Complete subvarieties of moduli spaces of algebraic curves

Zaal, C.G.

Publication date
2005

CHAPTER 4

A complete surface in M_6 in characteristic > 2

This chapter has been published as *Compositio Mathematica* **199**: 209–212 (1999) and appears here with minor modifications.

Abstract. *In all characteristics $p > 2$ we construct a complete surface in the moduli space of smooth genus 6 curves. The surface is contained in the locus of curves with automorphisms.*

We consider the following question: 'What is the maximal number of essential parameters on which a complete family of smooth curves of genus g depends?' or equivalently, 'What is the maximal dimension of a complete subvariety of M_g, the moduli space of smooth curves of genus g?' In [14] Diaz provided an upper bound for the dimension of such a subvariety: for $g \geq 2$ this dimension is at most $g-2$. The moduli space M_g itself is irreducible, quasi-projective of dimension $3g-3$ ($g \geq 2$). Diaz proved his result in characteristic 0, but his bound also holds in characteristic > 0 (see [35]).

In order to see how good Diaz's bound is one has to construct complete subvarieties of M_g. This turns out to be a difficult problem, in any characteristic. Only in genus ≤ 3 Diaz's bound is known to be sharp, since it is known that M_g contains complete curves if g is at least 3 (see [16]). In higher genera almost nothing is known. The best result we know is a construction of complete subvarieties of arbitrary dimension $d \geq 1$ in M_g with $g \geq 2^{d+1}$. This construction gives a complete surface in M_8. For $g = 4$, 5, 6 and 7 we still do not know whether a complete surface in M_g exists.

4.1 The construction

Starting from a complete curve in M_3, we construct a complete surface in M_6. However, this construction only works in characteristic $\neq 0, 2$. Our result is:

Theorem 4.1 *In any characteristic $p > 2$ the moduli space M_6 of smooth genus 6 curves contains a complete surface.*

To construct in characteristic 0 a complete surface in M_6 seems mor difficult. This is more or less similar to the fact that the moduli space A_g of principally polarized abelian varieties of dimension g contains in characteristic $p > 0$ complete subvarieties of rather high dimension [45]. The corresponding situation in characteristic 0 is completely unknown.†

† However, see Keel-Sadun [32].