Complete subvarieties of moduli spaces of algebraic curves

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Chapter 3

Explicit complete curves in the moduli space of curves of genus three


Consider $M_g$, the moduli space of smooth curves of genus $g$ over the field of complex numbers $\mathbb{C}$. For $g \geq 2$, $M_g$ is a quasi-projective variety of dimension $3g - 3$. Note that $M_g$ is not complete as smooth curves can degenerate to singular ones. In fact, $M_2$ is affine. However, $M_g$ contains complete curves if $g \geq 3$. This follows from the existence of a projective compactification $\overline{M}_g$ of $M_g$ in which the boundary $\overline{M}_g \setminus M_g$ has codimension $\geq 2$ (take the closure of the image of $M_g$ in the Satake compactification of $A_g$, the moduli space of principally polarized abelian varieties).

The complete curves are obtained by cutting $\overline{M}_g$ with sufficiently many hypersurfaces in general position. An upper bound for the dimension of a complete subvariety of $M_g$ is $g - 2$ if $g \geq 2$ (see [14]). So these complete curves achieve this bound if $g = 3$.

Harris (see [25]) notes that these curves are not very explicit, although constructions of explicit complete families of smooth curves are known. In [17] an explicit one-dimensional family is given for every genus $g \geq 4$, but for $g = 3$ a less explicit family is exhibited. The aim of this note is to produce explicit examples of complete families of smooth curves of genus 3 having a moduli theoretic interpretation.

In short the construction is the following. Fix a smooth base curve $C_3$ of genus 3 and fix a complete curve $F \subset C_3 \times C_3 \setminus \Delta$. Construct a complete family of smooth double covers $(C_f \to C_3)_{f \in F}$, where $C_f$ is branched over the two points determined by $f \in F$ (in fact such a family may exist only over a finite cover of $F$ due to monodromy obstructions). To the covers $C_f \to C_3$ we can associate their Prym varieties. We obtain a complete family of 3-dimensional principally polarized abelian varieties $\text{Prym}(C_f/C_3)_{f \in F}$, which turn out to be Jacobians of smooth curves if $C_3$ is not hyperelliptic. So we get a complete family of Jacobians of smooth genus 3 curves.

In the first section we present a specific construction of complete families of 3-dimensional principally polarized abelian varieties, depending on five parameters. The base is 1-dimensional and the fibers are Prym varieties of branched double covers of curves. We will see in section 2 that these fibers are actually Jacobians of smooth curves. The corresponding families of curves are then constructed via the so-called “trigonal construction”. Section 3 contains a calculation of invariants of these families, in particular the number of hyperelliptic fibers. Finally, we show that these families are general in the following sense: the generic smooth curve of genus 3 occurs as a fiber of one of the families.
3.1 The construction

Fix an elliptic curve $E = (E, 0)$. Let $\phi : C_3 \to E$ be a double cover ramified at 4 points of $E$, so that the genus $C_3$ is 3. We want $C_3$ to be non-hyperelliptic. This turns out to be an open condition on the four branch points: let $B$ the branch divisor, let $\tilde{B}$ the ramification divisor of $\phi$, and let $L$ denote the unique line bundle on $E$ satisfying $\phi^* L \cong \mathcal{O}_{C_3}(\tilde{B})$. Then $C_3$ is hyperelliptic if and only if $B = B_1 + B_2$ for some $B_1, B_2 \in [L]$ (if $C_3$ is hyperelliptic, then $\sigma$ acts freely on $\text{Supp}(B)$ and $L = \phi_*(x + \sigma x)$ for an $x \in \text{Supp}(B)$).

Fix a point $t \neq 0$ in $E$. Set $\Delta_t = \{(x, x + t) \mid x \in E\} \subset E \times E$. Let $F = F(t) \subset C_3 \times C_3$ be the inverse image of $\Delta_t$ under the natural map $\phi : C_3 \times C_3 \to E \times E$. Then in $C_3 \times C_3$ we have that $F \cap \Delta = \emptyset$, thus $F$ parametrizes pairs of distinct points on $C_3$. If we suppose—as we shall do in the following—that $t \notin \{\phi(p) - \phi(q) \mid p, q \text{ ramification points of } \phi\}$, then one easily verifies that $F$ is smooth of genus 9.

Denote by $\pi_1, \pi_2 : F \to C_3$ the maps induced by the projections of $C_3 \times C_3$ onto the first, respectively the second coordinate, and denote by $\Gamma_{\pi_i} \subset F \times C_3$ the graph of $\pi_i$, for $i = 1, 2$. We want to have a double cover of $F \times C_3$ ramified precisely over $\Gamma_{\pi_1} + \Gamma_{\pi_2}$. Such a cover may not exist due to monodromy obstructions. To overcome these, we consider the natural map $F \to \text{Pic}^2(C_3)$, $x \mapsto [\pi_1(x) + \pi_2(x)]$, and the squaring map $sq : \text{Pic}^1(C_3) \to \text{Pic}^2(C_3)$, $L \mapsto L \otimes L$. Define the curve $B$ as the fibered product $B = F \times_{\text{Pic}^2(C_3)} \text{Pic}^1(C_3)$.

$$
\begin{array}{ccc}
B & \to & \text{Pic}^1(C_3) \\
\downarrow f & & \downarrow \text{sq} \\
F & \to & \text{Pic}^2(C_3)
\end{array}
$$

Denote the map $B \to F$ by $f$, and set $\Gamma = \Gamma_{\pi_1} + \Gamma_{\pi_2} \subset B \times C_3$. The pull-back to $B \times C_3$ of the universal line bundle over $\text{Pic}^1(C_3) \times C_3$ gives a line bundle $L$ on $B \times C_3$ satisfying $L^2 \cong \mathcal{O}_{B \times C_3}(\Gamma)$.

