Complete subvarieties of moduli spaces of algebraic curves

Zaal, C.G.

Publication date
2005

CHAPTER 6
Some results on complete surfaces

As we have seen, the geometry of M_g allows us to say something about the existence of complete curves: for all $g \geq 3$ they do exist. Moreover, we can construct explicit examples, in any genus $g \geq 3$.

As for the existence of complete surfaces, the situation is much more mysterious. Complete surfaces in M_g do not exist for $g \leq 3$. For $g = 2$ this follows from the fact that M_2 is affine. For $g = 3$, any complete surface contained in M_3 would intersect the hyperelliptic locus H_3 in a complete curve (H_3 is ample on M_3—on M_3 the class of H_3 equals 9λ, which is ample). Since H_3 is affine, this is impossible.

Diaz's upper bound (Theorem 1.6) tells us that for $g \geq 4$, the moduli space M_g may contain complete surfaces. In Chapter 1 and 2 we have shown examples, but they live in M_g, $g \geq 8$ (Fact 1.12 and Theorem 2.3). For $g = 4$, 5, 6 and 7 this problem is completely open. However, in Chapter 4 we demonstrated the existence of a complete surface in M_6 in odd characteristic.

In the following we state some general facts which can be deduced for complete surfaces in M_g or, correspondingly, for non-degenerate families of smooth curves over a complete surface, in characteristic 0.

6.1 Complete surfaces in M_g and the classification of surfaces

For $g \geq 4$ complete surfaces may exist in M_g. Such a complete surface gives rise to a complete family $X \to S$ of smooth curves, where S is a smooth algebraic surface, in the following way.

1. Suppose $S_0 \subset M_g$ is a complete surface. In particular, S_0 is projective. The space M_g does not admit a universal curve, but after adding a level n structure ($n \geq 3$), a universal curve exists. So after a finite base change, we get a family of smooth curves $X_1 \to S_1$, with S_1 a finite cover of S_0. So S_1 is projective as well.
2. Desingularizing S_1 and base changing, we get a family $X_2 \to S_2$ of smooth curves, where S_2 is smooth and projective. Note that the composite map $S_2 \to M_g$ is generically finite, but it may have 1-dimensional fibers.

In the case of complete families of curves over a 1-dimensional base, we saw that that the base has genus ≥ 2. In particular, the base is a curve of general type. We claim that for 2-dimensional families a similar result holds, i.e., the base of a 2-dimensional complete family of smooth curves is of general type. We prove this in the following two theorems.

Theorem 6.1 *Let $f : X \to S$ be a non-degenerate family of smooth curves. If S is a smooth projective surface which is minimal, then S is of general type.*

PROOF. We recall the classification of smooth, projective minimal surfaces. For a smooth, projective variety X the Kodaira dimension is the dimension of $\phi_{nK}(X)$ for $n >> 0$. In the case $|nK|$ is empty for all n we set $\kappa = -\infty$. For surfaces, we have $\kappa = -\infty$, 0, 1 or 2. If $\kappa = 2$, the surface S is called of general type. We use the classification of minimal smooth projective surfaces (see [8]):

$\kappa = -\infty$ In this case S is birationally equivalent to a $\mathbf{P}^1$-bundle over a curve. Then S is swept out by rational curves. The restriction of $X \to S$ to these rational curves is isotrivial, since every family of smooth algebraic curves over $\mathbf{P}^1$ is isotrivial. This implies that the image of S in M_g is at most a curve, which is absurd.

$\kappa = 0$ To this case correspond four different types of surfaces.

Abelian surfaces ($p_g = 1$ and $q = 2$).

Since the universal cover of an abelian surface S is $\mathbf{C}^2$, a complete family of smooth curves has to be constant: a map $S \to M_g$ induces a holomorphic map $\mathbf{C}^2 \to H_g$, the Siegel upper half plane, which is analytically equivalent to a bounded domain. Therefore, the image of S in M_g is a point. This contradicts the assumption that $f : X \to S$ is a non-degenerate family.

K3 surfaces ($p_g = 1$ and $q = 0$).

Consider the map $S \to M_g$. Because $\pi_1(S, \star)$ is trivial, we can trivialize $R^1 f_* \mathbf{Z}$ over S. This yields a morphism $S \to H_g$ which must be constant, since S is a projective surface and H_g is equivalent to a bounded domain. Hence the image of S in M_g is a point.

Enriques surfaces ($p_g = 0$ and $q = 0$).

Every Enriques surface S admits a double étale cover $\tilde{S} \to S$, with $\tilde{S}$ a K3 surface. So this case reduces to the previous case.

Bi-elliptic or hyperelliptic surfaces ($p_g = 0$ and $q = 1$).

