Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 6
Some results on complete surfaces

As we have seen, the geometry of $M_g$ allows us to say something about the existence of complete curves: for all $g \geq 3$ they do exist. Moreover, we can construct explicit examples, in any genus $g \geq 3$.

As for the existence of complete surfaces, the situation is much more mysterious. Complete surfaces in $M_g$ do not exist for $g \leq 3$. For $g = 2$ this follows from the fact that $M_2$ is affine. For $g = 3$, any complete surface contained in $M_3$ would intersect the hyperelliptic locus $H_3$ in a complete curve ($H_3$ is ample on $M_3$—on $M_3$ the class of $H_3$ equals $9A$, which is ample). Since $H_3$ is affine, this is impossible.

Diaz's upper bound (Theorem 1.6) tells us that for $g \geq 4$, the moduli space $M_g$ may contain complete surfaces. In Chapter 1 and 2 we have shown examples, but they live in $M_g$, $g \geq 8$ (Fact 1.12 and Theorem 2.3). For $g = 4, 5, 6$ and $7$ this problem is completely open. However, in Chapter 4 we demonstrated the existence of a complete surface in $M_6$ in odd characteristic.

In the following we state some general facts which can be deduced for complete surfaces in $M_g$ or, correspondingly, for non-degenerate families of smooth curves over a complete surface, in characteristic 0.

6.1 Complete surfaces in $M_g$ and the classification of surfaces

For $g \geq 4$ complete surfaces may exist in $M_g$. Such a complete surface gives rise to a complete family $X \to S$ of smooth curves, where $S$ is a smooth algebraic surface, in the following way.

1. Suppose $S_0 \subset M_g$ is a complete surface. In particular, $S_0$ is projective. The space $M_g$ does not admit a universal curve, but after adding a level $n$ structure ($n \geq 3$), a universal curve exists. So after a finite base change, we get a family of smooth curves $X_1 \to S_1$, with $S_1$ a finite cover of $S_0$. So $S_1$ is projective as well.

2. Desingularizing $S_1$ and base changing, we get a family $X_2 \to S_2$ of smooth curves, where $S_2$ is smooth and projective. Note that the composite map $S_2 \to M_g$ is generically finite, but it may have 1-dimensional fibers.

In the case of complete families of curves over a 1-dimensional base, we saw that that the base has genus $\geq 2$. In particular, the base is a curve of general type. We claim that for 2-dimensional families a similar result holds, i.e., the base of a 2-dimensional complete family of smooth curves is of general type. We prove this in the following two theorems.
**Theorem 6.1** Let \( f : X \to S \) be a non-degenerate family of smooth curves. If \( S \) is a smooth projective surface which is minimal, then \( S \) is of general type.

**Proof.** We recall the classification of smooth, projective minimal surfaces. For a smooth, projective variety \( X \) the Kodaira dimension is the dimension of \( \phi_{nK}(X) \) for \( n >> 0 \). In the case \( |nK| \) is empty for all \( n \) we set \( \kappa = -\infty \). For surfaces, we have \( \kappa = -\infty, 0, 1 \) or \( 2 \). If \( \kappa = 2 \), the surface \( S \) is called of general type. We use the classification of minimal smooth projective surfaces (see [8]):

\( \kappa = -\infty \) In this case \( S \) is birationally equivalent to a \( \mathbb{P}^1 \)-bundle over a curve. Then \( S \) is swept out by rational curves. The restriction of \( X \to S \) to these rational curves is isotrivial, since every family of smooth algebraic curves over \( \mathbb{P}^1 \) is isotrivial. This implies that the image of \( S \) in \( M_g \) is at most a curve, which is absurd.

\( \kappa = 0 \) To this case correspond four different types of surfaces.

*Abelian surfaces* (\( p_g = 1 \) and \( q = 2 \)).

Since the universal cover of an abelian surface \( S \) is \( \mathbb{C}^2 \), a complete family of smooth curves has to be constant: a map \( S \to M_g \) induces a holomorphic map \( \mathbb{C}^2 \to H_g \), the Siegel upper half plane, which is analytically equivalent to a bounded domain. Therefore, the image of \( S \) in \( M_g \) is a point. This contradicts the assumption that \( f : X \to S \) is a non-degenerate family.

*K3 surfaces* (\( p_g = 1 \) and \( q = 0 \)).

Consider the map \( S \to M_g \). Because \( \pi_1(S,*) \) is trivial, we can trivialize \( R^1f_*\mathbb{Z} \) over \( S \). This yields a morphism \( S \to H_g \) which must be constant, since \( S \) is a projective surface and \( H_g \) is equivalent to a bounded domain. Hence the image of \( S \) in \( M_g \) is a point.

*Enriques surfaces* (\( p_g = 0 \) and \( q = 0 \)).

Every Enriques surface \( S \) admits a double étale cover \( \tilde{S} \to S \), with \( \tilde{S} \) a K3 surface. So this case reduces to the previous case.

*Bi-elliptic or hyperelliptic surfaces* (\( p_g = 0 \) and \( q = 1 \)).