Using this $L$ we make a double cover $C \to B \times C_3$, ramified precisely along $\Gamma$. This is a well-known construction: embed $B \times C_3$ in the total space of the line bundle corresponding to $\mathcal{O}_{B \times C_3}(\Gamma)$ with a section $s$ of $\mathcal{O}_{B \times C_3}(\Gamma)$ which vanishes precisely once along $\Gamma$. Define $\mathcal{C}$ as the inverse image of $s(B \times C_3)$ under the natural squaring map on local sections $L \xrightarrow{sq} L^2 \cong \mathcal{O}_{B \times C_3}(\Gamma)$. Here $\mathcal{L}$ denotes the total space of the line bundle $L$, by abuse of notation.

$$
\begin{array}{cccc}
\mathcal{C} & \to & \mathcal{L} & \to \mathcal{L}^2 \xrightarrow{\cong} \mathcal{O}_{B \times C_3}(\Gamma) \\
\downarrow & & \downarrow & \downarrow \\
B \times C_3 & = & B \times C_3 & = & B \times C_3
\end{array}
$$

The composition with the projection $B \times C_3 \to B$ yields a complete family of curves $\pi_C : \mathcal{C} \to B$. Let $C_b$ denote the fiber of $\pi_C : \mathcal{C} \to B$ over $b \in B$. Each $C_b$ is smooth of genus 6 and maps $2 : 1 \to C_3$ with two ramification points, via the composition $\mathcal{C} \to B \times C_3 \to C_3$. Note that the map $f : B \to F$ is étale of degree 64, so that $B$ is a smooth curve, but possibly reducible. Thus $\pi_C : \mathcal{C} \to B$ is a complete, possibly reducible family of smooth curves.
We can associate to every double covering \( C_b \to C_3 \) its Prym variety (see [39]). This is a 3-dimensional, principally polarized abelian variety \( \text{Prym}(C_b/C_3) = (P, \Xi) \). \( P \) is defined as the kernel of the norm map \( \text{Nm} : \text{Jac}(C_b) \to \text{Jac}(C_3) \) (which is connected since \( C_b \to C_3 \) is ramified). The Jacobian of \( C_3 \) injects in the Jacobian of \( C_b \), so up to isogeny \( \text{Jac}(C_b) \) splits as the product of \( \text{Jac}(C_3) \) and another factor, which is \( P \). The polarization on \( \text{Jac}(C_b) \) gives rise to a polarization of \( P \times \text{Jac}(C_3) : P \times \text{Jac}(C_3) \to \text{Jac}(C_b) \to \tilde{P} \times \text{Jac}(C_3) \). This polarization splits and induces on \( P \) twice a principal polarization \( \Xi \).

It is straightforward to globalize this construction. The family \( \pi_C : C \to B \) has sections. Thus we have the associated family \( \pi_C : C \to B \) of Jacobians \( JC \to B \). Consider the norm map \( \text{Nm} : JC 	o B \times \text{Jac}(C_3) \), induced by the usual norm map \( \text{Nm} : JC \to b \times \text{Jac}(C_3) \). Define \( P \) as \( \text{Nm}^{-1}(B \times 0) \subset JC \). The projection \( JC \to B \) yields a family \( \pi_P : P \to B \) of 3-dimensional principally polarized abelian varieties.

We need Theorem 2.1. The theorem says that the 1-dimensional family of Pryms \( \mathcal{P} \to B \) is in fact a family of Jacobians of smooth curves. Note that here we use that \( C_3 \) is non-hyperelliptic. For readability the statement and proof of this theorem is postponed to the next section. Summarizing, we have the following result:

**Corollary 3.1** The above construction yields a complete, one-dimensional family \( \pi_C : \mathcal{P} \to B \) of 3-dimensional Jacobians of smooth curves.

Note that the construction depends on five parameters: namely on \( E \), on the double cover \( C_3 \to E \) and on the point \( t \in E \). Thus we obtain in fact a five-dimensional family of complete, one-dimensional families of 3-dimensional Jacobians.

### 3.2 The corresponding family of curves

We will exhibit the family of smooth curves corresponding to the family of Jacobians \( \pi_P : \mathcal{P} \to B \), using the so-called trigonal construction. We explain this first for a single Prym variety \( \text{Prym}(C_b/C_3) \). For the details and proofs we refer to [18]. The map \( C_b \to C_3 \) has two branch points, which we denote by \( p, p' \in C_3 \). Consider \( C_3 \) canonically embedded in \( |K_{C_3}|^* \cong \mathbb{P}^2 \). The line through \( p \) and \( p' \) cuts out on \( C_3 \) two other points. Choose one of them and call it \( p'' \). Let \( \mathbb{P}^1 \) denote the linear system \( |K_{C_3} - p''| \). Projecting from \( p'' \) determines a three-to-one map \( C_3 \to \mathbb{P}^1 \cong \mathbb{P}^1 \).

Assume that this map does not ramify at \( p \) or \( p' \). Let \( C_b^{(3)} \) (respectively \( C_3^{(3)} \)) denote the threefold symmetric product of \( C_b \) (respectively \( C_3 \)). The line \( \mathbb{P}^1 \) embeds naturally in \( C_3^{(3)} \). Consider the inverse image of \( \mathbb{P}^1 \subset C_3^{(3)} \) under the natural map \( C_b^{(3)} \to C_3^{(3)} \). This is a curve with two smooth, isomorphic components \( T_0, T_1 \). Their intersection consists of two distinct points \( q, q' \), in which they meet transversally. Both curves map with degree four to \( \mathbb{P}^1 \) and \( q, q' \) are simple ramification points on each curve mapping to the same point on \( \mathbb{P}^1 \). Apart from \( q \) and \( q' \) the ramification on the tetragonal curves \( T_0 \) and \( T_1 \) arises from the ramification of the \( g_3 \) on \( C_3 \). Namely if \( C_3 \to \mathbb{P}^1 \) is simply branched over \( P \in \mathbb{P}^1 \), then also the maps \( T_i \to \mathbb{P}^1 \) are simply branched over \( P \). If \( C_3 \to \mathbb{P}^1 \) is completely branched over \( P \in \mathbb{P}^1 \) then the maps \( T_i \to \mathbb{P}^1 \) have a point of ramification index 3 over \( P \). In particular the genera of the \( T_i \) are 3.
Section 3.2. The corresponding family of curves

If on the other hand the triple cover \( C_3 \to \mathbb{P}^1 \) is ramified at \( p \) or \( p' \), then the situation is different. The points \( q \) and \( q' \) coincide, i.e., the inverse image still has two smooth isomorphic components \( T_0 \) and \( T_1 \) which meet in one point \( q \), and the fourfold cover \( T_i \to \mathbb{P}^1 \) is totally ramified in \( q \).

