Complete subvarieties of moduli spaces of algebraic curves

Zaal, C.G.

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criterion $\pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B})$ is ample on $C \times_B C$. Since $L = \pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B}) + 2\Delta$, L is the sum of an ample and an effective divisor, hence big by one of the equivalent criteria for bigness. By (i) L is nef.

Moreover, if we restrict L to any positive dimensional subvariety X not contained in Δ, then $L|_X$ is the sum of the restriction of an ample divisor plus an effective divisor, hence $L|_X$ also big.

(iii) To show that L is eventually free, we use a result of Keel which states that in characteristic $p > 0$ a nef and big line bundle L is eventually free iff $L|_{E(L)}$ is eventually free [31][Theorem 1.2]. Here $E(L)$ is the exceptional locus of L; this is the union of all subvarieties along which L is not big. By (ii) L is nef and big. Together (i) and (ii) imply $E(L) = \Delta$. By (i) L restricted to Δ is free. Hence L is eventually free. □

Proof of Theorem 4.1. From Theorem 4.2 we conclude that for some n the global sections of $L^{\otimes n}$ yield a morphism $\phi : C \times_B C \to \mathbf{P}^N$ which contracts the diagonal to a point. In $\mathbf{P}^N$ choose a hyperplane not meeting the image of Δ. Then

$$T = \phi^{-1}(H) \subset C \times_B C \setminus \Delta$$

is a surface which parametrizes pairs of *distinct* points on the fibers of the family $C \to B$. By standard arguments one constructs a complete family $X \to S$, each fiber being a double cover of a fiber of $C \to B$ ramified in the two distinct points determined by $t \in T$, see [49][Section 1]. Locally we take square roots, so we have to exclude the case that the characteristic is 2. Since $C \to B$ is a family of smooth genus 3 curves, $X \to S$ is a family of smooth genus 6 curves. The base S is a finite cover of T. It is needed to overcome the monodromy arising from the fact that for one pair of distinct branch points one can choose a finite number of distinct coverings. The base S maps into the locus of curves in M_6 having non-trivial automorphisms. We claim that this image is 2-dimensional. To prove this, note that the structural map from S to M_6 factors as:

$$S \to \mathcal{R}_{3,2} \to M_6,$$

where $\mathcal{R}_{3,2}$ parametrizes double coverings of genus 3 curves ramified in two distinct points. The image of S in $\mathcal{R}_{3,2}$ is clearly 2-dimensional: S maps to M_3 with 1-dimensional fibers and 1-dimensional image. Moreover, the map $\mathcal{R}_{3,2} \to M_6$ is quasi-finite, since the image parametrizes smooth genus 6 curves with an involution with a genus 3 quotient and a genus 6 curve admits only finitely many involutions. □

Remark 4.3 Consider the *difference* map $C \times_B C \to \mathrm{Jac}(C/B)$, $(x, y) \mapsto [x - y]$. This map contracts the diagonal $\Delta \subset C \times_B C$ to a curve. Any hypersurface in $\mathrm{Jac}(C/B)$ not meeting this curve pulls back to a complete 2-dimensional subvariety T in $C \times_B C$ not meeting the diagonal. Starting from such a subvariety one can, as in the proof of Theorem 1, construct a family of smooth genus 6 curves. This would give a different construction of a complete 2-dimensional family of smooth genus 6 curves. But such a hypersurface is hard to find, as the following result of E. Colombo and P. Pirola [11] shows: Let $\pi : \mathcal{A} \to B$ be a family of abelian varieties of relative dimension g over a smooth complete curve B, with zero section $e : B \to \mathcal{A}$ and not isogenous to a family $\mathcal{A}_1 \times_B \mathcal{A}_2$ with $\mathcal{A}_1$ isotrivial. Let Z be an effective relative ample divisor on $\mathcal{A}$ and C a curve on $\mathcal{A}$. Then $C \cap Z \neq \emptyset$.