) \to \text{Prym}(C_b/C_3)$. The trigonal construction goes back to Recillas and was generalized by Donagi.

To globalize this construction reconsider the curve $F \subset C_3 \times C_3$. Every point $x \in F = F(t)$ determines two distinct points $\pi_1(x), \pi_2(x) \in C_3$, and the line spanned by these points in the canonical embedding of $C_3$ we denote by $\ell(x)$. Let $K(x)$ denote the hyperplane section $C_3 \cdot \ell(x)$. Consider the curve

$$\tilde{F} = \tilde{F}(t) = \{ (x,c) \in F \times C_3 \mid c \in \text{Supp}(K(x) - \pi_1(x) - \pi_2(x)) \}.$$  

Then $\tilde{F}$ is smooth since $F$ and $C_3$ are. Clearly $\tilde{F}$ maps with degree two onto $F$. This map is ramified in 64 points (see Lemma 3.3 for the computation), hence $\tilde{F}$ is irreducible of genus 49. A point $(x, p'')$ of $\tilde{F}$ determines on $C_3$ a $g^1_4$, namely $|K_{C_3} - p''|$, plus two unordered, distinct points contained in a divisor of this $g^1_3$, namely $\pi_1(x), \pi_2(x)$. $\tilde{F}$ maps to $\text{Pic}^2(C_3)$ via the composition $\tilde{F} \to F \to \text{Pic}^3(C_3)$, and also $\text{Pic}^1(C_3)$ does via the squaring map $\text{Pic}^1(C_3) \to \text{Pic}^2(C_3)$. Let $\tilde{B} = \tilde{F} \times_{\text{Pic}^3(C_3)} \text{Pic}^1(C_3)$. Note that $\tilde{B}$ maps two-to-one to $B$.

The pull-back of $C \to B$ to $\tilde{B}$ yields a double cover $\tilde{C} \to \tilde{B} \times C_3$. The composition $\tilde{C} \to \tilde{B} \times C_3 \to \tilde{B}$ is a one-dimensional family of double covers. The extra structure is a three-to-one map $\tilde{B} \times C_3 \to S$, where $S$ is a ruled surface $S \to \tilde{B}$ with fiber $S_b = |K_{C_3} - p''|$. Then $S$ embeds in $\tilde{B} \times C_3^{(3)}$ and its inverse image along the natural map $\tilde{C}_{\tilde{B}}^{(3)} \to (\tilde{B} \times C_3)^{(3)}_{\tilde{B}} = \tilde{B} \times C_3^{(3)}$ we denote by $T'$. The (singular) surface $T'$ has an involution which preserves the fibers of $T' \to S$ setwise, and its quotient by this involution we denote by $T$. The surface $T$ is smooth. The natural map $T \to S$ composed with the projection $S \to \tilde{B}$ yields a family $T \to \tilde{B}$ of smooth, tetragonal curves of genus 3.

**Corollary 3.2** The above construction gives a complete, one-dimensional family $T \to S \to \tilde{B}$ of smooth, tetragonal curves of genus 3.
### 3.3 Pryms of ramified double coverings of genus 3 curves

This section is devoted to the proof of the theorem below. It shows that the Prym variety associated to a ramified double covering $\tilde{Z}/Z$ ramified in two points, where the genus of $Z$ is 3, is a Jacobian of a smooth curve as long as the base curve is not hyperelliptic.

In the following let $Z$ be a smooth genus 3 curve, and let $\pi_Z : \tilde{Z} \to Z$ be a double cover of smooth curves, with two branch points $p, p' \in Z$ and ramification points $\tilde{p}, \tilde{p}' \in \tilde{Z}$. Associated to this cover is a (unique) line bundle $L_{\tilde{Z}/Z}$ satisfying $(\pi_Z)^*(L_{\tilde{Z}/Z}) \cong \mathcal{O}_Z(\tilde{p} + \tilde{p}')$. The data $(Z, p + p', L_{\tilde{Z}/Z})$ determine the cover $\pi_Z : \tilde{Z} \to Z$ up to isomorphism. The Prym variety $P = \text{Prym}(\tilde{Z}/Z)$ is a 3-dimensional abelian variety which comes with a natural principal polarization, denoted by $\Xi$.

