Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 6
Some results on complete surfaces

As we have seen, the geometry of $M_g$ allows us to say something about the existence of complete curves: for all $g \geq 3$ they do exist. Moreover, we can construct explicit examples, in any genus $g \geq 3$.

As for the existence of complete surfaces, the situation is much more mysterious. Complete surfaces in $M_g$ do not exist for $g \leq 3$. For $g = 2$ this follows from the fact that $M_2$ is affine. For $g = 3$, any complete surface contained in $M_3$ would intersect the hyperelliptic locus $H_3$ in a complete curve ($H_3$ is ample on $M_3$—on $M_3$ the class of $H_3$ equals $9\lambda$, which is ample). Since $H_3$ is affine, this is impossible.

Diaz's upper bound (Theorem 1.6) tells us that for $g \geq 4$, the moduli space $M_g$ may contain complete surfaces. In Chapter 1 and 2 we have shown examples, but they live in $M_g$, $g \geq 8$ (Fact 1.12 and Theorem 2.3). For $g = 4, 5, 6$ and 7 this problem is completely open. However, in Chapter 4 we demonstrated the existence of a complete surface in $M_6$ in odd characteristic.

In the following we state some general facts which can be deduced for complete surfaces in $M_g$ or, correspondingly, for non-degenerate families of smooth curves over a complete surface, in characteristic 0.

6.1 Complete surfaces in $M_g$ and the classification of surfaces

For $g \geq 4$ complete surfaces may exist in $M_g$. Such a complete surface gives rise to a complete family $X \to S$ of smooth curves, where $S$ is a smooth algebraic surface, in the following way.

1. Suppose $S_0 \subset M_g$ is a complete surface. In particular, $S_0$ is projective. The space $M_g$ does not admit a universal curve, but after adding a level $n$ structure ($n \geq 3$), a universal curve exists. So after a finite base change, we get a family of smooth curves $X_1 \to S_1$, with $S_1$ a finite cover of $S_0$. So $S_1$ is projective as well.

2. Desingularizing $S_1$ and base changing, we get a family $X_2 \to S_2$ of smooth curves, where $S_2$ is smooth and projective. Note that the composite map $S_2 \to M_g$ is generically finite, but it may have 1-dimensional fibers.

In the case of complete families of curves over a 1-dimensional base, we saw that that the base has genus $\geq 2$. In particular, the base is a curve of general type. We claim that for 2-dimensional families a similar result holds, i.e., the base of a 2-dimensional complete family of smooth curves is of general type. We prove this in the following two theorems.
**Theorem 6.1** Let \( f : X \to S \) be a non-degenerate family of smooth curves. If \( S \) is a smooth projective surface which is minimal, then \( S \) is of general type.

**Proof.** We recall the classification of smooth, projective minimal surfaces. For a smooth, projective variety \( X \) the Kodaira dimension is the dimension of \( \phi_{nK}(X) \) for \( n \gg 0 \). In the case \( |nK| \) is empty for all \( n \) we set \( \kappa = -\infty \). For surfaces, we have \( \kappa = -\infty, 0, 1 \) or \( 2 \). If \( \kappa = 2 \), the surface \( S \) is called of general type. We use the classification of minimal smooth projective surfaces (see [8]):

\( \kappa = -\infty \) In this case \( S \) is birationally equivalent to a \( \mathbb{P}^1 \)-bundle over a curve. Then \( S \) is swept out by rational curves. The restriction of \( X \to S \) to these rational curves is isotrivial, since every family of smooth algebraic curves over \( \mathbb{P}^1 \) is isotrivial. This implies that the image of \( S \) in \( M_g \) is at most a curve, which is absurd.

\( \kappa = 0 \) To this case correspond four different types of surfaces.

