Exclusion and cooperation in networks

Ule, A.

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Chapter 2

Preliminaries

This chapter introduces the formal concepts that we use in subsequent chapters. In section 2.2 we describe the elements of the non-cooperative game theory. In section 2.3 we describe the elements of the mathematical theory of networks. In section 2.4 we describe the basic concepts and results of the theory of Markov chains.

2.1 Notation

The following notation is used throughout the thesis. For any set $S$ and any element $s \in S$ let $S \setminus \{s\}$ be the set obtained by removing $s$ from $S$. For any set $S$ and any subset $R \subseteq S$ let $S \setminus R$ be the set obtained by removing from $S$ all elements in $R$. For any two sets $S_1$ and $S_2$ let

$$S_1 \times S_2 = \{(s_1, s_2) \mid s_1 \in S_1, s_2 \in S_2\}$$

be their Cartesian product. Let $N = \{1, \ldots, n\}$. For any sets $S_1, \ldots, S_n$ let

$$\times_{i \in N} S_i = S_1 \times \ldots \times S_n = \{(s_1, \ldots, s_n) \mid s_1 \in S_1, \ldots, s_n \in S_n\}.$$ 

For any positive integer $t$ and any set $S$ let

$$\times_t S = \underbrace{S \times \ldots \times S}_{t} = \{(s_1, \ldots, s_t) \mid s_1, \ldots, s_t \in S\}.$$ 

Finally, let

$$\{g^t\}_{i \in N} = \{g^1, \ldots, g^n\}, \quad (p_i)_{i \in N} = (p_1, \ldots, p_n),$$

$$(p_{ij})_{j \in N} = (p_{i1}, \ldots, p_{in}), \text{ and } (t^n)_{\eta=0}^{m} = (t^0, \ldots, t^m).$$

2.2 Elements of game theory

A non-cooperative game is a description of an interaction between a set of agents, called players. The most basic class of non-cooperative games is that
of strategic games. A strategic game describes a situation in which each of the agents chooses his action or a plan of actions once and for all, and these choices are made simultaneously. Each possible profile of actions, one for each player, describes one possible outcome of such game. A solution is a systematic selection criterion over the outcomes.

The definition of a strategic game describes the players, the actions that players can take, and the players' preferences over the possible outcomes of the game.

**Definition 2.1** A finite strategic game is a triple \((N, (A_i)_{i \in N}, (\succ_i)_{i \in N})\), where

- \(N = \{1, 2, ..., n\}\) is the finite set of players,
- for every player \(i\), \(A_i\) is the finite set of actions available to \(i\),
- for every player \(i\), \(\succ_i\) is a preference relation on \(A = \times_{i \in N} A_i\), reflecting her preferences over outcomes \(A\).

In a strategic game each player \(i\) chooses her action \(a_i\) simultaneously with all other players. The outcome of the game is given by the profile of all chosen actions \(a = (a_1, ..., a_n) \in A\). The set \(A\) is called the set of action profiles and can be seen as the set of the outcomes of the game.

Players \(N \setminus \{i\} = \{1, ..., i-1, i+1, ..., n\}\) are called opponents of player \(i\). For each action profile \(a\) let

\[a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)\]

be the profile of actions taken by \(i\)'s opponents. Let \(A_{-i} = \times_{j \in N \setminus \{i\}} A_j\) be the set of all such action profiles. The profile of actions such that player \(i\) chooses \(a_i\) and her opponents choose \(a_{-i}\) is denoted by \((a_i, a_{-i})\).

Often the preference relation \(\succ_i\) of player \(i\) can be represented by a payoff function \(\pi_i : A \to \mathbb{R}\), in the sense that

\[\pi_i(a) \geq \pi_i(b) \iff a \succ_i b\]

for all pairs of outcomes \(a, b \in A\). The payoffs are meant to represent player's psychological utilities for various outcomes. Therefore, the payoff function is also called utility function. Two outcomes \(a, b \in A\) are payoff-equivalent if \(\pi_i(a) = \pi_i(b)\) for each \(i\).

