Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 6
Some results on complete surfaces

As we have seen, the geometry of $M_g$ allows us to say something about the existence of complete curves: for all $g \geq 3$ they do exist. Moreover, we can construct explicit examples, in any genus $g \geq 3$.

As for the existence of complete surfaces, the situation is much more mysterious. Complete surfaces in $M_g$ do not exist for $g \leq 3$. For $g = 2$ this follows from the fact that $M_2$ is affine. For $g = 3$, any complete surface contained in $M_3$ would intersect the hyperelliptic locus $H_3$ in a complete curve ($H_3$ is ample on $M_3$—on $M_3$ the class of $H_3$ equals $9\lambda$, which is ample). Since $H_3$ is affine, this is impossible.

Diaz’s upper bound (Theorem 1.6) tells us that for $g \geq 4$, the moduli space $M_g$ may contain complete surfaces. In Chapter 1 and 2 we have shown examples, but they live in $M_g$, $g \geq 8$ (Fact 1.12 and Theorem 2.3). For $g = 4, 5, 6$ and 7 this problem is completely open. However, in Chapter 4 we demonstrated the existence of a complete surface in $M_6$ in odd characteristic.

In the following we state some general facts which can be deduced for complete surfaces in $M_g$ or, correspondingly, for non-degenerate families of smooth curves over a complete surface, in characteristic 0.

6.1 Complete surfaces in $M_g$ and the classification of surfaces

For $g \geq 4$ complete surfaces may exist in $M_g$. Such a complete surface gives rise to a complete family $X \rightarrow S$ of smooth curves, where $S$ is a smooth algebraic surface, in the following way.

1. Suppose $S_0 \subset M_g$ is a complete surface. In particular, $S_0$ is projective. The space $M_g$ does not admit a universal curve, but after adding a level $n$ structure $(n \geq 3)$, a universal curve exists. So after a finite base change, we get a family of smooth curves $X_1 \rightarrow S_1$, with $S_1$ a finite cover of $S_0$. So $S_1$ is projective as well.

2. Desingularizing $S_1$ and base changing, we get a family $X_2 \rightarrow S_2$ of smooth curves, where $S_2$ is smooth and projective. Note that the composite map $S_2 \rightarrow M_g$ is generically finite, but it may have 1-dimensional fibers.

In the case of complete families of curves over a 1-dimensional base, we saw that that the base has genus $\geq 2$. In particular, the base is a curve of general type. We claim that for 2-dimensional families a similar result holds, i.e., the base of a 2-dimensional complete family of smooth curves is of general type. We prove this in the following two theorems.
Theorem 6.1 Let \( f : X \to S \) be a non-degenerate family of smooth curves. If \( S \) is a smooth projective surface which is minimal, then \( S \) is of general type.

**Proof.** We recall the classification of smooth, projective minimal surfaces. For a smooth, projective variety \( X \) the Kodaira dimension is the dimension of \( \phi_nK(X) \) for \( n >> 0 \). In the case \( |nK| \) is empty for all \( n \) we set \( \kappa = -\infty \). For surfaces, we have \( \kappa = -\infty, 0, 1 \) or 2. If \( \kappa = 2 \), the surface \( S \) is called of general type. We use the classification of minimal smooth projective surfaces (see [8]):

\( \kappa = -\infty \) In this case \( S \) is birationally equivalent to a \( \mathbb{P}^1 \)-bundle over a curve. Then \( S \) is swept out by rational curves. The restriction of \( X \to S \) to these rational curves is isotrivial, since every family of smooth algebraic curves over \( \mathbb{P}^1 \) is isotrivial. This implies that the image of \( S \) in \( M_g \) is at most a curve, which is absurd.

\( \kappa = 0 \) To this case correspond four different types of surfaces.

- **Abelian surfaces** (\( p_g = 1 \) and \( q = 2 \)).
  Since the universal cover of an abelian surface \( S \) is \( \mathbb{C}^2 \), a complete family of smooth curves has to be constant: a map \( S \to M_g \) induces a holomorphic map \( C^2 \to H_g \), the Siegel upper half plane, which is analytically equivalent to a bounded domain. Therefore, the image of \( S \) in \( M_g \) is a point. This contradicts the assumption that \( f : X \to S \) is a non-degenerate family.