Hyperelliptic surfaces are fibered over \( \mathbb{P}^1 \) by a pencil of elliptic curves [29]. Hence such a surface contains infinitely many elliptic curves, over which the family of curves must be isotrivial. This is absurd.

\( \kappa = 1 \) Let \( S \) be a minimal surface with \( \kappa = 1 \). Then there is a smooth curve \( B \) and a surjective morphism \( S \to B \) whose general fiber is an elliptic curve. Hence \( S \) contains infinitely many elliptic curves, which is absurd. \( \square \)

In fact, in Theorem 6.1 we can get rid of the assumption that \( S \) is minimal. This is the content of the following theorem.

**Theorem 6.2** Let \( f : X \to S \) be a non-degenerate, complete family of smooth curves of genus \( g \geq 4 \). If \( S \) a smooth projective surface, then \( S \) is of general type.

**Proof.** Suppose that \( f : X \to S \) is a non-degenerate family of smooth curves, with \( S \) smooth and projective. If \( \kappa(S) = -\infty \), then \( S \) contains infinitely many rational curves, which is impossible (since \( f : X \to S \) is non-degenerate, the map \( S \to M_g \) has generically finite fibers, so only a finite number of curves in \( S \) get contracted—but any rational curve in \( S \) gets contracted.)
Therefore we may assume that $\kappa(X) \geq 0$. Let $S'$ be the minimal model of $S$ and let $\pi : S \to S'$ the natural map.

Suppose $S'$ is an abelian surface. The family $X \to S$ gives a functorial map $S \to M_g$. Since $S \to S'$ is birational, this map factors via $S \to S'$ (since $S \to M_g$ contracts $\mathbb{P}^1$'s). So we have a map $S' \to M_g$. We proceed as in the proof of Theorem 6.1.

If $S'$ is a K3 surface, then $\pi_1(S, \ast) \cong \pi_1(S', \ast) = 0$, so the argument given in the proof of Theorem 6.1 applies: the period map gives a morphism $S \to H_g$, which must be constant.

If $S'$ is an Enriques surface, then pulling back to the double K3 cover of $S'$ reduces to the case that $S'$ is a K3 surface.

If $S'$ is bielliptic, then $S'$ contains infinitely many elliptic curves, so the same holds for $S$. Again, this is absurd.

Finally, if $\kappa(S') = 1$, then also $S$ contains infinitely many elliptic curves, since $S$ is birational to $S'$. This is absurd. 

\begin{remark}
In her thesis from 1998, Emanuela Nicorestianu independently proves Theorem 6.2 (see [44]). Her proof of Theorem 6.1 is essentially the same as the one given above, but the way in which she deduces Theorem 6.2 from Theorem 6.1 is different. In [44] she shows that any family of smooth curves $X \to S$ over a non-singular surface $S$ descends to a family of smooth curves $X' \to S'$, with $S'$ the minimal model of $S$.
\end{remark}

### 6.2 Surfaces in the locus of curves with nontrivial automorphisms

A standard way to construct complete families of smooth curves is via cyclic coverings. Thus the base of such a family maps to the locus of curves with non-trivial automorphisms. For $g \geq 4$ this locus is precisely the singular locus of $M_g$.

For $g = 4$ or 5, it turns out that the locus of curves with automorphisms does not contain complete surfaces. The rest of this section contains a proof of this claim. The main result is Theorem 6.6. To prove this we need two lemmas.

\begin{lemma}
Let $C$ be a smooth curve and $\Delta \subset C^n$ the big diagonal of $n$-tuples $(p_1, \ldots, p_n)$ with $p_i = p_j$ for some $i \neq j$. Then $C^n \setminus \Delta$ does not contain a complete surface.
\end{lemma}

\begin{proof}
Suppose that $S \subset C^n \setminus \Delta$ is a complete surface, and $n \geq 3$. The projection of $S$ onto the first $n - 1$ coordinates of $C^n$ is a complete subvariety of $C^{n-1} \setminus \Delta$. The fiber over $(p_1, \ldots, p_{n-1})$ is finite, since it is contained in $C \setminus \{p_1, \ldots, p_{n-1}\}$. It follows that the projection of $S$ is a complete surface in $C^{n-1} \setminus \Delta$. A repeated application of this step yields a complete surface in $C^2 \setminus \Delta$, which is absurd.
\end{proof}
Lemma 6.5 If the moduli space of $n$-pointed genus $g$ curves $M_{g,n}$ contains a complete surface, then $g \geq 4$, or $g = 3$ and $n \geq 1$.

Proof. For $n \geq 3$ the space $M_{0,n}$ is isomorphic to $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \Delta$. Hence $M_{0,n}$ is affine and does not contain any complete subvarieties of dimension $>0$.

For $n \geq 1$ the moduli space $M_{1,n}$ maps to $M_{0,1}$ which is affine. Hence a complete surface would be completely contained in a fiber of $M_{1,n} \to M_{0,1}$. The fiber over $(E,0)$ is a finite quotient of $(E \setminus 0)^{n-1} \setminus \Delta$, that is quasi-affine and does not contain any complete subvarieties of dimension $>0$.

