Complete subvarieties of moduli spaces of algebraic curves

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Chapter 5

Families of curves without sections

Abstract. *In this chapter we give examples of one-dimensional families of non-rational curves without sections. This answers a question of Joe Harris.*

In his lectures on rationally connected varieties [26], Joe Harris showed that families $X \to B$ with X smooth, B a curve and rationally connected fibers admit sections. He then asked for examples of families of varieties which are not rationally connected and which do not admit sections. To be more specific, he asked for 1-dimensional families of non-rational curves without sections.

In this chapter we construct such families. We will work over an algebraically closed field $\mathbf{k} = \bar{\mathbf{k}}$ of characteristic $\neq 2, 3$. In the first section we will construct a family of *stable* curves over $\mathbf{P}^1$, which does not admit a section. In the second section we will construct a complete family of *smooth* curves without a section.

5.1 Basic construction

In the following we will construct a family of stable curves of genus 2 over $\mathbf{P}^1$. This family is constructed in such a way, that it does not admit a section.

Step 1. The starting point of this construction are two curves covering $\mathbf{P}^1$. The first curve is a cover $p : \mathbf{P}^1 \to \mathbf{P}^1$ of degree 4 which is simply ramified in 6 points. The second is a degree 2 cover $q : E \to \mathbf{P}^1$ ramified in four points, so that E is an elliptic curve. Moreover, on the base $\mathbf{P}^1$ we want the branch loci of p and q to be disjoint.

Then we form the fiber product $C = \mathbf{P}^1 \times_{\mathbf{P}^1} E$, which is naturally embedded in $\mathbf{P}^1 \times E$ as a non-singular curve, since p and q ramify over distinct points. The projection $\pi_{\mathbf{P}^1}$ onto the first factor has degree 2 and ramifies in 16 points. The projection π_E onto the second factor has degree 4 and ramifies simply in 12 points. By Riemann-Hurwitz, the genus of C equals 7.

$$\begin{tikzcd} & C \arrow[dl, "\pi_{\mathbf{P}^1}"'] \arrow[dr, "\pi_E"] & \\ \mathbf{P}^1 \arrow[dr, "p"'] & & E \arrow[dl, "q"] \\ & \mathbf{P}^1 & \end{tikzcd}$$

Lemma 5.1 *The class of C in* $\mathrm{Pic}(\mathbf{P}^1 \times E)$ *is even.*

Proof. In $\mathbf{P}^1 \times E$, the curve C is of bidegree $(2, 4)$. Hence $C \sim_{\text{num}} 2h + 4v$, where h stands for the class of a 'horizontal' divisor $\mathbf{P}^1 \times \{\star\}$ and v for the class of a 'vertical' divisor $\{\star\} \times E$. Since $\mathbf{P}^1 \times E$ is ruled, $\mathrm{Pic}(\mathbf{P}^1 \times E) \cong \mathbf{Z} \cdot v \oplus \pi_E^\star(\mathrm{Pic}(E))$

(see [8]). So the class of C is linearly equivalent to $4v + \pi_E^*(d)$, where d is a divisor class on E of even degree. So $[C]$ is even. $\square$

Remark 5.2 Note that C is the zero locus of the function $p - q$ on $\mathbf{P}^1 \times E$. The pole divisor of $p-q$ equals $p^*(\infty) \times E + \mathbf{P}^1 \times q^*(\infty)$. From this it is immediate that C is linearly equivalent to $4v + \pi_E^*(d)$, with $d = q^*(\infty)$. This gives another way to see that the class of C is even.

STEP 2. Since the class of C in $\mathrm{Pic}(\mathbf{P}^1 \times E)$ is even we can by standard methods construct a double cover $f : S \to \mathbf{P}^1 \times E$ which ramifies precisely along C. The surface S is non-singular since C is. By construction, S is the total space of a family of curves in two ways. We denote by $\xi_{\mathbf{P}^1}$ the projection $\pi_{\mathbf{P}^1} \circ f$ from S to $\mathbf{P}^1$, and by ξ_E the projection $\pi_E \circ f$ from S to E.

$$\begin{array}{ccc} S & & \\ f\downarrow & & \\ \mathbf{P}^1 \times E & \xrightarrow{\pi_E} & E \\ \pi_{\mathbf{P}^1}\downarrow & & \\ \mathbf{P}^1 & & \end{array}$$

Theorem 5.3 *For the surface S as constructed above the following holds.*

1. *The fibers of $\xi_E : S \to E$ are stable curves of genus 1. The generic fiber is a smooth elliptic curve. The singular fibers are irreducible nodal rational curves, mapping onto $\mathbf{P}^1$ with degree 2 via $\xi_{\mathbf{P}^1}$.*
2. *The fibers of $\xi_{\mathbf{P}^1} : S \to \mathbf{P}^1$ are stable curves of genus 2. The generic fiber is a smooth curve of genus 2.*
3. *The family $\xi_{\mathbf{P}^1} : S \to \mathbf{P}^1$ has no sections.*