**Theorem 3.3**

1. If $Z$ is hyperelliptic, and if $p, p'$ are interchanged by the hyperelliptic involution of $Z$, then $(P, \Xi)$ is the Jacobian of a smooth irreducible hyperelliptic curve, or a product of two such.
2. If $Z$ is hyperelliptic and $p, p'$ are not interchanged by the hyperelliptic involution of $Z$, then $(P, \Xi)$ is a Jacobian of a smooth irreducible hyperelliptic curve.
3. If $Z$ is not hyperelliptic, then $(P, \Xi)$ is a Jacobian of a smooth irreducible curve. This curve is hyperelliptic if and only if for some $p'' \in Z$, $p + p' + 2p''$ is a canonical divisor and $h^0(L_{\tilde{Z}/Z}^{-1}(p + p' + p'')) = 0$.

**Proof.** Let $W = Z/(p \sim p')$ be the curve obtained from $Z$ by identifying $p$ and $p'$, and let $\tilde{W} = \tilde{Z}/(\tilde{p} \sim \tilde{p}')$. Denote by $f_N : Z \to W$ the normalisation mapping. Note that $p_a(W) = 4$, $p_a(\tilde{W}) = 7$.

The idea is to use the theory from [6]: the induced map $\pi_W : \tilde{W} \to W$ is an admissible covering in the sense of Beauville, and $\text{Prym}(\tilde{Z}/Z) = \text{Prym}(\tilde{W}/W)$. Here the right hand side is the Prym variety associated by Beauville to the covering $\tilde{W}/W$. It lives canonically in

$$J\tilde{W}^* = \{\text{line bundles } L \text{ on } \tilde{W} \mid 2 \deg L = \deg \omega_{\tilde{W}}\}$$

as $P = \text{Prym}(\tilde{W}/W) = \{L \text{ on } \tilde{W} \mid \text{Nm}L \cong \omega_{W} h^0(L) \text{ even}\}$, and the polarization $\Xi$ then is $\Xi = \{L \in P \mid h^0(L) \geq 2\}$.

To prove the theorem, we will show that, unless we are in case (1), $\dim \text{Sing} \Xi \leq 0$. Since $\dim P = 3$, this shows that in the cases (2) and (3) $(P, \Xi)$ is a Jacobian. If $\text{Sing} \Xi$ is a point $L \in \Xi$—in which case $(P, \Xi)$ is a hyperelliptic Jacobian—we show how $L$ arises from the geometry of $Z$.

Note that the proof is an adaptation of the proof of Theorem 4.10 in [6]. Also note that case (1) is already treated by Beauville (see [6], p. 171).

Start by assuming that $\text{Sing} \Xi$ is non-empty, say $L \in \text{Sing}(\Xi)$. We will show that in the cases (2) and (3) $L \in \Xi$ is a uniquely determined point. By [6], Lemma 4.1, either $h^0(L)$ is $\geq 4$ and even or $h^0(L) = 2$, and $s^*s \otimes t = s \otimes t^*t$ for a base \{s, t\} of $H^0(W, L)$, where $t : \tilde{W} \to W$ is the involution interchanging the sheets of $\pi_W : \tilde{W} \to W$. 


First suppose that there is an \( L \in \text{Sing}(\Xi) \), with \( h^0(L) \geq 4 \) and \( h^0(L) \) even. Let \( L_1 \) be the pull-back of \( L \) to \( \tilde{Z} \). Then \( \deg(L_1) = 6 \), and \( h^0(L_1) \geq 4 \). From Clifford's theorem it follows that \( \tilde{Z} \) is hyperelliptic. The hyperelliptic involution \( \sigma \) of \( \tilde{Z} \) commutes with \( t \). Moreover, \( \sigma \) interchanges the fixed points of \( t \), since all sections of \( L_1 \) come from sections of \( L \). Hence \( \sigma \) descends and \( Z \) is hyperelliptic. If we denote the hyperelliptic involution of \( Z \) also with \( \sigma \), we see that \( \sigma \) interchanges \( p, p' \). So we are in case (1). This is the case in which \( W \) has a so-called non-singular \( g_2^1 \).