The curves \( T_0 \) and \( T_1 \) map naturally to \( \text{Pic}^3(C_b) \), since they live in \( C_b^{(3)} \). The norm map \( \text{Nm} : \text{Pic}^3(C_b) \to \text{Pic}^3(C_3) \) maps their images to the image of \( \mathbb{P}^1 \) in \( \text{Pic}^3(C_3) \), which is a point. The images of \( T_0 \) and \( T_1 \) in \( \text{Pic}^3(C_b) \) lie thus in a suitable translate of \( P = \ker(\text{Nm} : \text{Jac}(C_b) \to \text{Jac}(C_3)) \). Moreover, the maps \( T_i \to \text{Pic}^3(C_b) \), \( i = 0, 1 \), induce isomorphisms of principal polarized abelian varieties \( \text{Jac}(T_i) \to \text{Prym}(C_b/C_3) \). The trigonal construction goes back to Recillas and was generalized by Donagi.

To globalize this construction reconsider the curve \( F \subset C_3 \times C_3 \). Every point \( x \in F = F(t) \) determines two distinct points \( \pi_1(x), \pi_2(x) \in C_3 \), and the line spanned by these points in the canonical embedding of \( C_3 \) we denote by \( \ell(x) \). Let \( K(x) \) denote the hyperplane section \( C_3 \cdot \ell(x) \). Consider the curve

\[
\tilde{F} = \tilde{F}(t) = \{ (x, c) \in F \times C_3 \mid c \in \text{Supp}(K(x) - \pi_1(x) - \pi_2(x)) \}.
\]

Then \( \tilde{F} \) is smooth since \( F \) and \( C_3 \) are. Clearly \( \tilde{F} \) maps with degree two onto \( F \). This map is ramified in 64 points (see Lemma 3.3 for the computation), hence \( \tilde{F} \) is irreducible of genus 49. A point \( (x, p'') \) of \( \tilde{F} \) determines on \( C_3 \) a \( g_3^1 \), namely \( |K_{C_3} - p''| \), plus two unordered, distinct points contained in a divisor of this \( g_3^1 \), namely \( \pi_1(x), \pi_2(x) \). \( \tilde{F} \) maps to \( \text{Pic}^2(C_3) \) via the composition \( \tilde{F} \to F \to \text{Pic}^2(C_3) \), and also \( \text{Pic}^1(C_3) \) does via the squaring map \( \text{Pic}^1(C_3) \to \text{Pic}^2(C_3) \). Let \( \tilde{B} = \tilde{F} \times_{\text{Pic}^3(C_3)} \text{Pic}^1(C_3) \). Note that \( \tilde{B} \) maps two-to-one to \( B \).

The pull-back of \( C \to B \) to \( \tilde{B} \) yields a double cover \( \tilde{C} \to \tilde{B} \times C_3 \). The composition \( \tilde{C} \to \tilde{B} \times C_3 \to \tilde{B} \) is a one-dimensional family of double covers. The extra structure is a three-to-one map \( \tilde{B} \times C_3 \to S \), where \( S \) is a ruled surface \( S \to \tilde{B} \) with fiber \( S_b = |K_{C_3} - p''| \). Then \( S \) embeds in \( \tilde{B} \times C_3^{(3)} \) and its inverse image along the natural map \( \tilde{C}^{(3)}_{\tilde{B}} \to (\tilde{B} \times C_3)^{(3)}_{\tilde{B}} = \tilde{B} \times C_3^{(3)} \) we denote by \( T' \). The (singular) surface \( T' \) has an involution which preserves the fibers of \( T' \to S \) setwise, and its quotient by this involution we denote by \( T \). The surface \( T \) is smooth. The natural map \( T \to S \) composed with the projection \( S \to \tilde{B} \) yields a family \( T \to \tilde{B} \) of smooth tetragonal curves.

**Corollary 3.2** The above construction gives a complete, one-dimensional family \( T \to S \to \tilde{B} \) of smooth, tetragonal curves of genus 3.
3.3 Pryms of ramified double coverings of genus 3 curves

This section is devoted to the proof of the theorem below. It shows that the Prym variety associated to a ramified double covering \( \tilde{Z}/Z \) ramified in two points, where the genus of \( Z \) is 3, is a Jacobian of a smooth curve as long as the base curve is not hyperelliptic.

In the following let \( Z \) be a smooth genus 3 curve, and let \( \pi_X : \tilde{Z} \to Z \) be a double cover of smooth curves, with two branch points \( p, p' \in Z \) and ramification points \( \tilde{p}, \tilde{p}' \in \tilde{Z} \). Associated to this cover is a (unique) line bundle \( L_{\tilde{Z}/Z} \) satisfying \((\pi_X)^*(L_{\tilde{Z}/Z}) \cong \mathcal{O}_\tilde{Z}(\tilde{p} + \tilde{p}')\). The data \((Z, p + p', L_{\tilde{Z}/Z})\) determine the cover \( \pi_X : \tilde{Z} \to Z \) up to isomorphism. The Prym variety \( P = \text{Prym}(\tilde{Z}/Z) \) is a 3-dimensional abelian variety which comes with a natural principal polarization, denoted by \( \Xi \).