- **Abelian surfaces** \((p_g = 1 \text{ and } q = 2)\).
  Since the universal cover of an abelian surface \( S \) is \( \mathbb{C}^2 \), a complete family of smooth curves has to be constant: a map \( S \to M_g \) induces a holomorphic map \( \mathbb{C}^2 \to H_g \), the Siegel upper half plane, which is analytically equivalent to a bounded domain. Therefore, the image of \( S \) in \( M_g \) is a point. This contradicts the assumption that \( f : X \to S \) is a non-degenerate family.

- **K3 surfaces** \((p_g = 1 \text{ and } q = 0)\).
  Consider the map \( S \to M_g \). Because \( \pi_1(S,*) \) is trivial, we can trivialize \( R^1f_*\mathbb{Z} \) over \( S \). This yields a morphism \( S \to H_g \) which must be constant, since \( S \) is a projective surface and \( H_g \) is equivalent to a bounded domain. Hence the image of \( S \) in \( M_g \) is a point.

- **Enriques surfaces** \((p_g = 0 \text{ and } q = 0)\).
  Every Enriques surface \( S \) admits a double étale cover \( \tilde{S} \to S \), with \( \tilde{S} \) a K3 surface. So this case reduces to the previous case.

- **Bi-elliptic or hyperelliptic surfaces** \((p_g = 0 \text{ and } q = 1)\).
  Hyperelliptic surfaces are fibered over \( \mathbb{P}^1 \) by a pencil of elliptic curves [29]. Hence such a surface contains infinitely many elliptic curves, over which the family of curves must be isotrivial. This is absurd.

\( \kappa = 1 \) Let \( S \) be a minimal surface with \( \kappa = 1 \). Then there is a smooth curve \( B \) and a surjective morphism \( S \to B \) whose general fiber is an elliptic curve. Hence \( S \) contains infinitely many elliptic curves, which is absurd. \( \square \)

In fact, in Theorem 6.1 we can get rid of the assumption that \( S \) is minimal. This is the content of the following theorem.

**Theorem 6.2** Let \( f : X \to S \) be a non-degenerate, complete family of smooth curves of genus \( g \geq 4 \). If \( S \) a smooth projective surface, then \( S \) is of general type.

**Proof.** Suppose that \( f : X \to S \) is a non-degenerate family of smooth curves, with \( S \) smooth and projective. If \( \kappa(S) = -\infty \), then \( S \) contains infinitely many rational curves, which is impossible (since \( f : X \to S \) is non-degenerate, the map \( S \to M_g \) has generically finite fibers, so only a finite number of curves in \( S \) get contracted—but any rational curve in \( S \) gets contracted.)
Therefore we may assume that $\kappa(X) \geq 0$. Let $S'$ be the minimal model of $S$ and let $\pi : S \to S'$ the natural map.

Suppose $S'$ is an abelian surface. The family $X \to S$ gives a functorial map $S \to M_g$. Since $S \to S'$ is birational, this map factors via $S \to S'$ (since $S \to M_g$ contracts $\mathbb{P}^1$s). So we have a map $S' \to M_g$. We proceed as in the proof of Theorem 6.1.

If $S'$ is a K3 surface, then $\pi_1(S,*) \cong \pi_1(S',*) = 0$, so the argument given in the proof of Theorem 6.1 applies: the period map gives a morphism $S \to H_g$, which must be constant.

If $S'$ is an Enriques surface, then pulling back to the double K3 cover of $S'$ reduces to the case that $S'$ is a K3 surface.

If $S'$ is bielliptic, then $S'$ contains infinitely many elliptic curves, so the same holds for $S$. Again, this is absurd.

Finally, if $\kappa(S') = 1$, then also $S$ contains infinitely many elliptic curves, since $S$ is birational to $S'$. This is absurd.

**Remark 6.3** In her thesis from 1998, Emanuela Nicorestianu independently proves Theorem 6.2 (see [44]). Her proof of Theorem 6.1 is essentially the same as the one given above, but the way in which she deduces Theorem 6.2 from Theorem 6.1 is different. In [44] she shows that any family of smooth curves $X \to S$ over a non-singular surface $S$ descends to a family of smooth curves $X' \to S'$, with $S'$ the minimal model of $S$.