Non-singular means that the \( g_2^1 \) contains a divisor with non-singular support. It follows that \((P, \Xi)\) is a product of hyperelliptic Jacobians (see [6], p. 171).

Secondly, suppose that there is an \( L \in \text{Sing}(\Xi) \) with \( h^0(L) = 2 \), with a base \( \{s, t\} \) for \( h^0(L) \) satisfying \( s \otimes t = t^*s \otimes t \).

We first assume that \( s, t \) have the property that either \( s \) or \( t \) is not zero at the singular point. By [6], lemma 4.4, \( L \) is of the form \((\pi_W)^*M(E)\). Here \( M \) is a line bundle on \( W \), \( h^0(M) \geq 2 \), \( \deg((\pi_W)^*M(E)) = \deg(L) = \deg(\omega_W) = 6 \). \( M \) is non-singular, and \( E \) is an effective divisor on \( W \) with non-singular support such that \((\pi_W)_*E \subset \omega_W \otimes M^-2\).

Let \( M_1 \) be the pull-back of \( M \) to \( \tilde{Z} \). Then \( M_1 \) gives an \( g_2^1 \) or a \( g_3^1 \) on \( Z \).

If \( Z \) is hyperelliptic, a \( g_3^1 \) on \( Z \) is the unique \( g_2^1 \) plus a point. We see that also in this case the two branch points \( p, p' \) of \( \tilde{Z} \to Z \) are interchanged by the hyperelliptic involution, and we are in case (1) again.

If \( Z \) is not hyperelliptic, then \(|M_1|\) is a \( g_3^1 \), \( E = 0 \), and the branch divisor \( p + p' \) is contained in an element of the system \(|M_1|\), say \( p + p' + p'' \in |M_1| \). The relation \( \text{Nm}_W((\pi_W)^*M) = \omega_W \) gives by pull-back \( \text{Nm}_Z((\pi_Z)^*M_1) = \omega_Z(p + p') \). It follows that \( p + p' + 2p'' \) is a canonical divisor, and \(|M_1|\) is the pencil \([K_Z - p'']\). So we are in case (3) and \( L = (\pi_W)^*M \).

Finally assume that the two sections \( s, t \) of \( L \) vanish simultaneously at the singular point. Let \( L_1 \) be the pull-back to \( \tilde{Z} \) of \( L \), and set \( L_2 = L_1(-\tilde{p} - \tilde{p}') \). Then \( \text{Nm}(L_2) = \omega_Z \), and \( L_1 \) has sections induced by \( s, t \), which we will also denote by \( s, t \), and which still satisfy \( s \otimes t = t^*s \otimes t \). Then [6], lemma 4.4 applies, giving that \( L_2 = (\pi_Z)^*(M) \), where \( M \) is a line bundle on \( Z \), \( h^0(M) \geq 2 \). It follows that \( Z \) is hyperelliptic and that \( M \) is the line bundle associated with the unique \( g_2^1 \) on \( Z \). So we are in case (2) and \( L \) is an extension of \( \pi_Z^*(M)(\tilde{p} + \tilde{p}') \) to \( \tilde{W} \).

This finishes the first part of the proof. We will show in the second part that the above procedure can be reversed to obtain precisely one singularity of \( \Xi \) if we are in case (2). If \( p \) and \( p' \) satisfy the condition of case (3) then we also obtain precisely one singularity of \( \Xi \).

First assume that we are in case (2), i.e., \( Z \) is hyperelliptic and that \( p, p' \) are not interchanged by the hyperelliptic involution. The \( g_3^1 \) on \( Z \) pulls back to a line bundle \( L_2 \) on \( Z \). All extensions of the line bundle \( L_1 = L_2(\tilde{p} + \tilde{p}') \) to a line bundle \( L \) on \( \tilde{W} \) admit sections \( s, t \) satisfying \( s \otimes t = t^*s \otimes t \). There are precisely two extensions \( L \) such that \( \text{Nm}_L = \omega_W \), and only one of these is even (see [6], Prop. 3.5). This takes care of the second case of the theorem, since we obtain indeed precisely one singularity \( L \) of \( \Xi \), showing that \((P, \Xi)\) is the Jacobian of a smooth, hyperelliptic curve.