**Theorem 3.3**

1. If \( Z \) is hyperelliptic, and if \( p, p' \) are interchanged by the hyperelliptic involution of \( Z \), then \((P, \Xi)\) is the Jacobian of a smooth irreducible hyperelliptic curve, or a product of two such.
2. If \( Z \) is hyperelliptic and \( p, p' \) are not interchanged by the hyperelliptic involution of \( Z \), then \((P, \Xi)\) is a Jacobian of a smooth irreducible hyperelliptic curve.
3. If \( Z \) is not hyperelliptic, then \((P, \Xi)\) is a Jacobian of a smooth irreducible curve. This curve is hyperelliptic if and only if for some \( p'' \in Z \), \( p + p' + 2p'' \) is a canonical divisor and \( h^0(L_{\tilde{Z}/Z}^{-1}(p + p' + p'')) = 0 \).

**Proof.** Let \( W = Z/(p \sim p') \) be the curve obtained from \( Z \) by identifying \( p \) and \( p' \), and let \( \tilde{W} = \tilde{Z}/(\tilde{p} \sim \tilde{p}') \). Denote by \( f_N : Z \to W \) the normalisation mapping. Note that \( p_\alpha(W) = 4, p_\alpha(W) = 7 \).

The idea is to use the theory from [6]: the induced map \( \pi_W : \tilde{W} \to W \) is an admissible covering in the sense of Beauville, and \( \text{Prym}(\tilde{Z}/Z) = \text{Prym}(\tilde{W}/W) \). Here the right hand side is the Prym variety associated by Beauville to the covering \( \tilde{W}/W \). It lives canonically in

\[
J\tilde{W}^\ast = \{ \text{line bundles } L \text{ on } \tilde{W} \mid 2 \deg L = \deg \omega_{\tilde{W}} \}
\]

as \( P = \text{Prym}(\tilde{W}/W) = \{ L \text{ on } \tilde{W} \mid \Nm L \cong \omega_W h^0(L) \text{ even} \} \), and the polarization \( \Xi \) then is \( \Xi = \{ L \in P \mid h^0(L) \geq 2 \} \).

To prove the theorem, we will show that, unless we are in case (1), \( \dim \text{Sing} \Xi \leq 0 \). Since \( \dim P = 3 \), this shows that in the cases (2) and (3) \((P, \Xi)\) is a Jacobian. If \( \text{Sing} \Xi \) is a point \( L \in \Xi \)---in which case \((P, \Xi)\) is a hyperelliptic Jacobian---we show how \( L \) arises from the geometry of \( Z \).

Note that the proof is an adaptation of the proof of Theorem 4.10 in [6]. Also note that case (1) is already treated by Beauville (see [6], p. 171).

Start by assuming that \( \text{Sing} \Xi \) is non-empty, say \( L \in \text{Sing} \Xi \). We will show that in the cases (2) and (3) \( L \in \Xi \) is a uniquely determined point. By [6], Lemma 4.1, either \( h^0(L) \geq 4 \) and even or \( h^0(L) = 2 \), and \( \ast s \otimes t = s \otimes \ast t \) for a base \( \{ s, t \} \) of \( H^0(W, L) \), where \( \ast : W \to \tilde{W} \) is the involution interchanging the sheets of \( \pi_W : \tilde{W} \to W \).


First suppose that there is an $L \in \text{Sing}(\Xi)$, with $h^0(L) \geq 4$ and $h^0(L)$ even. Let $L_1$ be the pull-back of $L$ to $\tilde{Z}$. Then $\deg(L_1) = 6$, and $h^0(L_1) \geq 4$. From Clifford's theorem it follows that $\tilde{Z}$ is hyperelliptic. The hyperelliptic involution $\sigma$ of $\tilde{Z}$ commutes with $\iota$. Moreover, $\sigma$ interchanges the fixed points of $\iota$, since all sections of $L_1$ come from sections of $L$. Hence $\sigma$ descends and $Z$ is hyperelliptic. If we denote the hyperelliptic involution of $Z$ also with $\sigma$, we see that $\sigma$ interchanges $p, p'$. So we are in case (1). This is the case in which $W$ has a so-called non-singular $g_2^1$. Non-singular means that the $g_2^1$ contains a divisor with non-singular support. It follows that $(P, \Xi)$ is a product of hyperelliptic Jacobians (see [6], p. 171).

Secondly, suppose that there is an $L \in \text{Sing}(\Xi)$ with $h^0(L) = 2$, with a base $\{s, t\}$ for $h^0(L)$ satisfying $s \otimes \iota^* t = \iota^* s \otimes t$.

We first assume that $s, t$ have the property that either $s$ or $t$ is not zero at the singular point. By [6], lemma 4.4, $L$ is of the form $(\pi_W)^* M(E)$. Here $M$ is a line bundle on $W$, $h^0(M) \geq 2$, $\deg((\pi_W)^* M(E)) = \deg(L) = \deg(\omega_W) = 6$. $M$ is non-singular, and $E$ is an effective divisor on $W$ with non-singular support such that $(\pi_W^*)E \in \omega_W \otimes M^{-2}$. Let $M_1$ be the pull-back of $M$ to $\tilde{Z}$. Then $M_1$ gives an $g_2^1$ or a $g_3^1$ on $Z$.

If $Z$ is hyperelliptic, a $g_3^1$ on $Z$ is the unique $g_2^1$ plus a point. We see that also in this case the two branch points $p, p'$ of $\tilde{Z} \rightarrow Z$ are interchanged by the hyperelliptic involution, and we are in case (1) again.

If $Z$ is not hyperelliptic, then $|M_1|$ is a $g_3^1$, $E = 0$, and the branch divisor $p + p'$ is contained in an element of the system $|M_1|$, say $p + p' + p'' \in |M_1|$. The relation $\text{Nm}_W((\pi_W)^* M) = \omega_W$ gives by pull-back $\text{Nm}_Z((\pi_Z)^* M_1) = \omega_Z(p + p')$. It follows that $p + p' + 2p''$ is a canonical divisor, and $|M_1|$ is the pencil $LZ - p''$. So we are in case (3) and $L = (\pi_W)^* M$.