For $g \geq 2$, let $[C]$ be a point of $M_g$. The fiber of $M_{g,n} \to M_g$ over $[C]$ is a finite quotient of $C^n \setminus \Delta$. By Lemma 6.4 these fibers contain at most complete curves. So the image of a complete surface under $M_{g,n} \to M_g$ has dimension at least one, i.e., $M_g$ contains a complete curve. Thus $g \geq 3$. Finally, $M_3$ does not contain a complete surface (Theorem 1.6).

Theorem 6.6 Suppose that the locus in $M_g(\mathbb{C})$ of curves with non-trivial automorphisms contains a complete surface. Then $g \geq 6$.

Proof. In [12], Cornalba has classified (over $\mathbb{C}$) the irreducible components of the locus of curves with automorphisms. Each component is a family $S(p, \gamma; a_1, \ldots, a_n)$ of curves $C$ having an automorphism $\sigma$ of prime order $p$, such that $C/\sigma$ has genus $\gamma$ and $C \to C/\sigma$ has $n$ branch points $q_1, \ldots, q_n$ and can be defined by a $p$th root of $\sum_{i=1}^n a_i q_i$, where the $a_i$ are integers between 1 and $p-1$. Modulo $p$, the $n$-tuple $(a_i)_i$ is up to permutation and up to multiplication by an integer uniquely determined by the cover. If $C$ has genus $g$, then

$$2g - 2 = p(2\gamma - 2) + n(p - 1)$$

by the Riemann-Hurwitz formula.

Choose $m \geq 3$ and denote by $M_g[m]$ the moduli space of curves of genus $g$ with level $m$ structure. Over $M_g[m]$ we have a universal curve $C_g[m]$. By [13], thm. 1.11, there exists a scheme $A = \text{Aut}_{M_g[m]}(C_g[m])$, finite and unramified over $M_g[m]$, representing the functor of automorphism sets. Denote by $A_p$ the closed subscheme of $A$ representing automorphisms of order dividing $p$. Let $A_p' = A_p \setminus A_1$. It is easy to see that this is finite over $M_g[m]$. The image of $A_p'$ in $M_g$ is a union of several $S(p, \gamma; a_1, \ldots, a_n)$. Let $T(p, \gamma; a_1, \ldots, a_n)$ be the inverse image of $S(p, \gamma; a_1, \ldots, a_n)$ in $A_p'$. The space $T(p, \gamma; a_1, \ldots, a_n)$ maps onto $M_{\gamma,n}$. So a complete surface in $S(p, \gamma; a_1, \ldots, a_n)$ would via pull-back to $A_p'$ give a complete surface in $M_{\gamma,n}$.

For $g \leq 3$ we know already that $M_g$ itself does not contain any complete surfaces. To prove the theorem we have to deal only with the cases $g = 4$ and $g = 5$. 

\[\square\]
In $M_4$ the locus of curves with nontrivial automorphisms has nine irreducible components:

- Dimension 1:
  - $S(5,0;1,1,1,2)$ maps to $M_{0,4}$
  - $S(5,0;1,2,3,4)$ maps to $M_{0,4}$
  - $S(3,0;1,1,1,1,1,1)$ maps to $M_{0,6}$
  - $S(3,0;1,1,1,2,2,2)$ maps to $M_{0,6}$
  - $S(3,1;1,1,1)$ maps to $M_{1,3}$
  - $S(2,2;1,1)$ maps to $M_{2,2}$
  - $S(2,1;1,1,1,1,1,1)$ maps to $M_{1,6}$
  - $S(2,0;1,1,1,1,1,1,1,1,1)$ maps to $M_{0,10}$

A complete surface in the locus of curves with automorphisms of $M_4$ would map to a complete surface in one of the moduli spaces $M_{\gamma,n}$ in the list. But by Lemma 6.5 no such $M_{\gamma,n}$ contains a complete surface.

In $M_5$ the locus of curves with nontrivial automorphisms has seven irreducible components. Again by Lemma 6.5 it follows that no component of the locus of curves with automorphisms of $M_5$ can contain a complete surface:

- Dimension 0:
  - $S(11,0;1,2,8)$ maps to $M_{0,3}$
  - $S(3,0;1,1,1,1,1,1,2,2)$ maps to $M_{0,7}$
  - $S(3,1;1,1,1,2,2)$ maps to $M_{1,4}$
  - $S(2,3)$ maps to $M_{3}$
  - $S(2,2;1,1,1,1)$ maps to $M_{2,4}$
  - $S(2,1;1,1,1,1,1,1,1,1)$ maps to $M_{1,8}$
  - $S(2,0;1,1,1,1,1,1,1,1,1,1,1)$ maps to $M_{0,12}$

**Remark 6.7** Whereas for $g = 4$ or $5$ the moduli space $M_g$ cannot contain complete surfaces in the singular locus (in characteristic 0 or greater than 11), $M_6$ can in characteristic greater than 2.

One component of the singular locus of $M_6$ is $S(2,3;1,1)$. This is a cover of $M_{3,2}$. In characteristic $p > 2$ we can show that $M_{3,2}$ contains a complete surface, which in turn gives rise to a complete surface in $M_6$. This is the content of Chapter 4.