Secondly, suppose that we are in case (3), and that we have a line bundle \( M_1 \) on \( Z \), with \( h^0(M_1) = 2 \) and \( M_1 \cong \mathcal{O}_Z(p + p' + p'') \cong \mathcal{O}_Z(K_Z - p'') \). This \( M_1 \) has a unique extension \( M \) to \( W \) such that \( h^0(M) = 2 \). Then \( \text{Nm}_W((\pi_W)^*(M)) = \omega_W \).
For $\text{Nm}_W((\pi_W)^*(M)) = M^2$ is an extension of $M^2_i = \omega_Z(p + p')$ to $W$ having a section non-zero at $p = p'$, and $\omega_W$ is the unique such extension.

The pullback $(\pi_W)^*(M)$ will yield an element of $\text{Sing}(\Xi)$ if $h^0_W((\pi_W)^*M) = 2$.

We have the decomposition

$$H^0_Z((\pi_Z)^*M_1) \cong H^0_Z(M_1) \oplus H^0_Z(M_1 \otimes L^{-1}_{2/2})$$

which is the decomposition into even and odd parts with respect to the action of the involution $\iota$ on $H^0_Z((\pi_Z)^*M_1)$. We already have seen that all sections of $H^0_Z((\pi_Z)^*M_1)^+ \cong H^0_Z(M_1)$ extend to $\tilde{W}$. Also all sections of $H^0_Z((\pi_Z)^*M_1)^- \cong H^0_Z(M_1 \otimes L^{-1}_{2/2})$ extend to $\tilde{W}$, since such a section is zero at $\tilde{p}$ and $\tilde{p}'$. It follows that $h^0_\tilde{W}((\pi_W)^*M) = h^0_\tilde{W}((\pi_Z)^*M_1)$, and that $h^0_\tilde{W}((\pi_W)^*M) = 2$ if and only if $h^0(Z_1 \otimes L^{-1}_{2/2}) = 0$. So we obtain precisely one singularity $(\pi_W)^*M$ of $\Xi$ if and only if $h^0(Z_1 \otimes L^{-1}_{2/2}) = 0$. Note that $M_1 \otimes L^{-1}_{2/2}$ is a theta characteristic: $M^2_i \otimes L^{-2}_{2/2} \cong \mathcal{O}_Z(p + p' + 2p'') \cong \omega_Z$ by assumption.

\section{3.4 The degree of $\lambda$ and the number of hyperelliptic fibers}

Consider the family $\mathcal{P} \to B$ constructed in section 1. Our aim in this section is to compute the degree of $\lambda = c_1((\pi_\mathcal{P})_*(\Omega_{\mathcal{P}/B}))$ and the number of hyperelliptic fibers of the family $\pi_\mathcal{P} : \mathcal{P} \to B$.

\textbf{Proposition 3.4} The degree of $\lambda_{\mathcal{P}/B}$ equals $2^7$.

\textbf{Proof.} Denote by $p_B$ the projection $B \times C_3 \to C_3$, by $p_C$ the projection $F \times C_3 \to C_3$, and let $p_C$ be the composition $C \to B \times C_3 \to C_3$. The branch locus of $C \to B \times C_3$ is $\Gamma = \Gamma_{\pi_1 \circ f} + \Gamma_{\pi_2 \circ f}$. Here $\Gamma_{\pi \circ f}$ denotes the graph of $\pi \circ f$ in $B \times C_3$, $i = 1, 2$. We denote the ramification locus of $C \to B \times C_3$ by $\tilde{\Gamma} = \Gamma_{\pi_1 \circ f} + \Gamma_{\pi_2 \circ f} \subset C$. Let $\omega$ be a holomorphic 1-form on $C_3$. Let $\Omega = p_B^*\omega$ (respectively $\tilde{\Omega} = p_C^*\omega$) be its pull-back to $B \times C_3$ (respectively $C$), and consider it as a section of the line bundle $\omega_{B \times C_3/B}$ (respectively $\omega_C/B$).