Finally assume that the two sections $s, t$ of $L$ vanish simultaneously at the singular point. Let $L_1$ be the pull-back to $\tilde{Z}$ of $L$, and set $L_2 = L_1(-\tilde{p} - \tilde{p'})$. Then $\text{Nm}(L_2) = \omega_Z$, and $L_1$ has sections induced by $s, t$, which we will also denote by $s, t$, and which still satisfy $s \otimes \iota^* t = \iota^* s \otimes t$. Then [6], lemma 4.4 applies, giving that $L_2 = (\pi_Z)^* (M)$, where $M$ is a line bundle on $Z$, $h^0(M) \geq 2$. It follows that $Z$ is hyperelliptic and that $M$ is the line bundle associated with the unique $g_2^1$ on $Z$. So we are in case (2) and $L$ is an extension of $\pi_Z^* (M)(\tilde{p} + \tilde{p'})$ to $\tilde{W}$.

This finishes the first part of the proof. We will show in the second part that the above procedure can be reversed to obtain precisely one singularity of $\Xi$ if we are in case (2). If $p$ and $p'$ satisfy the condition of case (3) then we also obtain precisely one singularity of $\Xi$.

First assume that we are in case (2), i.e., $Z$ is hyperelliptic and that $p, p'$ are not interchanged by the hyperelliptic involution. The $g_2^1$ on $Z$ pulls back to a line bundle $L_2$ on $\tilde{Z}$. All extensions of the line bundle $L_1 = L_2(\tilde{p} + \tilde{p'})$ to a line bundle $L$ on $\tilde{W}$ admit sections $s, t$ satisfying $s \otimes \iota^* t = \iota^* s \otimes t$. There are precisely two extensions $L$ such that $\text{Nm}_W L = \omega_W$, and only one of these is even (see [6], Prop. 3.5). This takes care of the second case of the theorem, since we obtain indeed precisely one singularity $L$ of $\Xi$, showing that $(P, \Xi)$ is the Jacobian of a smooth, hyperelliptic curve.

Secondly, suppose that we are in case (3), and that we have a line bundle $M_1$ on $Z$, with $h^0(Z(M_1)) = 2$ and $M_1 \cong \mathcal{O}_Z(p + p' + p'') \cong \mathcal{O}_Z(K_Z - p'')$. This $M_1$ has a unique extension $M$ to $W$ such that $h^0(M) = 2$. Then $\text{Nm}_W((\pi_W)^*(M)) = \omega_W$. 

Section 3.3. Pryms of ramified double coverings of genus 3 curves
For $\text{Nm}_W((\pi_W)^*(M)) = M^2$ is an extension of $M_1^2 = \omega_Z(p + p')$ to $W$ having a section non-zero at $p = p'$, and $\omega_W$ is the unique such extension.

The pullback $(\pi_W)^*(M)$ will yield an element of $\text{Sing}(\Xi)$ if $h^0_W((\pi_W)^*M) = 2$.

We have the decomposition

$$H^0_Z((\pi_Z)^*M_1) \cong H^0_Z(M_1) \oplus H^0_Z(M_1 \otimes L^{-1}_{Z/2})$$

which is the decomposition into even and odd parts with respect to the action of the involution $\iota$ on $H^0_Z((\pi_Z)^*M_1)$. We already have seen that all sections of $H^0_Z((\pi_Z)^*M_1)^+ \cong H^0_Z(M_1)$ extend to $\tilde{W}$. Also all sections of $H^0_Z((\pi_Z)^*M_1)^- \cong H^0_Z(M_1 \otimes L^{-1}_{Z/2})$ extend to $\tilde{W}$, since such a section is zero at $\tilde{p}$ and $\tilde{p}'$. It follows that $h^0_W((\pi_W)^*M) = h^0_Z((\pi_Z)^*M_1)$, and that $h^0_W((\pi_W)^*M) = 2$ if and only if $h^0_Z(M_1 \otimes L^{-1}_{Z/2}) = 0$. So we obtain precisely one singularity $(\pi_W)^*M$ of $\Xi$ if and only if $h^0(Z_1 \otimes L^{-1}_{Z/2}) = 0$. Note that $M_1 \otimes L^{-1}_{Z/2}$ is a theta characteristic: $M_1^2 \otimes L^{-2}_{Z/2} \cong \mathcal{O}_Z(p + p' + 2p'') \cong \omega_Z$ by assumption.

### 3.4 The degree of $\lambda$ and the number of hyperelliptic fibers

Consider the family $\mathcal{P} \to B$ constructed in section 1. Our aim in this section is to compute the degree of $\lambda = c_1((\pi_\mathcal{P})_* (\Omega^1_{\mathcal{P}/B}))$ and the number of hyperelliptic fibers of the family $\pi_\mathcal{P} : \mathcal{P} \to B$.

**Proposition 3.4** The degree of $\lambda_{\mathcal{P}/B}$ equals $2^7$.

**Proof.** Denote by $p_B$ the projection $B \times C_3 \to C_3$, by $p_F$ the projection $F \times C_3 \to C_3$, and let $p_C$ be the composition $C \to B \times C_3 \to C_3$. The branch locus of $C \to B \times C_3$ is $\Gamma = \Gamma_{\pi_1 \circ f} + \Gamma_{\pi_2 \circ f}$. Here $\Gamma_{\pi_i \circ f}$ denotes the graph of $\pi_i \circ f$ in $B \times C_3$, $i = 1, 2$. We denote the ramification locus of $C \to B \times C_3$ by $\bar{\Gamma} = \bar{\Gamma}_{\pi_1 \circ f} + \bar{\Gamma}_{\pi_2 \circ f} \subset C$. Let $\omega$ be a holomorphic 1-form on $C_3$. Let $\Omega = p_B^* \omega$ (respectively $\Omega = p_C^* \omega$) be its pull-back to $B \times C_3$ (respectively $C$), and consider it as a section of the line bundle $\omega_{B \times C_3/B}$ (respectively $\omega_C/B$).

