As we have seen, the geometry of $M_g$ allows us to say something about the existence of complete curves: for all $g \geq 3$ they do exist. Moreover, we can construct explicit examples, in any genus $g \geq 3$.

As for the existence of complete surfaces, the situation is much more mysterious. Complete surfaces in $M_g$ do not exist for $g \leq 3$. For $g = 2$ this follows from the fact that $M_2$ is affine. For $g = 3$, any complete surface contained in $M_3$ would intersect the hyperelliptic locus $H_3$ in a complete curve ($H_3$ is ample on $M_3$—on $M_3$ the class of $H_3$ equals $9\lambda$, which is ample). Since $H_3$ is affine, this is impossible.

Diaz’s upper bound (Theorem 1.6) tells us that for $g \geq 4$, the moduli space $M_g$ may contain complete surfaces. In Chapter 1 and 2 we have shown examples, but they live in $M_g$, $g \geq 8$ (Fact 1.12 and Theorem 2.3). For $g = 4$, 5, 6 and 7 this problem is completely open. However, in Chapter 4 we demonstrated the existence of a complete surface in $M_6$ in odd characteristic.

In the following we state some general facts which can be deduced for complete surfaces in $M_g$ or, correspondingly, for non-degenerate families of smooth curves over a complete surface, in characteristic 0.

### 6.1 Complete surfaces in $M_g$ and the classification of surfaces

For $g \geq 4$ complete surfaces may exist in $M_g$. Such a complete surface gives rise to a complete family $X \to S$ of smooth curves, where $S$ is a smooth algebraic surface, in the following way.

1. Suppose $S_0 \subset M_g$ is a complete surface. In particular, $S_0$ is projective. The space $M_g$ does not admit a universal curve, but after adding a level $n$ structure ($n \geq 3$), a universal curve exists. So after a finite base change, we get a family of smooth curves $X_1 \to S_1$, with $S_1$ a finite cover of $S_0$. So $S_1$ is projective as well.

2. Desingularizing $S_1$ and base changing, we get a family $X_2 \to S_2$ of smooth curves, where $S_2$ is smooth and projective. Note that the composite map $S_2 \to M_g$ is generically finite, but it may have 1-dimensional fibers.

In the case of complete families of curves over a 1-dimensional base, we saw that that the base has genus $\geq 2$. In particular, the base is a curve of general type. We claim that for 2-dimensional families a similar result holds, i.e., the base of a 2-dimensional complete family of smooth curves is of general type. We prove this in the following two theorems.
Section 6.1. Complete surfaces in $M_g$ and the classification of surfaces

**Theorem 6.1** Let $f : X \to S$ be a non-degenerate family of smooth curves. If $S$ is a smooth projective surface which is minimal, then $S$ is of general type.

**Proof.** We recall the classification of smooth, projective minimal surfaces. For a smooth, projective variety $X$ the Kodaira dimension is the dimension of $\phi_nK(X)$ for $n \gg 0$. In the case $|nK|$ is empty for all $n$ we set $\kappa = -\infty$. For surfaces, we have $\kappa = -\infty$, $0$, $1$ or $2$. If $\kappa = 2$, the surface $S$ is called of general type. We use the classification of minimal smooth projective surfaces (see [8]):

- $\kappa = -\infty$: In this case $S$ is birationally equivalent to a $\mathbb{P}^1$-bundle over a curve. Then $S$ is swept out by rational curves. The restriction of $X \to S$ to these rational curves is isotrivial, since every family of smooth algebraic curves over $\mathbb{P}^1$ is isotrivial. This implies that the image of $S$ in $M_g$ is at most a curve, which is absurd.

- $\kappa = 0$: To this case correspond four different types of surfaces.
  - Abelian surfaces ($p_g = 1$ and $q = 2$).
    Since the universal cover of an abelian surface $S$ is $\mathbb{C}^2$, a complete family of smooth curves has to be constant: a map $S \to M_g$ induces a holomorphic map $\mathbb{C}^2 \to H_g$, the Siegel upper half plane, which is analytically equivalent to a bounded domain. Therefore, the image of $S$ in $M_g$ is a point. This contradicts the assumption that $f : X \to S$ is a non-degenerate family.
  - $K3$ surfaces ($p_g = 1$ and $q = 0$).
    Consider the map $S \to M_g$. Because $\pi_1(S,*)$ is trivial, we can trivialize $R^1f_*\mathbb{Z}$ over $S$. This yields a morphism $S \to H_g$ which must be constant, since $S$ is a projective surface and $H_g$ is equivalent to a bounded domain. Hence the image of $S$ in $M_g$ is a point.
  - Enriques surfaces ($p_g = 0$ and $q = 0$).
    Every Enriques surface $S$ admits a double étale cover $\tilde{S} \to S$, with $\tilde{S}$ a $K3$ surface. So this case reduces to the previous case.
  - Bi-elliptic or hyperelliptic surfaces ($p_g = 0$ and $q = 1$).
    Hyperelliptic surfaces are fibered over $\mathbb{P}^1$ by a pencil of elliptic curves [29]. Hence such a surface contains infinitely many elliptic curves, over which the family of curves must be isotrivial. This is absurd.