We compute the intersection number $(\tilde{\Omega})^2$, where by $(\tilde{\Omega})$ we denote the divisor of $\tilde{\Omega}$. Clearly, $(\tilde{\Omega}) = p_C^*(\omega) + \tilde{\Gamma}$. So

$$(\tilde{\Omega})^2 = 2\tilde{\Gamma} \cdot p_C^*(\omega) + \tilde{\Gamma}^2$$

$$= 2\Gamma \cdot p_B^*(\omega) + \frac{1}{2}\Gamma^2.$$
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Furthermore, we have

$$
\Gamma \cdot p_b^*(\omega) = \deg(f)(\Gamma_{x_1} + \Gamma_{x_2}) \cdot p_F^*(\omega)
$$

$$
= 2 \deg(f) \Gamma_{x_1} \cdot p_F^*(\omega)
$$

$$
= 2 \deg(f) \deg(x_1) \deg(\omega)
$$

$$
= 16 \deg(f),
$$

and

$$
\Gamma^2 = \deg(f)(\Gamma_{x_1} + \Gamma_{x_2})^2
$$

$$
= \deg(f)(2\Gamma_{x_1}^2 + 2\Gamma_{x_1} \cdot \Gamma_{x_2})
$$

$$
= \deg(f)(2\Gamma_{x_1}^2)
$$

$$
= 2 \deg(f)((\pi_1 \times \text{id})^* \Delta_{C_3 \times C_3})^2
$$

$$
= -16 \deg(f).
$$

Hence

$$
(\hat{\Omega})^2 = 32 \deg(f) - 8 \deg(f)
$$

$$
= 24 \deg(f).
$$

Thus for the family $\pi_C : C \to B$ we get $\deg((\pi_C)_*(\omega_{C/B})^2)) = 24 \deg(f)$. The degree of $\lambda_{C/B}$ on $B$ follows from the relation $12 \lambda = (\pi_C)_*(\omega_{C/B})^2)$, which is a consequence of the Grothendieck-Riemann-Roch formula. Hence

$$
\deg(\lambda_{C/B}) = 2 \deg(f) = 2^7.
$$

We show that this is equal to the degree of the class $\lambda_{P/B}$ corresponding to the family of Pryms $\pi_P : P \to B$. By definition, $\lambda_{P/B}$ is the first Chern class of the vectorbundle $(\pi_P)_*(\Omega_{P/B})^1)$ on $B$. We have the splitting $\Omega_{J_C/B}^1 = \Omega_{P/B}^1 \oplus \Omega_{\text{Jac}(C_3)/B}^1$, so that

$$(\pi_J)_*(\Omega_{J_C/B}^1) = (\pi_P)_*(\Omega_{P/B}^1) \oplus H^0(C_3, \omega_{C_3}).$$

On the other hand we have $(\pi_J)_*(\Omega_{J_C/B}^1) \cong (\pi_C)_*(\omega_{C/B})$. We conclude that $\lambda_{P/B} = \lambda_{C/B}$. In particular $\deg(\lambda_{P/B}) = 2^7$.

**Proposition 3.5** The number of hyperelliptic fibers of $\pi_P : P \to B$ is $2^8 \cdot 9$.

**Proof.** Let $L_b = L|_{B \times C_3}$. Here $L$ is the line bundle on $B \times C_3$ used to construct the cover $C \to B \times C_3$. The cover $C_b/C_3$ is determined by the data $(C_3, f(x_1(b)) + f(x_2(b)), L_b)$. By Theorem 2.1 Prym$(C_b/C_3)$ is hyperelliptic if and only if there is a $p \in C_3$ for which the divisor $D = f(x_1(b)) + f(x_2(b)) + 2p$ is canonical and the theta characteristic $L_b^{-1}(D - p)$ is even.

The first condition depends only on the image $f(b) \in F$. Let

$$
T = \{ x \in F \mid O_{C_3}(x_1) + x_2 + 2p \cong \omega_{C_3} \text{ for some } p \in C_3 \}.
$$

For $x \in T$ let $D$ be the unique canonical divisor $x_1(b) + x_2(b) + 2p$. The set $\{ L_b \mid y \in f^{-1}(x) \}$ consists of all $2^{2q} = 64$ distinct roots of $x_1 + x_2$ in Pic$(C_3)$. It follows that the set $\{ L_b^{-1}(D - p) \mid b \in f^{-1}(x) \}$ consists of 64 distinct theta characteristics, of which 36 are even and 28 are odd. Hence $h = 36 \cdot \#T$. The proposition follows from the following lemma. \[\square\]
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Lemma 3.6 The cardinality of $T$ equals 64.