We compute the intersection number $(\bar{\Omega})^2$, where by $(\bar{\Omega})$ we denote the divisor of $\bar{\Omega}$. Clearly, $(\bar{\Omega}) = p^*_C(\omega) + \bar{\Gamma}$. So

$$\left(\bar{\Omega}\right)^2 = 2\bar{\Gamma} \cdot p^*_C(\omega) + \bar{\Gamma}^2$$

$$= 2\Gamma \cdot p^*_B(\omega) + \frac{1}{2} \Gamma^2.$$
Section 3.4. The degree of $\lambda$ and the number of hyperelliptic fibers

Furthermore, we have
\[
\Gamma \cdot p_b^*(\omega) = \deg(f)(\Gamma_{\pi_1} + \Gamma_{\pi_2}) \cdot p_F^*(\omega) \\
= 2 \deg(f)\Gamma_{\pi_1} \cdot p_F^*(\omega) \\
= 2 \deg(f) \deg(\pi_1) \deg(\omega) \\
= 16 \deg(f),
\]
and
\[
\Gamma^2 = \deg(f)(\Gamma_{\pi_1} + \Gamma_{\pi_2})^2 \\
= \deg(f)(2\Gamma_{\pi_1}^2 + 2\Gamma_{\pi_1} \cdot \Gamma_{\pi_2}) \\
= \deg(f)(2\Gamma_{\pi_1}^2) \\
= 2 \deg(f)((\pi_1 \times \text{id})^*\Delta_{C_3 \times C_3})^2 \\
= -16 \deg(f).
\]
Hence
\[
(\bar{\Omega})^2 = 32 \deg(f) - 8 \deg(f) \\
= 24 \deg(f).
\]
Thus for the family $\pi_C : C \to B$ we get $\deg((\pi_C)_*(c_1(\omega_{C/B}))^2) = 24 \deg(f)$. The degree of $\lambda_{C/B}$ on $B$ follows from the relation $12\lambda = (\pi_C)_*(c_1(\omega_{C/B}))^2$, which is a consequence of the Grothendieck-Riemann-Roch formula. Hence
\[
\deg(\lambda_{C/B}) = 2 \deg(f) = 2^7.
\]
We show that this is equal to the degree of the class $\lambda_{P/B}$ corresponding to the family of Pryms $\pi_P : P \to B$. By definition, $\lambda_{P/B}$ is the first Chern class of the vectorbundle $(\pi_P)_*(\Omega^1_{P/B})$ on $B$. We have the splitting $\Omega^1_{J_C/B} = \Omega^1_{P/B} \otimes \Omega^1_{\text{Jac}(C_3)/B}$, so that
\[
(\pi_J)_*(\Omega^1_{J_C/B}) = (\pi_P)_*(\Omega^1_{P/B}) \oplus H^0(C_3, \omega_{C_3}).
\]
On the other hand we have $(\pi_J)_*(\Omega^1_{J_C/B}) \cong (\pi_C)_*(\omega_{C/B})$. We conclude that $\lambda_{P/B} = \lambda_{C/B}$. In particular $\deg(\lambda_{P/B}) = 2^7$.

Proposition 3.5 The number of hyperelliptic fibers of $\pi_P : P \to B$ is $2^8 \cdot 9$.

PROOF. Let $L_b = L|_{b \times C_3}$. Here $L$ is the line bundle on $B \times C_3$ used to construct the cover $C \to B \times C_3$. The cover $C_b/C_3$ is determined by the data $(C_3, f(\pi_1(x)) + f(\pi_2(b)), L_b)$. By Theorem 2.1 Prym($C_b/C_3$) is hyperelliptic if and only if there is a $p \in C_3$ for which the divisor $D = f(\pi_1(b)) + f(\pi_2(b)) + 2p$ is canonical and the theta characteristic $L_b^{-1}(D - p)$ is even.

The first condition depends only on the image $f(b) \in F$. Let
\[
T = \{ x \in F \mid \mathcal{O}_{C_3}(\pi_1(x) + \pi_2(x) + 2p) \cong \omega_{C_3} \text{ for some } p \in C_3 \}.
\]
For $x \in T$ let $D$ be the unique canonical divisor $\pi_1(b) + \pi_2(b) + 2p$. The set $\{ L_b \mid y \in f^{-1}(x) \}$ consists of all $2^{2g} = 64$ distinct roots of $\pi_1(x) + \pi_2(x)$ in Pic($C_3$). It follows that the set $\{ L_b^{-1}(D - p) \mid b \in f^{-1}(x) \}$ consists of 64 distinct theta characteristics, of which 36 are even and 28 are odd. Hence $h = 36 \cdot \#T$. The proposition follows from the following lemma. □
Chapter 3. Explicit complete curves in $M_3$

**Lemma 3.6** The cardinality of $T$ equals 64.

**Proof.** Consider the curve $\tilde{F} = \tilde{F}(t) = \{(x, p) \mid p \in \text{Supp}(K(x) - \pi_1(x) - \pi_2(x))\}$ in $F \times C_3$. $\tilde{F}$ is smooth, since $F$ and $C_3$ are. Clearly $T$ is the branch locus of the projection $\tilde{F} \to F$. We compute $\#T$ by computing the genus of $\tilde{F}$.

Let $d$ be the degree of the projection of $F$ onto $F$ and $d'$ be the degree of the projection of $\tilde{F}$ onto $C_3$. By definition $d = 2$.

We compute $d'$ first by computing this degree not for $\tilde{F}(t)$ but for $\tilde{F}(0)$ instead. Note that $\tilde{F}(0) \subset C_3 \times C_3$ is reducible: $\tilde{F}(0) = \Delta + \Delta'$. Here $\Delta' \subset C_3 \times C_3$ denotes the graph of $i_\phi$, the involution interchanging the sheets of $\phi : C_3 \to E$. Then $\tilde{F}(0)$ consists of the disjoint union of the tangential correspondence

$$\{(p, q) \mid q \text{ on tangent line to } q\} \subset C_3 \times C_3$$

and the correspondence

$$\{(p, q) \mid p + i_\phi(p) + q + i_\phi(q) \in |K_{C_3}|\} \subset C_3 \times C_3.$$  

It follows that $d' = 10 + 2 = 12$.