Throughout the thesis we consider non-cooperative finite games with preferences represented by payoff functions. For simplicity we will refer to a finite

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1 This may be an unjust term as it gives an impression that players are necessarily adversaries. In a game the player's interests may be conflicting but need not be.
2 von Neumann and Morgenstern (1944) have shown that this is possible under remarkably weak assumptions about consistency and regularity of preferences. For a comprehensive discussion see also Myerson (1991).
Figure 2.1: An example of a payoff matrix representing a two-player strategic game. This is an instance of a prisoner’s dilemma game.

strategic game with set of players \( N \), action sets \( A_i \), and payoff functions \( \pi_i \) simply as game \( \langle N, A, \pi \rangle \), where \( \pi = (\pi_1, ..., \pi_n) \).

A game with two players can be described conveniently using a table, called payoff matrix, like the one illustrated in Figure 2.1. Player 1 is identified with the rows and player two with the columns. The column to the left of the table lists the actions available to player 1. The row at the top of the table lists actions available to player 2. Each cell of the table is associated with an action of player 1 and an action of player 2 and describes the payoffs that the players earn when each plays the corresponding action. The first number in a cell is the payoff to player 1 and the second number the payoff to player 2.

In the game described by the payoff matrix in Figure 2.1 each of the players has two possible actions, \( C \) and \( D \). If, for example, player 1 chooses \( C \) and player 2 chooses \( D \), the payoff to player 1 is 1 and the payoff to player 2 is 7. This game is an instance of prisoner’s dilemma games described below.

A standard assumption in game theory is that players are rational in the sense that they always select an action that maximizes their psychological utility given their beliefs about the actions being taken by the other players. Rationality is embedded in the notion of a best response mapping. For each player \( i \) her best response mapping \( B_i : A_{-i} \rightarrow A_i \) is defined by

\[
B_i(a_{-i}) = \{ a_i^* \in A_i | \pi_i(a_i^*, a_{-i}) \geq \pi_i(a_i, a_{-i}) \text{ for all } a_i \in A_i \} = \arg \max_{a_i \in A_i} \pi_i(a_i, a_{-i}).
\]

For each profile \( a_{-i} \in A_{-i} \) the set \( B_i(a_{-i}) \) consists of those actions of player \( i \) which maximize her payoff given that her opponents play \( a_{-i} \). An action \( a_i^* \in B_i(a_{-i}) \) is a best response of player \( i \) to actions \( a_{-i} \) of her opponents. For notational convenience we define also \( B_i(a) = B_i(a_{-i}) \) for \( a = (a_i, a_{-i}) \). If \( a_i \in B_i(a) \) we say that player \( i \) is playing her best response in \( a \).

The commonly used solution concept for strategic games is that of the Nash equilibrium (Nash, 1951). It characterizes action profiles which may be seen as steady states of the game. Each rational player \( i \) who knows that her opponents are playing \( a_{-i} \) is willing to play \( a_i \) if and only if it is a best response to \( a_{-i} \). In this sense rational players are willing to conform to an outcome \( a \in A \) if each of them is playing her best response in \( a \) and each knows that \( a \) will be chosen.

**Definition 2.2** A Nash equilibrium of a strategic game \( \langle N, A, \pi \rangle \) is an action profile \( a^* \in A \) such that

\[
a_i^* \in B_i^*(a) \text{ for every player } i.
\]
We may restate the definition as follows: \( a^* \in A \) is a Nash equilibrium if and only if for each \( i \),
\[
\pi_i(a^*_i, a^*_{-i}) \geq \pi_i(a'_i, a^*_{-i}) \quad \text{for all } a'_i \in A_i.
\]
Profile \( a^* \in A \) is a strict Nash equilibrium if and only if for each \( i \),
\[
\pi_i(a^*_i, a^*_{-i}) > \pi_i(a'_i, a^*_{-i}) \quad \text{for all } a'_i \in A_i, \ a'_i \neq a^*_i.
\]

### 2.2.1 Finitely repeated games

A non-cooperative game in which not all decisions are made simultaneously can be described as an extensive game. In this thesis we restrict our attention to extensive games obtained when a strategic game is repeated finitely many times. Such games are called finitely repeated games. We assume perfect recall, that is, at each period of the repeated game, each player is perfectly informed about all actions taken by all players in all previous periods. In game-theoretic terms, we assume complete information.

In each period of the repeated game all players simultaneously choose an action for the constituent strategic game. In each period they may make their decisions using the information about all past choices. Rational players take into account the effect of their current actions on the future behavior of the other players. Repeated games may thus capture phenomena like reputation, threats, and revenge.