Proof. Consider the curve $\tilde{F} = \tilde{F}(t) = \{(x, p) \mid p \in \text{Supp}((K(x) - \pi_1(x) - \pi_2(x))\}$ in $F \times C_3$. $\tilde{F}$ is smooth, since $F$ and $C_3$ are. Clearly $T$ is the branch locus of the projection $\tilde{F} \to F$. We compute $\#T$ by computing the genus of $\tilde{F}$.

Let $d$ be the degree of the projection of $F$ onto $F$ and $d'$ be the degree of the projection of $\tilde{F}$ onto $C_3$. By definition $d = 2$.

We compute $d'$ first by computing this degree not for $\tilde{F}(t)$ but for $\tilde{F}(0)$ instead. Note that $\tilde{F}(0) \subset C_3 \times C_3$ is reducible: $\tilde{F}(0) = \Delta + \Delta'$. Here $\Delta' \subset C_3 \times C_3$ denotes the graph of $i_\phi$, the involution interchanging the sheets of $\phi : C_3 \to E$. Then $\tilde{F}(0)$ consists of the disjoint union of the tangential correspondence

$$\{(p, q) \mid q \text{ on tangent line to } q\} \subset C_3 \times C_3$$

and the correspondence

$$\{(p, q) \mid p + i_\phi(p) + q + i_\phi(q) \in |K_{C_3}|\} \subset C_3 \times C_3.$$ 

It follows that $d' = 10 + 2 = 12$.

Denote by $E$ a 'horizontal' fiber $F \times \{c\}$ and by $F$ a 'vertical' fiber $\{y\} \times C_3$ in $F \times C_3$. Let $\Gamma_{\pi_i}$ be the graph of the maps $\pi_i : F \to C_3$, $i = 1, 2$. Then we have (see [24], p. 285):

$$\tilde{F} \sim (d + 2)E + (d' + 4)F - \Gamma_{\pi_1} - \Gamma_{\pi_2} = 4E + 16F - \Gamma_{\pi_1} - \Gamma_{\pi_2}.$$ 

Here $\sim$ denotes linear equivalence. Furthermore, $\Gamma_{\pi_1}^2 = \Gamma_{\pi_2}^2 = 2 \cdot \Delta^2_{C_3} = -8$ and $\Gamma_{\pi_1} \cdot \Gamma_{\pi_2} = 0$ by construction, and $K_{F \times C_3} \sim 4E + 16F$. From the adjunction formula it follows that $2p_a(\tilde{F}) - 2 = 96$, so in particular $\tilde{F}$ is irreducible. Applying the Riemann-Hurwitz formula to the projection $\tilde{F} \to F$ yields the number of points of $T$. It equals $\deg(K_{\tilde{F}}) - d \cdot \deg(K_{C_3}) = 96 - 2 \cdot 16 = 64$. \qed

These results enable us to determine the intersection of the families $T \to \tilde{B}$ with the hyperelliptic locus. In the Chow group $A^1(M_3)$ (with $\mathbb{Q}$-coefficients) denote by $[H_3]_Q$ the $\mathbb{Q}$-class of the hyperelliptic locus, and by $\lambda$ the class $c_1(\pi_*(\omega_{C_3/M_3}))$ (see [41]). In $A^1(M_3)$ we have that $[H_3]_Q = 9\lambda$ (see [25]). Let $[\tilde{B}]_Q$ be the $\mathbb{Q}$-class of the image of $\tilde{B}$ in $M_3$. Then

$$\deg([\tilde{B}]_Q \cdot [\lambda]_Q) = \deg([\lambda]_\tilde{B}) = 2 \deg([\lambda]_B) = 2^8.$$ 

The relation $[H_3]_Q = 9\lambda$ implies that $\deg([\tilde{B}]_Q \cdot [H_3]_Q) = 9 \cdot 2^8$. We have $9 \cdot 2^8$ hyperelliptic fibers on $\tilde{B}$ by Corollary 3.3. If we let $b \in \tilde{B}$ be a point such that the fiber $T_b$ is hyperelliptic, it follows that the intersection multiplicity $i([\tilde{B}]_Q, [H_3]_Q; b) = 1$.

