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Complete subvarieties of moduli spaces of algebraic curves

Zaal, C.G.

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Our construction depends heavily on a theorem of Keel. The starting point is a complete family $C \to B$ of smooth genus 3 curves. The idea is to construct a family of double covers ramified in two *distinct* points over the fibers of $C \to B$. To parametrize pairs of distinct branch points, we consider $C \times_B C$. On $C \times_B C$ we want to contract Δ to a point. To achieve this, we define on $C \times_B C$ a nef and big line bundle L with the property that $L|_\Delta$ is trivial. To show that the global sections of a power of L do not have a base locus we use a theorem of Keel which is only valid in positive characteristic. Keel's theorem relies on a lemma, which roughly states that if $L|_{\Delta_k}$ is trivial, then $L^{\otimes p}|_{\Delta_{kp}}$ is trivial, see [31][Lemma 1.7]. Here Δ_i is the ith order neighborhood of Δ, the subscheme of $C \times_B C$ defined by I^{k+1}, where I is the ideal sheaf of Δ. This is where the Frobenius map is used.

The line bundle L exists in all characteristics, but we do not know how to prove its eventual freeness in characterictic 0. Instead of using Keel's theorem, we tried to prove by direct methods that L is free on $C \times_B C$. To prove this, we need that L is trivial on Δ_i for every $i > 0$. Unfortunately, we don't know how to establish this.

4.2 Blowing down the diagonal

Let $C \to B$ be a family of smooth genus 3 curves over a complete 1-dimensional base B, having the property that the induced map $B \to M_3$ has finite fibers. We consider the fiber product $C \times_B C$. Let $\Delta \subset C \times_B C$ be the relative diagonal and $\pi_1, \pi_2 : C \times_B C \to C$ the projections on the first and second coordinate. On $C \times_B C$ consider the line bundle L associated to the divisor

$$(\pi_1^* + \pi_2^*)(K_{C/B}) + 2\Delta.$$

In characteristic $p > 0$ we can prove that a sufficiently high power of L is free:

Theorem 4.2 *Let L be the line bundle associated to the divisor $(\pi_1^* + \pi_2^*)(K_{C/B}) +$ 2Δ. Then L satisfies on $C \times_B C$:*

(i) the restriction of L to Δ is trivial;

(ii) L is big and nef on $C \times_B C$ and big on any subvariety not contained in Δ;

(iii) in characteristic $p > 0$ a sufficiently high multiple of L is free and defines a birational morphism of $C \times_B C$ to a projective threefold. Under this morphism, Δ is contracted to a point.

PROOF. (i) Let $\Delta : C \to C \times_B C$ be the diagonal map $c \mapsto (c, c)$. Then according to the adjunction formula $\Delta^*(K_{C\times_B C} + \Delta) \cong K_C$. Now $\Delta^*(K_{C\times_B C}) \cong K_{C/B} + K_C$, so it follows that $\Delta^*(\Delta) \cong -K_{C/B}$. Hence $\Delta^*(L) \cong \mathcal{O}_C$ and the restriction of L to Δ is trivial.

(ii) Let X be a subvariety of $C \times_B C$. If X has dimension 1, then

$$\left(\pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B})\right) \cdot X = K_{C/B} \cdot (\pi_{1,*}[X] + \pi_{2,*}[X]) > 0,$$

since $K_{C/B}$ is ample on C (see [1]) and since $\pi_{1,*}(X)$ and $\pi_{2,*}(X)$ cannot both be zero-dimensional. If X has dimension $s > 1$, then using a similar argument one proves that $\left(\pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B})\right)^s \cdot X > 0$. Hence by the Nakai-Moishezon