Complete subvarieties of moduli spaces of algebraic curves

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CHAPTER 7

Speculations

Whereas the previous chapters contain results on complete subvarieties of M_g, this chapter contains only questions. Questions which possibly might give rise to further research.

7.1 The genus of the base curve

If one tries to write down a naive example of a non-degenerate complete family of smooth curves of low genus, there are a few things one could try. For instance, one could try to construct a family of double, ramified covers of $\mathbf{P}^1$ ramified in six points—yielding a family of curves of arithmetic genus 2. Or one could start off with a curve in the $\mathbf{P}^{14}$ of degree 4 curves in $\mathbf{P}^2$—which yields a family of curves of arithmetic genus 3.

In either way, one discovers that the limiting fibers in the resulting families of curves do acquire singularities. In the first case because of the collapsing of branch points. In the second case because of the fact that the locus of singular degree 4 curves in $\mathbf{P}^2$ is a hypersurface in $\mathbf{P}^{14}$. And indeed constructing complete families of smooth curves is not straightforward.

As a matter of fact, complete families of smooth curves of genus g do not exist for $g = 0$, 1 or 2. This follows from the fact that the corresponding moduli spaces M_0, M_1 and M_2 are 'too small': M_0 is a point, M_1 is isomorphic to the affine line and M_2 is a 3-dimensional variety which is *affine* [30], hence does not contain complete curves. However, in Chapter 1 we have seen that for $g \geq 3$ complete families of smooth algebraic curves of genus g do exist if $g \geq 3$.

What about the base curves of such families? Over $\mathbf{C}$ the base of a complete family of smooth algebraic curves i not $\mathbf{P}^1$ or an elliptic curve, since such families necessarily have singular fibers. This is a consequence of the following facts:

1. Every non-isotrivial family of curves of $g \geq 1$ over $\mathbf{P}^1$ admits at least three singular fibers.
2. Every non-isotrivial family of curves of genus $g \geq 1$ over an elliptic curve has at least one singular fiber.

To see this, suppose $C \to B$ is a family of curves of genus $g \geq 1$. Let B^0 be the (non-empty) subset of B over which the fibers are smooth curves. The functorial map $B^0 \to M_g$ can be composed with the Torelli map $M_g \to A_g$. The composition $B^0 \to A_g$ can be lifted to the universal cover H_g, the Siegel upper half space of symmetric g by g matrices with positive definite imaginary part. The Siegel upper half space is equivalent to a bounded domain.

The universal covers of $\mathbf{P}^1$, $\mathbf{C}$ and $\mathbf{C}^*$ are $\mathbf{P}^1$ and $\mathbf{C}$ which are unbounded. By Liouville's theorem, every map of $\mathbf{P}^1$, $\mathbf{C}$ and $\mathbf{C}^*$ into H_g is constant. So if $B = \mathbf{P}^1$, then $\mathbf{P}^1 \setminus B^0$ must consist of at least three points. This proves (1). In the same vein, the universal cover of an elliptic curve E is $\mathbf{C}$. So if $B = E$, then $B \setminus B^0$ consists of at least one point. This proves (2).

For families of semi-stable curves over $\mathbf{P}^1$ or an elliptic curve we have the following results:

3. Every non-isotrivial family of semi-stable curves of genus $g \geq 1$ over $\mathbf{P}^1$ admits at least 4 singular fibers. (Beauville [7])
4. Every non-isotrivial family of semi-stable curves of genus $g \geq 2$ over $\mathbf{P}^1$ admits at least 5 singular fibers. (Beauville [7], Tan [48])
5. Every non-isotrivial family of semi-stable curves of genus 2 over an elliptic curve has at least two singular fibers.

Proof of 5. We apply a theorem of Tan [48], which states that:

$$\deg \lambda < \frac{g}{2}\big(2g(C) - 2 + s\big),$$

where C is the base curve of the family, s the number of singular fibers, and g the genus of the fiber. The theorem is a consequence of the so-called canonical class inequality. Applying this theorem to the case in which C is an elliptic curve and $g = 2$, we get $\deg \lambda < s$. Since $\deg \lambda \geq 1$ for a non-isotrivial family, we get $s \geq 2$. □

Remark 7.1 In characteristic $p > 0$ the so-called Moret-Bailly families are examples of non-isotrivial families of principally polarized abelian varieties over $\mathbf{P}^1$ [37]. However, in positive characteristic families of *semi-stable* curves of genus 1 over $\mathbf{P}^1$ have at least four singular fibers [43].

A natural question is: what is in characteristic 0 the minimal genus γ_g of the base curve of a 1-dimensional complete family of smooth curves of genus g? We have seen the following examples:

(a) Kodaira's construction gives a complete family of smooth curves of genus 6 over a base curve of genus 9 (Theorem 1.17);

(b) the example of González-Díez and Harvey yields a complete family of smooth curves of genus 4 over a base curve of genus 9 (Theorem 1.19).