Denote by $E$ a ‘horizontal’ fiber $F \times \{c\}$ and by $F$ a ‘vertical’ fiber $\{y\} \times C_3$ in $F \times C_3$. Let $\Gamma_{\pi_i}$ be the graph of the maps $\pi_i : F \to C_3$, $i = 1, 2$. Then we have (see [24], p. 285):

$$\tilde{F} \sim (d + 2)E + (d' + 4)F - \Gamma_{\pi_1} - \Gamma_{\pi_2} = 4E + 16F - \Gamma_{\pi_1} - \Gamma_{\pi_2}.$$  

Here $\sim$ denotes linear equivalence. Furthermore, $\Gamma_{\pi_1}^2 = \Gamma_{\pi_2}^2 = 2 \cdot \Delta_{C_3}^2 = -8$ and $\Gamma_{\pi_1} \cdot \Gamma_{\pi_2} = 0$ by construction, and $K_{F \times C_3} \sim 4E + 16F$. From the adjunction formula it follows that $2p_a(\tilde{F}) - 2 = 96$, so in particular $\tilde{F}$ is irreducible. Applying the Riemann-Hurwitz formula to the projection $\tilde{F} \to F$ yields the number of points of $T$. It equals $\deg(K_{\tilde{F}}) - d \cdot \deg(K_{C_3}) = 96 - 2 \cdot 16 = 64$.  

These results enable us to determine the intersection of the families $T \to \tilde{B}$ with the hyperelliptic locus. In the Chow group $A^1(M_3)$ (with $\mathbb{Q}$-coefficients) denote by $[\mathcal{H}_3]_{\mathbb{Q}}$ the $\mathbb{Q}$-class of the hyperelliptic locus, and by $\lambda$ the class $c_1(\pi_*(\omega_{C_3/M_3}))$ (see [41]). In $A^1(M_3)$ we have that $[\mathcal{H}_3]_{\mathbb{Q}} = 9\lambda$ (see [25]). Let $[\tilde{B}]_{\mathbb{Q}}$ be the $\mathbb{Q}$-class of the image of $\tilde{B}$ in $M_3$. Then

$$\deg([\tilde{B}]_{\mathbb{Q}} : [\mathcal{H}_3]_{\mathbb{Q}}) = \deg(\lambda|\tilde{B}) = 2 \deg(\lambda|B) = 2^8.$$  

The relation $[\mathcal{H}_3]_{\mathbb{Q}} = 9\lambda_{\mathbb{Q}}$ implies that $\deg([\tilde{B}]_{\mathbb{Q}} : [\mathcal{H}_3]_{\mathbb{Q}}) = 9 \cdot 2^8$. We have $9 \cdot 2^8$ hyperelliptic fibers on $\tilde{B}$ by Corollary 3.3. If we let $b \in \tilde{B}$ be a point such that the fiber $\tilde{T}_b$ is hyperelliptic, it follows that the intersection multiplicity $i([\tilde{B}]_{\mathbb{Q}}, [\mathcal{H}_3]_{\mathbb{Q}}; b) = 1$.

This multiplicity is computed on the base of the versal deformation space of $T_\mathbb{B}$. The tangent space to the versal deformation space in the point corresponding to $T_b$ is $H^1(T_b, \Theta_b)$, where $\Theta_b$ is the tangent bundle of $T_b$. The tangent direction of $\tilde{B}$ in $H^1(T_b, \Theta_b)$ is the image of the Kodaira-Spencer map $T_{\tilde{B},b} \to H^1(T_b, \Theta_b)$, and the tangent space to the hyperelliptic locus is the subspace of $H^1(T_b, \Theta_b)$ invariant under the hyperelliptic involution on $H^1(T_b, \Theta_b)$. So we conclude that the image of the Kodaira-Spencer map in $H^1(T_b, \Theta_b)$ is not contained in the invariant subspace of the hyperelliptic involution acting on $H^1(T_b, \Theta_b)$. However, we don't know how to prove this directly.
3.5 The surjectivity of the Prym map

Let $\mathcal{R}_{3,2}$ be the moduli space of tuples $(C, p + p', L)$, where $C$ is a smooth curve of genus 3, $p + p'$ is an effective divisor of degree two on $C$ such that $p \neq p'$, and $L$ is a line bundle on $C$ such that $L^2 \cong \mathcal{O}_C(p + p')$. Such a tuple $(C, p + p', L)$ determines a double cover of smooth curves $\tilde{C} \to C$ to which we can associate its Prym variety $\text{Prym}(C, p + p', L)$. This operation yields a morphism $\mathcal{P}_r : \mathcal{R}_{3,2} \to \mathcal{A}_3$, where $\mathcal{A}_3$ is the moduli space of principally polarized abelian varieties of dimension 3. We call this the Prym map.

Let $\mathcal{R}_{3,2}^{\text{ell}} \subset \mathcal{R}_{3,2}$ be the locus of tuples $(C, p + p', L)$ with the property that $C$ admits a map of degree 2 to an elliptic curve. We will denote the restriction of the Prym map to $\mathcal{R}_{3,2}^{\text{ell}}$ also by $\mathcal{P}_r$. The aim of this section is to show that $\mathcal{P}_r : \mathcal{R}_{3,2}^{\text{ell}} \to \mathcal{A}_3$ is dominant. Note that the dimension of $\mathcal{R}_{3,2}$ is 8 and that both $\mathcal{R}_{3,2}^{\text{ell}}$ and $\mathcal{A}_3$ are of dimension 6. Hence if the morphism $\mathcal{P}_r : \mathcal{R}_{3,2}^{\text{ell}} \to \mathcal{A}_3$ is dominant, it is also generically finite.