Hyperelliptic surfaces are fiberd over $\mathbf{P}^1$ by a pencil of elliptic curves [29]. Hence such a surface contains infinitely many elliptic curves, over which the family of curves must be isotrivial. This is absurd.

$\kappa = 1$ Let S be a minimal surface with $\kappa = 1$. Then there is a smooth curve B and a surjective morphism $S \to B$ whose general fiber is an elliptic curve. Hence S contains infinitely many elliptic curves, which is absurd. □

In fact, in Theorem 6.1 we can get rid of the assumption that S is minimal. This is the content of the following theorem.

Theorem 6.2 *Let $f : X \to S$ be a non-degenerate, complete family of smooth curves of genus $g \geq 4$. If S a smooth projective surface, then S is of general type.*

PROOF. Suppose that $f : X \to S$ is a non-degenerate family of smooth curves, with S smooth and projective. If $\kappa(S) = -\infty$, then S contains infinitely many rational curves, which is impossible (since $f : X \to S$ is non-degenerate, the map $S \to M_g$ has generically finite fibers, so only a finite number of curves in S get contracted—but any rational curve in S gets contracted.)

Therefore we may assume that $\kappa(X) \geq 0$. Let S' be the minimal model of S and let $\pi : S \to S'$ the natural map.

Suppose S' is an abelian surface. The family $X \to S$ gives a functorial map $S \to M_g$. Since $S \to S'$ is birational, this map factors via $S \to S'$ (since $S \to M_g$ contracts $\mathbf{P}^1$s). So we have a map $S' \to M_g$. We proceed as in the proof of Theorem 6.1.

If S' is a K3 surface, then $\pi_1(S,*) \cong \pi_1(S',*) = 0$, so the argument given in the proof of Theorem 6.1 applies: the period map gives a morphism $S \to H_g$, which must be constant.

If S' is an Enriques surface, then pulling back to the double K3 cover of S' reduces to the case that S' is a K3 surface.

If S' is bielliptic, then S' contains infinitely many elliptic curves, so the same holds for S. Again, this is absurd.

Finally, if $\kappa(S') = 1$, then also S contains infinitely many elliptic curves, since S is birational to S'. This is absurd. □

Remark 6.3 In her thesis from 1998, Emanuela Nicorestianu independently proves Theorem 6.2 (see [44]). Her proof of Theorem 6.1 is essentially the same as the one given above, but the way in which she deduces Theorem 6.2 from Theorem 6.1 is different. In [44] she shows that any family of smooth curves $X \to S$ over a non-singular surface S descends to a family of smooth curves $X' \to S'$, with S' the minimal model of S.

6.2 Surfaces in the locus of curves with nontrivial automorphisms

A standard way to construct complete families of smooth curves is via cyclic coverings. Thus the base of such a family maps to the locus of curves with non-trivial automorphisms. For $g \geq 4$ this locus is precisely the singular locus of M_g.

For $g = 4$ or 5, it turns out that the locus of curves with automorphisms does not contain complete surfaces. The rest of this section contains a proof of this claim. The main result is Theorem 6.6. To prove this we need two lemmas.

Lemma 6.4 *Let C be a smooth curve and $\Delta \subset C^n$ the big diagonal of n-tuples $(p_1, \ldots, p_n)$ with $p_i = p_j$ for some $i \neq j$. Then $C^n \setminus \Delta$ does not contain a complete surface.*

PROOF. Suppose that $S \subset C^n \setminus \Delta$ is a complete surface, and $n \geq 3$. The projection of S onto the first $n-1$ coordinates of C^n is a complete subvariety of $C^{n-1} \setminus \Delta$. The fiber over $(p_1, \ldots, p_{n-1})$ is finite, since it is contained in $C \setminus \{p_1, \ldots, p_{n-1}\}$. It follows that the projection of S is a complete surface in $C^{n-1} \setminus \Delta$. A repeated application of this step yields a complete surface in $C^2 \setminus \Delta$, which is absurd. □

Lemma 6.5 *If the moduli space of n-pointed genus g curves $M_{g,n}$ contains a complete surface, then $g \geq 4$, or $g = 3$ and $n \geq 1$.*

PROOF. For $n \geq 3$ the space $M_{0,n}$ is isomorphic to $(\mathbf{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$. Hence $M_{0,n}$ is affine and does not contain any complete subvarieties of dimension > 0.

For $n \geq 1$ the moduli space $M_{1,n}$ maps to $M_{1,1}$ which is affine. Hence a complete surface would be completely contained in a fiber of $M_{1,n} \to M_{1,1}$. The fiber over $(E, 0)$ is a finite quotient of $(E \setminus 0)^{n-1} \setminus \Delta$, that is quasi-affine and does not contain any complete subvarieties of dimension > 0.