- $\kappa = 1$: Let $S$ be a minimal surface with $\kappa = 1$. Then there is a smooth curve $B$ and a surjective morphism $S \to B$ whose general fiber is an elliptic curve. Hence $S$ contains infinitely many elliptic curves, which is absurd. □

In fact, in Theorem 6.1 we can get rid of the assumption that $S$ is minimal. This is the content of the following theorem.

**Theorem 6.2** Let $f : X \to S$ be a non-degenerate, complete family of smooth curves of genus $g \geq 4$. If $S$ a smooth projective surface, then $S$ is of general type.

**Proof.** Suppose that $f : X \to S$ is a non-degenerate family of smooth curves, with $S$ smooth and projective. If $\kappa(S) = -\infty$, then $S$ contains infinitely many rational curves, which is impossible (since $f : X \to S$ is non-degenerate, the map $S \to M_g$ has generically finite fibers, so only a finite number of curves in $S$ get contracted—but any rational curve in $S$ gets contracted.)
Therefore we may assume that $\kappa(X) \geq 0$. Let $S'$ be the minimal model of $S$ and let $\pi : S \to S'$ the natural map.

Suppose $S'$ is an abelian surface. The family $X \to S$ gives a functorial map $S \to M_g$. Since $S \to S'$ is birational, this map factors via $S \to S'$ (since $S \to M_g$ contracts $\mathbb{P}^1$s). So we have a map $S' \to M_g$. We proceed as in the proof of Theorem 6.1.

If $S'$ is a K3 surface, then $\pi_1(S, \ast) \cong \pi_1(S', \ast) = 0$, so the argument given in the proof of Theorem 6.1 applies: the period map gives a morphism $S \to H_g$, which must be constant.

If $S'$ is an Enriques surface, then pulling back to the double K3 cover of $S'$ reduces to the case that $S'$ is a K3 surface.

If $S'$ is bielliptic, then $S'$ contains infinitely many elliptic curves, so the same holds for $S$. Again, this is absurd.

Finally, if $\kappa(S') = 1$, then also $S$ contains infinitely many elliptic curves, since $S$ is birational to $S'$. This is absurd. \hfill $\square$

**Remark 6.3** In her thesis from 1998, Emanuela Nicoreistianu independently proves Theorem 6.2 (see [44]). Her proof of Theorem 6.1 is essentially the same as the one given above, but the way in which she deduces Theorem 6.2 from Theorem 6.1 is different. In [44] she shows that any family of smooth curves $X \to S$ over a non-singular surface $S$ descends to a family of smooth curves $X' \to S'$, with $S'$ the minimal model of $S$.

### 6.2 Surfaces in the locus of curves with nontrivial automorphisms

A standard way to construct complete families of smooth curves is via cyclic coverings. Thus the base of such a family maps to the locus of curves with non-trivial automorphisms. For $g \geq 4$ this locus is precisely the singular locus of $M_g$.

For $g = 4$ or 5, it turns out that the locus of curves with automorphisms does not contain complete surfaces. The rest of this section contains a proof of this claim. The main result is Theorem 6.6. To prove this we need two lemmas.

**Lemma 6.4** Let $C$ be a smooth curve and $\Delta \subset C^n$ the big diagonal of $n$-tuples $(p_1, \ldots, p_n)$ with $p_i = p_j$ for some $i \neq j$. Then $C^n \setminus \Delta$ does not contain a complete surface.

**Proof.** Suppose that $S \subset C^n \setminus \Delta$ is a complete surface, and $n \geq 3$. The projection of $S$ onto the first $n - 1$ coordinates of $C^n$ is a complete subvariety of $C^{n-1} \setminus \Delta$. The fiber over $(p_1, \ldots, p_{n-1})$ is finite, since it is contained in $C \setminus \{p_1, \ldots, p_{n-1}\}$. It follows that the projection of $S$ is a complete surface in $C^{n-1} \setminus \Delta$. A repeated application of this step yields a complete surface in $C^2 \setminus \Delta$, which is absurd. \hfill $\square$
Section 6.2. Surfaces in the locus of curves with nontrivial automorphisms

**Lemma 6.5** If the moduli space of $n$-pointed genus $g$ curves $M_{g,n}$ contains a complete surface, then $g \geq 4$, or $g = 3$ and $n > 1$.

**Proof.** For $n \geq 3$ the space $M_{0,n}$ is isomorphic to $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \Delta$. Hence $M_{0,n}$ is affine and does not contain any complete subvarieties of dimension $> 0$.

For $n \geq 1$ the moduli space $M_{1,n}$ maps to $M_{1,1}$ which is affine. Hence a complete surface would be completely contained in a fiber of $M_{1,n} \to M_{1,1}$. The fiber over $(E,0)$ is a finite quotient of $(E \setminus 0)^{n-1} \setminus \Delta$, that is quasi-affine and does not contain any complete subvarieties of dimension $> 0$.

For $g \geq 2$, let $[C]$ be a point of $M_g$. The fiber of $M_{g,n} \to M_g$ over $[C]$ is a finite quotient of $C^n \setminus \Delta$. By Lemma 6.4 these fibers contain at most complete curves.