- **K3 surfaces** (\( p_g = 1 \) and \( q = 0 \)).
  Consider the map \( S \to M_g \). Because \( \pi_1(S,*) \) is trivial, we can trivialize \( R^1f_*\mathbb{Z} \) over \( S \). This yields a morphism \( S \to H_g \) which must be constant, since \( S \) is a projective surface and \( H_g \) is equivalent to a bounded domain. Hence the image of \( S \) in \( M_g \) is a point.

- **Enriques surfaces** (\( p_g = 0 \) and \( q = 0 \)).
  Every Enriques surface \( S \) admits a double étale cover \( \tilde{S} \to S \), with \( \tilde{S} \) a K3 surface. So this case reduces to the previous case.

- **Bi-elliptic or hyperelliptic surfaces** (\( p_g = 0 \) and \( q = 1 \)).
  Hyperelliptic surfaces are fibered over \( \mathbb{P}^1 \) by a pencil of elliptic curves [29]. Hence such a surface contains infinitely many elliptic curves, over which the family of curves must be isotrivial. This is absurd.

\( \kappa = 1 \) Let \( S \) be a minimal surface with \( \kappa = 1 \). Then there is a smooth curve \( B \) and a surjective morphism \( S \to B \) whose general fiber is an elliptic curve. Hence \( S \) contains infinitely many elliptic curves, which is absurd. \( \Box \)

In fact, in Theorem 6.1 we can get rid of the assumption that \( S \) is minimal. This is the content of the following theorem.

**Theorem 6.2** Let \( f : X \to S \) be a non-degenerate, complete family of smooth curves of genus \( g \geq 4 \). If \( S \) a smooth projective surface, then \( S \) is of general type.

**Proof.** Suppose that \( f : X \to S \) is a non-degenerate family of smooth curves, with \( S \) smooth and projective. If \( \kappa(S) = -\infty \), then \( S \) contains infinitely many rational curves, which is impossible (since \( f : X \to S \) is non-degenerate, the map \( S \to M_g \) has generically finite fibers, so only a finite number of curves in \( S \) get contracted—but any rational curve in \( S \) gets contracted.)
Therefore we may assume that \( \kappa(X) \geq 0 \). Let \( S' \) be the minimal model of \( S \) and let \( \pi : S \to S' \) the natural map.

Suppose \( S' \) is an abelian surface. The family \( X \to S \) gives a functorial map \( S \to M_g \). Since \( S \to S' \) is birational, this map factors via \( S \to S' \) (since \( S \to M_g \) contracts \( \mathbb{P}^1 \)s). So we have a map \( S' \to M_g \). We proceed as in the proof of Theorem 6.1.

If \( S' \) is a K3 surface, then \( \pi_1(S,*) \cong \pi_1(S',*) = 0 \), so the argument given in the proof of Theorem 6.1 applies: the period map gives a morphism \( S \to H_g \), which must be constant.

If \( S' \) is an Enriques surface, then pulling back to the double K3 cover of \( S' \) reduces to the case that \( S' \) is a K3 surface.

If \( S' \) is bielliptic, then \( S' \) contains infinitely many elliptic curves, so the same holds for \( S \). Again, this is absurd.

Finally, if \( \kappa(S') = 1 \), then also \( S \) contains infinitely many elliptic curves, since \( S \) is birational to \( S' \). This is absurd. \( \square \)

**Remark 6.3** In her thesis from 1998, Emanuela Nicorestianu independently proves Theorem 6.2 (see [44]). Her proof of Theorem 6.1 is essentially the same as the one given above, but the way in which she deduces Theorem 6.2 from Theorem 6.1 is different. In [44] she shows that any family of smooth curves \( X \to S \) over a non-singular surface \( S \) descends to a family of smooth curves \( X' \to S' \), with \( S' \) the minimal model of \( S \).

### 6.2 Surfaces in the locus of curves with nontrivial automorphisms

A standard way to construct complete families of smooth curves is via cyclic coverings. Thus the base of such a family maps to the locus of curves with non-trivial automorphisms. For \( g \geq 4 \) this locus is precisely the singular locus of \( M_g \).