This multiplicity is computed on the base of the versal deformation space of $T_b$. The tangent space to the versal deformation space in the point corresponding to $T_b$ is $H^1(T_b, \Theta_b)$, where $\Theta_b$ is the tangent bundle of $T_b$. The tangent direction of $\tilde{B}$ in $H^1(T_b, \Theta_b)$ is the image of the Kodaira-Spencer map $T_{\tilde{B}, b} \to H^1(T_b, \Theta_b)$, and the tangent space to the hyperelliptic locus is the subspace of $H^1(T_b, \Theta_b)$ invariant under the hyperelliptic involution on $H^1(T_b, \Theta_b)$. So we conclude that the image of the Kodaira-Spencer map in $H^1(T_b, \Theta_b)$ is not contained in the invariant subspace of the hyperelliptic involution acting on $H^1(T_b, \Theta_b)$. However, we don't know how to prove this directly.
3.5 The surjectivity of the Prym map

Let $R_{3,2}$ be the moduli space of tuples $(C, p + p', L)$, where $C$ is a smooth curve of genus 3, $p + p'$ is an effective divisor of degree two on $C$ such that $p \neq p'$, and $L$ is a line bundle on $C$ such that $L^2 \cong \mathcal{O}_C(p + p')$. Such a tuple $(C, p + p', L)$ determines a double cover of smooth curves $\tilde{C} \to C$ to which we can associate its Prym variety $\text{Prym}(C, p + p', L)$. This operation yields a morphism $\text{Pr} : R_{3,2} \to A_3$, where $A_3$ is the moduli space of principally polarized abelian varieties of dimension 3. We call this the Prym map.

Let $R_{3,2}^{\text{ell}} \subset R_{3,2}$ be the locus of tuples $(C, p + p', L)$ with the property that $C$ admits a map of degree 2 to an elliptic curve. We will denote the restriction of the Prym map to $R_{3,2}^{\text{ell}}$ also by $\text{Pr}$. The aim of this section is to show that $\text{Pr} : R_{3,2}^{\text{ell}} \to A_3$ is dominant. Note that the dimension of $R_{3,2}$ is 8 and that both $R_{3,2}^{\text{ell}}$ and $A_3$ are of dimension 6. Hence if the morphism $\text{Pr} : R_{3,2}^{\text{ell}} \to A_3$ is dominant, it is also generically finite.

We compute the codifferential of $\text{Pr}$. After taking appropriate finite level structures, we may assume that $R_{3,2}^{\text{ell}}$ and $A_3$ are smooth. Since $R_{3,2}$ is an unramified cover of $M_{3,2}$, we identify the cotangent spaces at the points $(C, p + p', L) \in R_{3,2}$ and $(C, p + p') \in M_{3,2}$. The identification of the cotangent space $T^*_M(C, p + p')$ with $H^0(C, \omega_C^2(p + p'))$ thus gives us $T^*_M(C, p + p', L) \cong H^0(C, \omega_C^2(p + p'))$.

Let $(A, \Theta)$ be a point of $A_3$. We identify its cotangent space $T^*_A(A, \Theta)$ with $S^2T^*_{A,0}$, the symmetric square of the cotangent space to $A$ at the origin, by standard identifications. In our case $(A, \Theta) = \text{Prym}(C, p + p', L)$. From the definition of Prym($\tilde{C}/C$) as the odd part of $\text{Jac}(\tilde{C})$ (see [39]) and the splitting $H^0(C, \omega_C) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes L)$ it follows that $T^*_\text{Prym}(C, p + p', L, 0) \cong H^0(C, \omega_C \otimes L)$.

Using these identifications, the codifferential

$$\text{Pr}^* : S^2H^0(C, \omega_C \otimes L) \to H^0(C, \omega_C^2(p + p'))$$

is the composition of two natural maps: the cup-product $S^2H^0(C, \omega_C \otimes L) \to H^0(C, \omega_C^2 \otimes L^2)$ followed by the identification $H^0(C, \omega_C^2 \otimes L^2) \cong H^0(C, \omega_C^2(p + p'))$ induced by the isomorphism $L^2 \cong \mathcal{O}_C(p + p')$.