Thus $\gamma_4 \leq 9$ and $\gamma_6 \leq 9$. By (1) and (2), we have $\gamma_g \geq 2$ for all $g > 2$. So we have that $2 \leq \gamma_4 \leq 9$ and $2 \leq \gamma_6 \leq 9$. The main question is to find good lower and upper bounds for γ_g.

7.2 Can M_4 contain a complete surface?

According to Diaz's bound, M_g does not contain any complete surfaces for $g \leq 3$. So M_4 is the first place to look for complete surfaces. In the rest of this section we try to tackle this question from several directions.

There are two approaches: (a) try to construct such a surface, or (b) try to disprove the existence of a complete surface in M_4.

7.2.1 A computation in the Chow ring of M_4

If a complete surface would exist in M_4, then it would determine a dimension 2 class in the Chow ring of M_4.

Faber has determined the structure of the Chow ring of M_4 completely (see [20]). Faber showed that $A^*(M_4) \cong \mathbf{Q}[\kappa_1]/\kappa_1^3$. In particular:

$$\begin{aligned} A^1(M_4) &= \mathbf{Q} \cdot \kappa_1 \neq 0, \\ A^2(M_4) &= \mathbf{Q} \cdot \kappa_1^2 \neq 0, \\ A^k(M_4) &= 0 \quad \text{for } k > 2. \end{aligned}$$

From this we conclude that the class of a complete surface in M_4 would be zero.

On the other hand, the class of a complete surface in the Chow ring of $\overline{M}_4$ is not zero. Faber has suggested a natural candidate for the class of a complete surface in M_4, namely $\lambda_3\lambda_4$ (see [22]). This is a dimension 2 class which has the following properties:

$$\begin{aligned} &\lambda_3\lambda_4 \cdot \alpha = 0 \text{ for all } \alpha \in A_*(\overline{M}_4 \setminus M_4), \\ &\lambda_3\lambda_4\kappa_1^2 \neq 0. \end{aligned}$$

A complete surface in M_4 would have precisely the same properties.

Question 7.2 A complete surface in M_4 determines a dimension two class in the Chow ring of $\overline{M}_4$. Is this class necessarily a multiple of $\lambda_3\lambda_4$?

One way to obtain complete curves in M_g is to embed M_g via a multiple of λ in a projective space and then to take hyperplane sections. Can one use the same trick to arrive at a complete surface in M_4, by using a different line bundle?

To be more precise: the Picard group of the stack M_4 (with $\mathbf{Q}$-coefficients) is generated by λ, δ_0, δ_1 and δ_2. For $a, b, c, d \in \mathbf{Z}$ write $H = a\lambda + b\delta_0 + c\delta_1 + d\delta_2$. Suppose that H is sufficiently positive. Since the dimension of M_4 is 9, we can ask ourselves whether H^7 could be a surface which has intersection zero with all codimension 2 boundary classes. In part, this comes down to an explicit computation, since all intersection numbers of monomials in λ, δ_0, δ_1 and δ_2 are explicitly computed by Faber [21]. This computation leads to the following.

Theorem 7.3 *The only* $\mathbf{Z}$*-linear combination of* λ, δ_0, δ_1 *and* δ_2 *whose 7th power has zero intersection with the boundary classes* $\lambda\delta_0$, $\lambda\delta_1$, $\lambda\delta_2$, δ_0^2, $\delta_0\delta_1$, $\delta_0\delta_2$, δ_1^2, $\delta_1\delta_2$ *and* δ_2^2 *is the zero combination.*

PROOF. Write $H = a\lambda + b\delta_0 + c\delta_1 + d\delta_2$. Intersecting H^7 with the denoted boundary classes and putting the resulting intersection numbers equal to zero, we get nine equations with rational coefficients in a, b, c and d, homogeneous of degree 7. Taking the primitive part of these equations, we look for solutions modulo a prime. Using Maple, one finds non-trivial solutions modulo 2 and 5, but only the trivial solution modulo 3 and 7. □

Remark 7.4 In [21] Faber gives algorithms for computing intersection numbers on moduli spaces of curves. In the programs Faber has developed, one uses tables of intersection numbers of the classes ψ_i on $\overline{M}_{g,n}$ made by the author of this thesis. The programs may be found at **http://math.stanford.edu/~vakil**

Remark 7.5 Since $\lambda_3\lambda_4$ is a candidate for the class of a complete surface in M_4 in the Chow ring of $\overline{M}_4$, one can ask oneself if one can get a complete surface by intersecting $\overline{M}_4$ with two different hypersurfaces K and L. For instance, one can study intersections of type K^4L^3.

Consider two $\mathbf{Z}$-linear combinations K and L of the divisor classes λ, δ_0, δ_1 and δ_2. Suppose $K^4 \cdot L^3$ intersects the boundary classes $\lambda\delta_0$, $\lambda\delta_1$, $\lambda\delta_2$, δ_0^2, $\delta_0\delta_1$, $\delta_0\delta_2$, δ_1^2, $\delta_1\delta_2$ and δ_2^2 trivially.