For \( g = 4 \) or 5, it turns out that the locus of curves with automorphisms does not contain complete surfaces. The rest of this section contains a proof of this claim. The main result is Theorem 6.6. To prove this we need two lemmas.

**Lemma 6.4** Let \( C \) be a smooth curve and \( \Delta \subset C^n \) the big diagonal of \( n \)-tuples \( (p_1, \ldots, p_n) \) with \( p_i = p_j \) for some \( i \neq j \). Then \( C^n \setminus \Delta \) does not contain a complete surface.

**Proof.** Suppose that \( S \subset C^n \setminus \Delta \) is a complete surface, and \( n \geq 3 \). The projection of \( S \) onto the first \( n - 1 \) coordinates of \( C^n \) is a complete subvariety of \( C^{n-1} \setminus \Delta \). The fiber over \( (p_1, \ldots, p_{n-1}) \) is finite, since it is contained in \( C \setminus \{p_1, \ldots, p_{n-1}\} \). It follows that the projection of \( S \) is a complete surface in \( C^{n-1} \setminus \Delta \). A repeated application of this step yields a complete surface in \( C^2 \setminus \Delta \), which is absurd. \( \square \)
Lemma 6.5 If the moduli space of $n$-pointed genus $g$ curves $M_{g,n}$ contains a complete surface, then $g \geq 4$, or $g = 3$ and $n \geq 1$.

**Proof.** For $n \geq 3$ the space $M_{0,n}$ is isomorphic to $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \Delta$. Hence $M_{0,n}$ is affine and does not contain any complete subvarieties of dimension $> 0$.

For $n \geq 1$ the moduli space $M_{1,n}$ maps to $M_{1,1}$ which is affine. Hence a complete surface would be completely contained in a fiber of $M_{1,n} \rightarrow M_{1,1}$. The fiber over $(E,0)$ is a finite quotient of $(E \setminus 0)^{n-1} \setminus \Delta$, that is quasi-affine and does not contain any complete subvarieties of dimension $> 0$.

For $g \geq 2$, let $[C]$ be a point of $M_g$. The fiber of $M_{g,n} \rightarrow M_g$ over $[C]$ is a finite quotient of $C^n \setminus \Delta$. By Lemma 6.4 these fibers contain at most complete curves. So the image of a complete surface under $M_{g,n} \rightarrow M_g$ has dimension at least one, i.e., $M_g$ contains a complete curve. Thus $g \geq 3$. Finally, $M_3$ does not contain a complete surface (Theorem 1.6).

**Theorem 6.6** Suppose that the locus in $M_g(\mathbb{C})$ of curves with non-trivial automorphisms contains a complete surface. Then $g \geq 6$.

**Proof.** In [12], Cornalba has classified (over $\mathbb{C}$) the irreducible components of the locus of curves with automorphisms. Each component is a family $S(p, \gamma; a_1, \ldots, a_n)$ of curves $C$ having an automorphism $\sigma$ of prime order $p$, such that $C/\sigma$ has genus $\gamma$ and $C \rightarrow C/\sigma$ has $n$ branch points $q_1, \ldots, q_n$ and can be defined by a $p$th root of $\sum_{i=1}^n a_i q_i$, where the $a_i$ are integers between $1$ and $p-1$. Modulo $p$, the $n$-tuple $(a_i)_i$ is up to permutation and up to multiplication by an integer uniquely determined by the cover. If $C$ has genus $g$, then

$$2g - 2 = p(2\gamma - 2) + n(p - 1)$$

by the Riemann-Hurwitz formula.