For $g \geq 2$, let $[C]$ be a point of M_g. The fiber of $M_{g,n} \to M_g$ over $[C]$ is a finite quotient of $C^n \setminus \Delta$. By Lemma 6.4 these fibers contain at most complete curves. So the image of a complete surface under $M_{g,n} \to M_g$ has dimension at least one, i.e., M_g contains a complete curve. Thus $g \geq 3$. Finally, M_3 does not contain a complete surface (Theorem 1.6). □

Theorem 6.6 *Suppose that the locus in $M_g(\mathbf{C})$ of curves with non-trivial automorphisms contains a complete surface. Then $g \geq 6$.*

PROOF. In [12], Cornalba has classified (over $\mathbf{C}$) the irreducible components of the locus of curves with automorphisms. Each component is a family $S(p, \gamma; a_1, \ldots, a_n)$ of curves C having an automorphism σ of prime order p, such that C/σ has genus γ and $C \to C/\sigma$ has n branch points $q_1, \ldots, q_n$ and can be defined by a pth root of $\sum_{i=1}^{n} a_i q_i$, where the a_i are integers between 1 and $p-1$. Modulo p, the n-tuple $(a_i)_i$ is up to permutation and up to multiplication by an integer uniquely determined by the cover. If C has genus g, then

$$2g - 2 = p(2\gamma - 2) + n(p - 1)$$

by the Riemann-Hurwitz formula.

Choose $m \geq 3$ and denote by $M_g[m]$ the moduli space of curves of genus g with level m structure. Over $M_g[m]$ we have a universal curve $C_g[m]$. By [13], thm. 1.11, there exists a scheme $A = \mathrm{Aut}_{M_g[m]}(C_g[m])$, finite and unramified over $M_g[m]$, representing the functor of automorphism sets. Denote by A_p the closed subscheme of A representing automorphisms of order dividing p. Let $A'_p = A_p \setminus A_1$. It is easy to see that this is finite over $M_g[m]$. The image of A'_p in M_g is a union of several $S(p, \gamma, a_1, \ldots, a_n)$. Let $T(p, \gamma, a_1, \ldots, a_n)$ be the inverse image of $S(p, \gamma, a_1, \ldots, a_n)$ in A'_p. The space $T(p, \gamma, a_1, \ldots, a_n)$ maps onto $M_{\gamma,n}$. So a complete surface in $S(p, \gamma, a_1, \ldots, a_n)$ would via pull-back to A'_p give a complete surface in $M_{\gamma,n}$.

For $g \leq 3$ we know already that M_g itself does not contain any complete surfaces. To prove the theorem we have to deal only with the cases $g = 4$ and $g = 5$.

In M_4 the locus of curves with nontrivial automorphisms has nine irreducible components:

dimension 1:	$S(5,0;1,1,1,2)$	maps to:	$M_{0,4}$
	$S(5,0;1,2,3,4)$		$M_{0,4}$
3:	$S(3,0;1,1,1,1,1,1)$		$M_{0,6}$
	$S(3,0;1,1,1,2,2,2)$		$M_{0,6}$
	$S(3,1;1,1,1)$		$M_{1,3}$
5:	$S(2,2;1,1)$		$M_{2,2}$
6:	$S(2,1;1,1,1,1,1,1)$		$M_{1,6}$
7:	$S(2,0;1,1,1,1,1,1,1,1,1,1)$		$M_{0,10}$

A complete surface in the locus of curves with automorphisms of M_4 would map to a complete surface in one of the moduli spaces $M_{\gamma,n}$ in the list. But by Lemma 6.5 no such $M_{\gamma,n}$ contains a complete surface.

In M_5 the locus of curves with nontrivial automorphisms has seven irreducible components. Again by Lemma 6.5 it follows that no component of the locus of curves with automorphisms of M_5 can contain a complete surface:

dimension 0:	$S(11,0;1,2,8)$	maps to:	$M_{0,3}$
4:	$S(3,0;1,1,1,1,1,2,2)$		$M_{0,7}$
	$S(3,1;1,1,2,2)$		$M_{1,4}$
6:	$S(2,3)$		M_3
7:	$S(2,2,1,1,1,1)$		$M_{2,4}$
8:	$S(2,1;1,1,1,1,1,1,1,1)$		$M_{1,8}$
9:	$S(2,0;1,1,1,1,1,1,1,1,1,1,1,1)$		$M_{0,12}$

□

Remark 6.7 Whereas for $g = 4$ or 5 the moduli space M_g cannot contain complete surfaces in the singular locus (in characteristic 0 or greater than 11), M_6 can in characteristic greater than 2.

One component of the singular locus of M_6 is $S(2,3;1,1)$. This is a cover of $M_{3,2}$. In characteristic $p > 2$ we can show that $M_{3,2}$ contains a complete surface, which in turn gives rise to a complete surface in M_6. This is the content of Chapter 4.