PROOF. First we prove (1) and (2). Singular fibers occur when ramification points meet, i.e., when the branch curve C in $\mathbf{P}^1 \times E$ is tangent to a horizontal divisor $\mathbf{P}^1 \times \{\star\}$ or a vertical divisor $\{\star\} \times E$. The fact that $\pi_E : C \to E$ is simply ramified guarantees that C intersects a horizontal divisor $\mathbf{P}^1 \times \{\star\}$ in four distinct points or in one double point and two distinct simple ramification points. So the singular fibers of $\xi_E : S \to E$ are irreducible nodal curves of arithmetic genus 1 mapping with degree 2 to $\mathbf{P}^1$. This proves (1) and (2).

To prove (3), suppose $s : \mathbf{P}^1 \to S$ is a section. Composing with the projection $\xi_E : S \to E$, we get a map $\mathbf{P}^1 \to E$ which must be constant. So the image of s in S is contained in the fibers of $\xi_E : S \to E$. But the fibers of $\xi_E : S \to E$ are all irreducible and map with degree 2 to $\mathbf{P}^1$, contradicting the fact that $\xi_{\mathbf{P}^1} \circ s$ has degree 1. So $\xi_{\mathbf{P}^1} : S \to \mathbf{P}^1$ has no sections. $\square$

5.2 A family of smooth curves without sections

In the family constructed in the previous section, the total space is a smooth surface. However, not all the fibers are smooth. With a bit of work we can modify the construction to obtain a family of *smooth* curves without sections.

Start with two curves covering $\mathbf{P}^1$. The first curve is a cover $p : \mathbf{P}^1 \to \mathbf{P}^1$ of degree 4 which is simply ramified in 6 points. The second is a degree 2 cover $q : D \to \mathbf{P}^1$ ramified in 6 points, so that D has genus 2. Again, we require that on the base $\mathbf{P}^1$ the branch loci of p and q are disjoint.

Then we form the fiber product $B = \mathbf{P}^1 \times_{\mathbf{P}^1} D$, which is naturally embedded in $\mathbf{P}^1 \times D$ as a non-singular curve, since p and q ramify over distinct points. The projection $\pi_{\mathbf{P}^1}$ onto the first factor has degree 2 and ramifies in 24 points. The projection π_D onto the second factor has degree 4 and ramifies simply in 12 points. By Riemann-Hurwitz, the genus of B equals 11.

Then construct an unramified covering $F \to D$ of degree 12. By Riemann-Hurwitz, F has genus 13. Form the fiber product $C = B \times_D F$. This curve embeds naturally as a smooth curve in $B \times F$. The genus of C equals 121.

$$\begin{array}{ccc} C & \longrightarrow & F \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \\ \downarrow & & \downarrow q \\ \mathbf{P}^1 & \xrightarrow{p} & \mathbf{P}^1 \end{array}$$

Lemma 5.4 *The class of C in* $\mathrm{Pic}(B \times F)$ *is even.*

PROOF. Just as in the proof of Lemma 5.1, the class $[B]$ of B in $\mathrm{Pic}(\mathbf{P}^1 \times D)$ is even. The class $[C]$ is the pullback of $[B]$ via the base change $F \to D$. Hence $[C]$ is also even. $\square$

Since the class of C in $\mathrm{Pic}(B \times F)$ is even, we can construct a double cover $g : T \to B \times F$ ramified precisely along C. The surface T is smooth since the branch curve C is. Composing g with the projection onto B yields a family of curves $\pi_B : T \to B$.

Theorem 5.5 *For the family $\pi_B : T \to B$ constructed above we have:*

1. *$\pi_B : T \to B$ is a complete family of smooth curves of genus 31.*
2. *The fibers of $\pi_F : T \to F$ are stable curves of genus 23. The generic fiber is a smooth curve of genus 23. The singular fibers are irreducible curves of geometrical genus 22 with one ordinary node, mapping onto B with degree 2 via π_B.*
3. *The family $\pi_B : T \to B$ has no sections.*

PROOF. (1) Since C is a curve of bidegree $(12, 4)$ in $B \times F$, every fiber of $\pi_B : T \to B$ is a double cover of the curve F ramified in 12 points. Since $C \to B$ is unramified, these points are *distinct*. Hence the fibers of $\pi_B : T \to B$ are smooth.

There are no non-constant maps from B to F, since $g(B) = 11$ and $g(F) = 13$. Now (2) and (3) follow as in the proof of Theorem 5.3. $\square$

Remark 5.6 One can realize this result even with curves of lower genus. If one chooses $F \to D$ to have degree 6, then $g(F) = 7$ and $g(B) = 11$. A map $B \to F$ has to be constant, because it cannot be an isomorphism and covers of F of degree 2 or more have genus at least 13.