Choose $m \geq 3$ and denote by $M_g[m]$ the moduli space of curves of genus $g$ with level $m$ structure. Over $M_g[m]$ we have a universal curve $C_g[m]$. By [13], thm. 1.11, there exists a scheme $A = \text{Aut}_{M_g[m]}(C_g[m])$, finite and unramified over $M_g[m]$, representing the functor of automorphism sets. Denote by $A_p$ the closed subscheme of $A$ representing automorphisms of order dividing $p$. Let $A'_p = A_p \setminus A_1$. It is easy to see that this is finite over $M_g[m]$. The image of $A'_p$ in $M_g$ is a union of several $S(p, \gamma, a_1, \ldots, a_n)$. Let $T(p, \gamma, a_1, \ldots, a_n)$ be the inverse image of $S(p, \gamma, a_1, \ldots, a_n)$ in $A'_p$. The space $T(p, \gamma, a_1, \ldots, a_n)$ maps onto $M_{\gamma,n}$. So a complete surface in $S(p, \gamma, a_1, \ldots, a_n)$ would via pull-back to $A'_p$ give a complete surface in $M_{\gamma,n}$.

For $g \leq 3$ we know already that $M_g$ itself does not contain any complete surfaces. To prove the theorem we have to deal only with the cases $g = 4$ and $g = 5$. 


In $M_4$ the locus of curves with nontrivial automorphisms has nine irreducible components:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Curve</th>
<th>Maps to</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$S(5,0;1,1,1,2)$</td>
<td>$M_{0,4}$</td>
</tr>
<tr>
<td></td>
<td>$S(5,0;1,2,3,4)$</td>
<td>$M_{0,4}$</td>
</tr>
<tr>
<td>3</td>
<td>$S(3,0;1,1,1,1,1,1)$</td>
<td>$M_{0,6}$</td>
</tr>
<tr>
<td></td>
<td>$S(3,0;1,1,1,2,2,2)$</td>
<td>$M_{0,6}$</td>
</tr>
<tr>
<td></td>
<td>$S(3,1;1,1,1)$</td>
<td>$M_{1,3}$</td>
</tr>
<tr>
<td>5</td>
<td>$S(2,2;1,1)$</td>
<td>$M_{2,2}$</td>
</tr>
<tr>
<td>6</td>
<td>$S(2;1,1,1,1,1,1,1,1)$</td>
<td>$M_{1,6}$</td>
</tr>
<tr>
<td>7</td>
<td>$S(2;0;1,1,1,1,1,1,1,1,1,1,1)$</td>
<td>$M_{0,10}$</td>
</tr>
</tbody>
</table>

A complete surface in the locus of curves with automorphisms of $M_4$ would map to a complete surface in one of the moduli spaces $M_{g,n}$ in the list. But by Lemma 6.5 no such $M_{g,n}$ contains a complete surface.

In $M_5$ the locus of curves with nontrivial automorphisms has seven irreducible components. Again by Lemma 6.5 it follows that no component of the locus of curves with automorphisms of $M_5$ can contain a complete surface:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Curve</th>
<th>Maps to</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$S(11,0;1,2,8)$</td>
<td>$M_{0,3}$</td>
</tr>
<tr>
<td>4</td>
<td>$S(3,0;1,1,1,1,1,1,2,2)$</td>
<td>$M_{0,7}$</td>
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<tr>
<td></td>
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<td>$M_{1,4}$</td>
</tr>
<tr>
<td>6</td>
<td>$S(2,3)$</td>
<td>$M_{3}$</td>
</tr>
<tr>
<td>7</td>
<td>$S(2,2;1,1,1,1)$</td>
<td>$M_{2,4}$</td>
</tr>
<tr>
<td>8</td>
<td>$S(2;1,1,1,1,1,1,1,1,1)$</td>
<td>$M_{1,8}$</td>
</tr>
<tr>
<td>9</td>
<td>$S(2;0;1,1,1,1,1,1,1,1,1,1,1)$</td>
<td>$M_{0,12}$</td>
</tr>
</tbody>
</table>

**Remark 6.7** Whereas for $g = 4$ or $5$ the moduli space $M_g$ cannot contain complete surfaces in the singular locus (in characteristic 0 or greater than 11), $M_6$ can in characteristic greater than 2.

One component of the singular locus of $M_6$ is $S(2,3;1,1)$. This is a cover of $M_{3,2}$. In characteristic $p > 2$ we can show that $M_{3,2}$ contains a complete surface, which in turn gives rise to a complete surface in $M_6$. This is the content of Chapter 4.