The relation $\delta_2(10\lambda - \delta_0 - 2\delta_1) = 0$ (see [20]) gives two irrelevant solutions. Modulo 11 there are two different solutions, but these do not give solutions over $\mathbf{Z}$, as a computation shows. Moreover, it turns out that there are no solutions congruent modulo 11 to one of the irrelevant solutions, but different from it.

7.2.2 The Prym construction for M_4

Given the construction from Chapter 3, it is an obvious question to ask if this construction can be iterated to obtain a complete surface in M_4. Namely, consider a 2-dimensional family of smooth genus 8 curves as obtained in Chapter 2. Every fiber is a double cover of a genus 4 curve, which itself is a double cover of a bi-elliptic curve of genus 2. Applying fiberwise the Prym construction, we obtain a family of 4-dimensional Prym varieties, which are principally polarized abelian varieties. Can all these principally polarized abelian varieties be Jacobians of smooth genus 4 curves?

Theorem 7.6 *Consider a 2-dimensional complete family of smooth genus 8 curves as obtained in Chapter 2. Consider the 2-dimensional family of principally polarized abelian varieties of dimension 4 which arises by taking fiberwise Pryms. Then not all Pryms are Jacobians of smooth genus 4 curves.*

PROOF. For the Pryms to be Jacobians, the base curve in the double coverings has to be trigonal, with the varying pairs of two distinct branch points contained in elements of the g_3^1 (exercise 10, p. 407 in [9]; to apply the exercise, note that on the base curves we have to identify the branch points, so we consider nodal, genus 5 curves).

So we have to check this for the 1-dimensional family of genus 4 curves obtained in Chapter 2. Clearly, at least one of those curves is non-hyperelliptic. The case of a g^1_3 with a total ramification point is obviously excluded.

Let C be a curve in the family with a g^1_3 with only simple ramification points. It is not difficult to show that the inverse image in C^3 of the g^1_3 in $C^{(3)}$ is a smooth curve of genus 13 that is an unramified triple cyclic cover of a smooth hyperelliptic curve of genus 5 with the same twelve branch points as the g^1_3 on C (see [19], §6). This triple cover is necessarily connected. Hence it intersects the three components of the big diagonal in C^3. Therefore this case is excluded as well. $\square$

7.2.3 Diaz's filtration of M_4

In 1984 Diaz proved that $g-2$ is an upper bound for the dimension of a complete subvariety of M_g [16]. In the proof he used the following stratification of M_g:

$$\emptyset = H_g(g,1) \subset H_g(g,2) \subset H_g(g,3) \subset \cdots \subset H_g(g,g-1) \subset H_g(g,g) = M_g.$$

Here $H_g(i,j)$ denotes the closure in M_g of the image of the Hurwitz scheme of genus g coverings $\pi : C \to \mathbf{P}^1$ of degree at most i, with $|\pi^{-1}(\{0,\infty\})| \leq j$ and simple ramification elsewhere. Diaz proves that the differences $H_g(i,j) \setminus H_g(i,j-1)$ do not contain any complete curves. This fact combined with the length of the stratification gives the upper bound $g-2$.

Because the general genus 4 curve has a g^1_3, we have $H_4(3,4) = M_4$. So for M_4 one may take a slightly smaller stratification:

$$H_4(3,1) = \emptyset \subset H_4(3,2) \subset H_4(3,3) \subset H_4(3,4) = M_4.$$

The general element of $H_4(3,3)$ is a genus 4 curve which possesses a base point free g^1_3 with one point of total ramification, and $H_4(3,2)$ is the union of the hyperelliptic locus and the locus of curves which possess a base point free g^1_3 with at least two points of total ramification.

With this filtration we can make Diaz's upper bound explicit for M_4, as follows. Suppose T is a complete threefold in M_4. Consider $T \cap H_4(3,3)$: this intersection is complete. Moreover, it necessarily has dimension 2. For if $\dim(T \cap H_4(3,3)) \leq 1$, then by cutting T with two sufficiently general hyperplanes in a projective embedding of $\overline{M}_4$, one would obtain a complete curve in $T \setminus H_4(3,3)$, contradicting the fact that $H_4(3,4) \setminus H_4(3,3)$ does not contain any complete curves. The same argument shows that $T \cap H_4(3,2)$ is complete and of dimension at least 1. But $H_4(3,2)$ can not contain any complete curves (see [14]). So in M_4 there is no complete threefold.

Suppose there exists in M_4 a complete surface. In the way described above, such a surface would give rise to a complete curve in $H_4(3,3)$. Such a curve can be lifted to a one-dimensional family of smooth genus 4 curves. The generic fiber of this family is a smooth genus 4 curve having a base point free g^1_3 with precisely one point of total ramification. The special fibers of this family are either smooth hyperelliptic curves of genus 4, or smooth genus 4 curves with a base point free g^1_3 with at least two points of total ramification.

So the non-existence of such a family of curves would disprove the existence of a complete surface in M_4. We formulate this as a corollary.

Corollary 7.7 *Suppose one can prove that there does not exist a non-isotrivial family of genus 4 curves over a complete base, in such a way that the general fiber carries a g_3^1 with one point of total ramification. Then there does not exist a complete surface in M_4.*