So the image of a complete surface under $M_{g,n} \to M_g$ has dimension at least one, i.e., $M_g$ contains a complete curve. Thus $g \geq 3$. Finally, $M_3$ does not contain a complete surface (Theorem 1.6). \qed

**Theorem 6.6** Suppose that the locus in $M_g(C)$ of curves with non-trivial automorphisms contains a complete surface. Then $g \geq 6$.

**Proof.** In [12], Cornalba has classified (over $C$) the irreducible components of the locus of curves with automorphisms. Each component is a family $S(p, \gamma; a_1, \ldots, a_n)$ of curves $C$ having an automorphism $\sigma$ of prime order $p$, such that $C/\sigma$ has genus $\gamma$ and $C \to C/\sigma$ has $n$ branch points $q_1, \ldots, q_n$ and can be defined by a $p$th root of $\sum_{i=1}^n a_i q_i$, where the $a_i$ are integers between $1$ and $p - 1$. Modulo $p$, the $n$-tuple $(a_i)_i$ is up to permutation and up to multiplication by an integer uniquely determined by the cover. If $C$ has genus $g$, then

$$2g - 2 = p(2\gamma - 2) + n(p - 1)$$

by the Riemann-Hurwitz formula.

Choose $m \geq 3$ and denote by $M_g[m]$ the moduli space of curves of genus $g$ with level $m$ structure. Over $M_g[m]$ we have a universal curve $C_g[m]$. By [13], thm. 1.11, there exists a scheme $A = \text{Aut}_{M_g[m]}(C_g[m])$, finite and unramified over $M_g[m]$, representing the functor of automorphism sets. Denote by $A_p$ the closed subscheme of $A$ representing automorphisms of order dividing $p$. Let $A'_p = A_p \setminus A_1$. It is easy to see that this is finite over $M_g[m]$. The image of $A'_p$ in $M_g$ is a union of several $S(p, \gamma, a_1, \ldots, a_n)$. Let $T(p, \gamma, a_1, \ldots, a_n)$ be the inverse image of $S(p, \gamma, a_1, \ldots, a_n)$ in $A'_p$. The space $T(p, \gamma, a_1, \ldots, a_n)$ maps onto $M_{\gamma,n}$. So a complete surface in $S(p, \gamma, a_1, \ldots, a_n)$ would via pull-back to $A'_p$ give a complete surface in $M_{\gamma,n}$.

For $g \leq 3$ we know already that $M_g$ itself does not contain any complete surfaces. To prove the theorem we have to deal only with the cases $g = 4$ and $g = 5$. 


In $M_4$ the locus of curves with nontrivial automorphisms has nine irreducible components:

- **dimension 1:**
  - $S(5, 0; 1, 1, 1, 2)$ maps to $M_{0,4}$
  - $S(5, 0; 1, 2, 3, 4)$ maps to $M_{0,4}$
  - $S(3, 0; 1, 1, 1, 1, 1, 1)$ maps to $M_{0,6}$
  - $S(3, 0; 1, 1, 1, 1, 2, 2)$ maps to $M_{0,6}$
  - $S(3, 1; 1, 1, 1)$ maps to $M_{1,3}$
  - $S(2, 2; 1, 1)$ maps to $M_{2,2}$
  - $S(2, 1; 1, 1, 1, 1, 1, 1)$ maps to $M_{1,6}$
  - $S(2, 0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ maps to $M_{0,10}$

A complete surface in the locus of curves with automorphisms of $M_4$ would map to a complete surface in one of the moduli spaces $M_{g,n}$ in the list. But by Lemma 6.5 no such $M_{g,n}$ contains a complete surface.

In $M_5$ the locus of curves with nontrivial automorphisms has seven irreducible components. Again by Lemma 6.5 it follows that no component of the locus of curves with automorphisms of $M_5$ can contain a complete surface:

- **dimension 0:**
  - $S(11, 0; 1, 2, 8)$ maps to $M_{0,3}$
  - $S(3, 0; 1, 1, 1, 1, 1, 1, 2, 2)$ maps to $M_{0,7}$
  - $S(3, 1; 1, 1, 1, 2, 2)$ maps to $M_{1,4}$
  - $S(2, 3)$ maps to $M_3$
  - $S(2, 2; 1, 1, 1, 1)$ maps to $M_{2,4}$
  - $S(2, 1; 1, 1, 1, 1, 1, 1, 1, 1, 1)$ maps to $M_{1,8}$
  - $S(2, 0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ maps to $M_{0,12}$

**Remark 6.7** Whereas for $g = 4$ or $5$ the moduli space $M_g$ cannot contain complete surfaces in the singular locus (in characteristic 0 or greater than 11), $M_6$ can in characteristic greater than 2.

One component of the singular locus of $M_6$ is $S(2, 3; 1, 1)$. This is a cover of $M_{3,2}$. In characteristic $p > 2$ we can show that $M_{3,2}$ contains a complete surface, which in turn gives rise to a complete surface in $M_6$. This is the content of Chapter 4.