### 6.2 Surfaces in the locus of curves with nontrivial automorphisms

A standard way to construct complete families of smooth curves is via cyclic coverings. Thus the base of such a family maps to the locus of curves with non-trivial automorphisms. For $g \geq 4$ this locus is precisely the singular locus of $M_g$.

For $g = 4$ or 5, it turns out that the locus of curves with automorphisms does not contain complete surfaces. The rest of this section contains a proof of this claim. The main result is Theorem 6.6. To prove this we need two lemmas.

**Lemma 6.4** Let $C$ be a smooth curve and $\Delta \subset C^n$ the big diagonal of $n$-tuples $(p_1, \ldots, p_n)$ with $p_i = p_j$ for some $i \neq j$. Then $C^n \setminus \Delta$ does not contain a complete surface.

**Proof.** Suppose that $S \subset C^n \setminus \Delta$ is a complete surface, and $n \geq 3$. The projection of $S$ onto the first $n - 1$ coordinates of $C^n$ is a complete subvariety of $C^{n-1} \setminus \Delta$. The fiber over $(p_1, \ldots, p_{n-1})$ is finite, since it is contained in $C \setminus \{p_1, \ldots, p_{n-1}\}$. It follows that the projection of $S$ is a complete surface in $C^{n-1} \setminus \Delta$. A repeated application of this step yields a complete surface in $C^2 \setminus \Delta$, which is absurd. $\square$
Section 6.2. Surfaces in the locus of curves with nontrivial automorphisms

Lemma 6.5 If the moduli space of \( n \)-pointed genus \( g \) curves \( M_{g,n} \) contains a complete surface, then \( g \geq 4 \), or \( g = 3 \) and \( n \geq 1 \).

Proof. For \( n \geq 3 \) the space \( M_{0,n} \) is isomorphic to \( (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \Delta \). Hence \( M_{0,n} \) is affine and does not contain any complete subvarieties of dimension \( > 0 \).

For \( n \geq 1 \) the moduli space \( M_{1,n} \) maps to \( M_{1,1} \) which is affine. Hence a complete surface would be completely contained in a fiber of \( M_{1,n} \to M_{1,1} \). The fiber over \( (E,0) \) is a finite quotient of \((\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-1} \setminus \Delta\), that is quasi-affine and does not contain any complete subvarieties of dimension \( > 0 \).

For \( g \geq 2 \), let \( [C] \) be a point of \( M_g \). The fiber of \( M_{g,n} \to M_g \) over \( [C] \) is a finite quotient of \( C^n \setminus \Delta \). By Lemma 6.4 these fibers contain at most complete curves. So the image of a complete surface under \( M_{g,n} \to M_g \) has dimension at least one, i.e., \( M_g \) contains a complete curve. Thus \( g \geq 3 \). Finally, \( M_3 \) does not contain a complete surface (Theorem 1.6).

Theorem 6.6 Suppose that the locus in \( M_g(\mathbb{C}) \) of curves with non-trivial automorphisms contains a complete surface. Then \( g \geq 6 \).

Proof. In [12], Cornalba has classified (over \( \mathbb{C} \)) the irreducible components of the locus of curves with automorphisms. Each component is a family \( S(p, \gamma; a_1, \ldots, a_n) \) of curves \( C \) having an automorphism \( \sigma \) of prime order \( p \), such that \( C/\sigma \) has genus \( \gamma \) and \( C \to C/\sigma \) has \( n \) branch points \( q_1, \ldots, q_n \) and can be defined by a \( p \)-th root of \( \sum_{i=1}^{n} a_i q_i \), where the \( a_i \) are integers between 1 and \( p-1 \). Modulo \( p \), the \( n \)-tuple \((a_i)_i\) is up to permutation and up to multiplication by an integer uniquely determined by the cover. If \( C \) has genus \( g \), then

\[
2g - 2 = p(2\gamma - 2) + n(p - 1)
\]

by the Riemann-Hurwitz formula.