We compute the codifferential of $\mathcal{P}_r$. After taking appropriate finite level structures, we may assume that $\mathcal{R}_{3,2}^{\text{ell}}$ and $\mathcal{A}_3$ are smooth. Since $\mathcal{R}_{3,2}$ is an unramified cover of $\mathcal{M}_{3,2}$, we identify the cotangent spaces at the points $(C, p + p', L) \in \mathcal{R}_{3,2}$ and $(C, p + p') \in \mathcal{M}_{3,2}$. The identification of the cotangent space $T^*_\mathcal{M}_{3,2}(C, p + p')$ with $H^0(C, \omega_C^2(p + p'))$ thus gives us $T^*_{\mathcal{R}_{3,2}^{\text{ell}}(C, p + p', L)} \cong H^0(C, \omega_C^2(p + p'))$.

Let $(A, \Theta)$ be a point of $\mathcal{A}_3$. We identify its cotangent space $T^*_{\mathcal{A}_3(A, \Theta)}$ with $S^2T^*_A$, the symmetric square of the cotangent space to $A$ at the origin, by standard identifications. In our case $(A, \Theta) = \text{Prym}(C, p + p', L)$. From the definition of Prym$(\tilde{C}/C)$ as the odd part of $\text{Jac}(\tilde{C})$ (see [39]) and the splitting $H^0(C, \omega_C^2) \cong H^0(C, \omega_C^2) \oplus H^0(C, \omega_C \otimes L)$ it follows that $T^*_\text{Prym}(C, p + p', L, 0) \cong H^0(C, \omega_C \otimes L)$.

Using these identifications, the codifferential

$$\mathcal{P}_r^* : S^2H^0(C, \omega_C \otimes L) \to H^0(C, \omega_C^2(p + p'))$$

is the composition of two natural maps: the cup-product $S^2H^0(C, \omega_C \otimes L) \to H^0(C, \omega_C^2 \otimes L^2)$ followed by the identification $H^0(C, \omega_C^2 \otimes L^2) \cong H^0(C, \omega_C^2(p + p'))$ induced by the isomorphism $L^2 \cong \mathcal{O}_C(p + p')$.

The codifferential is computed in [6] for the Prym map $\overline{\mathcal{R}}_{g+1} \to \mathcal{A}_g$, where $\overline{\mathcal{R}}_{g+1}$ is the moduli space of “admissible” double covers. We obtain the above result by embedding $\mathcal{R}_{3,2}$ into $\overline{\mathcal{R}}_4$ via $(C, p + p', L) \mapsto (C/(p \sim p'), L)$.

Suppose that $C$ admits an elliptic involution, i.e., an automorphism of order two such that the quotient is an elliptic curve. Then $H^0(C, \omega_C^2)$ decomposes into $(\pm 1)$-eigenspaces $H^0(C, \omega_C^2)^\pm$. Embed the space $H^0(C, \omega_C^2)$ into $H^0(C, \omega_C^2(p + p'))$. Then the cotangent space $T^*_{\mathcal{R}_{3,2}^{\text{ell}}(C, p + p', L)}$ to $\mathcal{R}_{3,2}^{\text{ell}}$ at $(C, p + p', L)$ can be identified with the quotient $H^0(C, \omega_C^2(p + p'))/H^0(C, \omega_C^2)^\pm$. The codifferential of $\mathcal{R}_{3,2}^{\text{ell}} \to \mathcal{A}_3$ in the point $(C, p + p', L)$ is the composition

$$S^2(H^0(C, \omega_C \otimes L)) \to H^0(C, \omega_C^2(p + p')) \to H^0(C, \omega_C^2(p + p'))/H^0(C, \omega_C^2)^\pm$$

We compute this codifferential in one specific point of $\mathcal{R}_{3,2}^{\text{ell}}$. Let $C_{48}$ be the plane quartic curve given by the equation $X^4 + Y(Y^3 + Z^3) = 0$. It has an automorphism...
group of order 48 and admits the elliptic involution $X \mapsto -X, Y \mapsto Y, Z \mapsto Z$. This curve has 4 hyperflexes and 16 ordinary flexes. Let $q = (0,1,-1)$, $p = (0,0,1)$ and $p' = p'' = (1, \eta, 0)$, with $\eta$ a fourth root of $-1$. The points $p$ and $q$ are hyperflexes and $p' = p''$ is a ordinary flex. If we let $L$ be the line bundle $\mathcal{O}_{C_{48}}(2q - p'')$, the tuple $(C_{48}, p + p', L)$ is a point of $\mathcal{R}^{\text{ell}}_{3,2}$.

**Lemma 3.7** The codifferential of the Prym map $\mathcal{P}_r : \mathcal{R}^{\text{ell}}_{3,2} \to \mathcal{A}_3$ in the point $(C_{48}, p + p', L)$ has maximal rank.

**Proof.** The canonical embedding of $C_{48} \subset \mathbb{P}^2$ and the isomorphism $H^0(C_{48}, \omega \otimes L) \cong H^0(C_{48}, \omega^2(-2q-p''))$ enable us to write down an explicit base for the space $H^0(C_{48}, \omega \otimes L)$. This gives six generators for the image of $S^2H^0(C_{48}, \omega \otimes L)$ in $H^0(C_{48}, \omega^2(p + p'))$. Embedding $H^0(C_{48}, \omega^2(p + p')) \cong H^0(C_{48}, \omega^3(-2p''))$ in $H^0(C_{48}, \omega^3) \cong H^0(\mathbb{P}^2, O(3))$ we get six homogeneous forms of degree 3.

Also we embed $H^0(C_{48}, \omega^2)$ in $H^0(\mathbb{P}^2, O(3))$ and write down a base. In fact, if $L_{p''}$ is the equation of the flex line through $p''$ then the two forms $XYL_{p''}$ and $XZL_{p''}$) form a base. A straightforward calculation shows that these eight forms are linearly independent in $H^0(\mathbb{P}^2, O(3))$. \qed

**Corollary 3.8** The Prym map $\mathcal{P}_r : \mathcal{R}^{\text{ell}}_{3,2} \to \mathcal{A}_3$ is dominant and generically finite.

**Corollary 3.9** The generic curve of genus three occurs as a fiber of a family $\mathcal{T} \to \tilde{B}$ appearing in Corollary 1.2.