The codifferential is computed in [6] for the Prym map $\overline{\mathcal{R}}_{g+1} \to A_g$, where $\overline{\mathcal{R}}_{g+1}$ is the moduli space of “admissible” double covers. We obtain the above result by embedding $R_{3,2}$ into $\mathcal{M}_{3,2}$ via $(C, p + p', L) \to (C/(p \sim p'), L)$.

Suppose that $C$ admits an elliptic involution, i.e., an automorphism of order two such that the quotient is an elliptic curve. Then $H^0(C, \omega_C^2)$ decomposes into $(\pm 1)$-eigenspaces $H^0(C, \omega_C^2)^\pm$. Embed the space $H^0(C, \omega_C^2(p + p'))$ into $H^0(C, \omega_C^2(p + p'))$. Then the cotangent space $T^*(R_{3,2}^{\text{ell}})(C, p + p', L)$ to $R_{3,2}^{\text{ell}}$ at $(C, p + p', L)$ can be identified with the quotient $H^0(C, \omega_C^2(p + p'))/H^0(C, \omega_C^2)^-$. The codifferential of $R_{3,2}^{\text{ell}} \to A_3$ in the point $(C, p + p', L)$ is the composition

$$S^2(H^0(C, \omega_C \otimes L)) \to H^0(C, \omega_C^2(p + p')) \to H^0(C, \omega_C^2(p + p'))/H^0(C, \omega_C^2)^- \quad (*)$$

We compute this codifferential in one specific point of $R_{3,2}^{\text{ell}}$. Let $C_{48}$ be the plane quartic curve given by the equation $X^4 + Y(Y^3 + Z^3) = 0$. It has an automorphism
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group of order 48 and admits the elliptic involution $X \mapsto -X, Y \mapsto Y, Z \mapsto Z$. This curve has 4 hyperflexes and 16 ordinary flexes. Let $q = (0,1,-1), p = (0,0,1)$ and $p' = p'' = (1,\eta,0)$, with $\eta$ a fourth root of $-1$. The points $p$ and $q$ are hyperflexes and $p' = p''$ is a ordinary flex. If we let $L$ be the line bundle $O_{C_{48}}(2q - p'')$, the tuple $(C_{48}, P + p', L)$ is a point of $R_{3,2}^{ell}$.

**Lemma 3.7** The codifferential of the Prym map $\mathcal{P}r : R_{3,2}^{ell} \to \mathcal{A}_3$ in the point $(C_{48}, P + p', L)$ has maximal rank.

**Proof.** The canonical embedding of $C_{48} \subset \mathbb{P}^2$ and the isomorphism $H^0(C_{48}, \omega \otimes L) \cong H^0(C_{48}, \omega^2(-2q - p''))$ enable us to write down an explicit base for the space $H^0(C_{48}, \omega \otimes L)$. This gives six generators for the image of $S^2 H^0(C_{48}, \omega \otimes L)$ in $H^0(C_{48}, \omega^2(p + p'))$. Embedding $H^0(C_{48}, \omega^2(p + p')) \cong H^0(C_{48}, \omega^3(-2p''))$ in $H^0(C_{48}, \omega^3) \cong H^0(P^2, O(3))$ we get six homogeneous forms of degree 3.

Also we embed $H^0(C_{48}, \omega^2)$ in $H^0(P^2, O(3))$ and write down a base. In fact, if $L_{p''}$ is the equation of the flex line through $p''$ then the two forms $XY L_{p''}$ and $XZ L_{p''}$ form a base. A straightforward calculation shows that these eight forms are linearly independent in $H^0(P^2, O(3))$. 

**Corollary 3.8** The Prym map $\mathcal{P}r : R_{3,2}^{ell} \to \mathcal{A}_3$ is dominant and generically finite.

**Corollary 3.9** The generic curve of genus three occurs as a fiber of a family $\mathcal{T} \to \mathcal{B}$ appearing in Corollary 1.2.