CHAPTER 4
A complete surface in $M_6$ in characteristic $> 2$


Abstract. In all characteristics $p > 2$ we construct a complete surface in the moduli space of smooth genus 6 curves. The surface is contained in the locus of curves with automorphisms.

We consider the following question: ‘What is the maximal number of essential parameters on which a complete family of smooth curves of genus $g$ depends?’ or equivalently, ‘What is the maximal dimension of a complete subvariety of $M_g$, the moduli space of smooth curves of genus $g$?’ In [14] Diaz provided an upper bound for the dimension of such a subvariety: for $g \geq 2$ this dimension is at most $g - 2$. The moduli space $M_g$ itself is irreducible, quasi-projective of dimension $3g - 3$ ($g \geq 2$). Diaz proved his result in characteristic 0, but his bound also holds in characteristic $> 0$ (see [35]).

In order to see how good Diaz’s bound is one has to construct complete subvarieties of $M_g$. This turns out to be a difficult problem, in any characteristic. Only in genus $\leq 3$ Diaz’s bound is known to be sharp, since it is known that $M_g$ contains complete curves if $g$ is at least 3 (see [16]). In higher genera almost nothing is known. The best result we know is a construction of complete subvarieties of arbitrary dimension $d \geq 1$ in $M_g$ with $g \geq 2^{d+1}$. This construction gives a complete surface in $M_g$. For $g = 4, 5, 6$ and 7 we still do not know whether a complete surface in $M_g$ exists.

4.1 The construction

Starting from a complete curve in $M_3$, we construct a complete surface in $M_6$. However, this construction only works in characteristic $\neq 0, 2$. Our result is:

Theorem 4.1 In any characteristic $p > 2$ the moduli space $M_6$ of smooth genus 6 curves contains a complete surface.

To construct in characteristic 0 a complete surface in $M_6$ seems more difficult. This is more or less similar to the fact that the moduli space $A_g$ of principally polarized abelian varieties of dimension $g$ contains in characteristic $p > 0$ complete subvarieties of rather high dimension [45]. The corresponding situation in characteristic 0 is completely unknown.†

† However, see Keel-Sadun [32].