Choose \( m \geq 3 \) and denote by \( M_g[m] \) the moduli space of curves of genus \( g \) with level \( m \) structure. Over \( M_g[m] \) we have a universal curve \( C_g[m] \). By [13], thm. 1.11, there exists a scheme \( A = \text{Aut}_{M_g[m]}(C_g[m]) \), finite and unramified over \( M_g[m] \), representing the functor of automorphism sets. Denote by \( A_p \) the closed subscheme of \( A \) representing automorphisms of order dividing \( p \). Let \( A_p' = A_p \setminus A_1 \). It is easy to see that this is finite over \( M_g[m] \). The image of \( A_p' \) in \( M_g \) is a union of several \( S(p, \gamma, a_1, \ldots, a_n) \). Let \( T(p, \gamma, a_1, \ldots, a_n) \) be the inverse image of \( S(p, \gamma, a_1, \ldots, a_n) \) in \( A_p' \). The space \( T(p, \gamma, a_1, \ldots, a_n) \) maps onto \( M_{g,n} \). So a complete surface in \( S(p, \gamma, a_1, \ldots, a_n) \) would via pull-back to \( A_p' \) give a complete surface in \( M_{g,n} \).

For \( g \leq 3 \) we know already that \( M_g \) itself does not contain any complete surfaces. To prove the theorem we have to deal only with the cases \( g = 4 \) and \( g = 5 \).
In $M_4$ the locus of curves with nontrivial automorphisms has nine irreducible components:

- dimension 1: $S(5, 0; 1, 1, 1, 2)$ maps to: $M_{0,4}$
  $S(5, 0; 1, 2, 3, 4)$ maps to: $M_{0,4}$
  $S(3, 0; 1, 1, 1, 1, 1, 1)$ maps to: $M_{0,6}$
  $S(3, 0; 1, 1, 1, 2, 2, 2)$ maps to: $M_{0,6}$
  $S(3, 1; 1, 1, 1)$ maps to: $M_{1,3}$
  $S(2, 2; 1, 1)$ maps to: $M_{2,2}$
  $S(2, 1; 1, 1, 1, 1, 1)$ maps to: $M_{1,6}$
  $S(2, 0; 1, 1, 1, 1, 1, 1, 1)$ maps to: $M_{0,10}$

A complete surface in the locus of curves with automorphisms of $M_4$ would map to a complete surface in one of the moduli spaces $M_{7,n}$ in the list. But by Lemma 6.5 no such $M_{7,n}$ contains a complete surface.

In $M_5$ the locus of curves with nontrivial automorphisms has seven irreducible components. Again by Lemma 6.5 it follows that no component of the locus of curves with automorphisms of $M_5$ can contain a complete surface:

- dimension 0: $S(11, 0; 1, 2, 8)$ maps to: $M_{0,3}$
  $S(3, 0; 1, 1, 1, 1, 1, 1, 2, 2)$ maps to: $M_{0,7}$
  $S(3, 1; 1, 1, 1, 2, 2)$ maps to: $M_{1,4}$
  $S(2, 3)$ maps to: $M_3$
  $S(2, 2; 1, 1, 1, 1)$ maps to: $M_{2,4}$
  $S(2, 1; 1, 1, 1, 1, 1, 1, 1)$ maps to: $M_{1,8}$
  $S(2, 0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ maps to: $M_{0,12}$

**Remark 6.7** Whereas for $g = 4$ or 5 the moduli space $M_g$ cannot contain complete surfaces in the singular locus (in characteristic 0 or greater than 11), $M_6$ can in characteristic greater than 2.

One component of the singular locus of $M_6$ is $S(2, 3; 1, 1)$. This is a cover of $M_{3,2}$. In characteristic $p > 2$ we can show that $M_{3,2}$ contains a complete surface, which in turn gives rise to a complete surface in $M_6$. This is the content of Chapter 4.