Let \( \Delta = \langle N, A, \pi \rangle \) be a strategic game. Let \( T < \infty \) be the number of repetitions of the game \( \Delta \). We refer to the constituent strategic game \( \Delta \) as the *stage game*.

For each period \( t \in \{1, 2, \ldots, T\} \) the profile of actions at \( t \) is denoted by \( a^t \in A \). The action of player \( i \) in period \( t \) is \( a^t_i \in A_i \). A \( t \)-period *history* is a vector of action profiles in periods \( 1, 2, \ldots, t \),
\[
h^t = (a^1, a^2, \ldots, a^t).
\]
The space of all possible \( t \)-period histories is \( H^t = \times^t A \). Take \( t \) and \( \tau \) such that \( t + \tau \leq T \). For any \( t \)-period history \( h^t = (a^1, \ldots, a^t) \) and any \( \tau \)-period history \( \overline{h}^\tau = (\overline{a}^1, \ldots, \overline{a}^\tau) \), let
\[
h^t \oplus \overline{h}^\tau = (a^1, \ldots, a^t, \overline{a}^1, \ldots, \overline{a}^\tau)
\]
be the \((t + \tau)\)-period history given by actions \( h^t \) during the first \( t \) periods and \( \overline{h}^\tau \) during the following \( \tau \) periods.

The sequence of outcomes in each repetition of the stage game constitutes the outcome path of the repeated game. Let \( a^t \) be the period \( t \) outcome. The *outcome path* can be represented by the \( T \)-period history \( h^T = (a^1, \ldots, a^T) \). We assume throughout the thesis that players do not discount the future. The aim of a rational player \( i \) is to maximize her *total payoff*, given by
\[
\Pi_i(h^T) = \sum_{t=1}^{T} \pi_i(a^t).
\]
**Definition 2.3** A finitely repeated game is a tuple $\Delta^T = (N, A, \pi, T)$, representing a $T$-fold repetition of the stage game $\Delta = (N, A, \pi)$.

The behavior of a player in the repeated game is characterized by a strategy, which is a plan of actions that specifies, at the beginning of each period, how her action will depend on the most recent history of play. The behavior of players in repeated games is thus modeled by functions that map histories into actions.

A strategy for player $i$ in the game $\Delta^T$ is a function $\sigma_i$ which selects, for any history of play, an element of $A_i$. Formally,

$$\sigma_i = (\sigma_i^1, \sigma_i^2, ..., \sigma_i^T),$$

where $\sigma_i^t \in A_i$ and for $t > 1$,

$$\sigma_i^t : H^{t-1} \rightarrow A_i.$$

Let $\sigma^t = (\sigma_1^t, ..., \sigma_n^t)$. A profile of strategies $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ inductively defines an outcome path of the game $\Delta^T$ as follows: $a^1(\sigma) = \sigma^1$ and for $t \geq 1$,

$$a^{t+1}(\sigma) = \sigma^{t+1}(h^t(\sigma))$$

where

$$h^t(\sigma) = (a^1(\sigma), ..., a^t(\sigma)).$$

Let $S_i$ be the space of strategies of player $i$ and let $S = \times_{i \in N} S_i$. The repeated game $\Delta^T = (N, A, \pi, T)$ can be associated with the strategic game $(N, S, \Pi)$, called the normal form of the repeated game $\Delta$. The normal form describes the game in which each player chooses a strategy $\sigma_i \in S_i$ and earns the payoff, determined by the corresponding outcome path $h^T(\sigma)$,

$$\Pi_i(\sigma) = \Pi_i(h^T(\sigma)) = \sum_{t=1}^{T} \pi_i(a^t(\sigma)).$$

There is a crucial difference between the repeated game and its normal form. In the repeated game the players may modify their strategy in each period. In the normal form, however, the strategy is chosen once and for all at the outset of the game. The normal form is thus not the perfect representation of the repeated game. However, we can relate its solutions, such as its Nash equilibria, to the solutions for the repeated game.

In short, a strategy profile $\sigma \in S$ is a Nash equilibrium of the repeated game if and only if it is a Nash equilibrium of the corresponding normal form. Formally, let $(\sigma'_i, \sigma_{-i})$ denote the strategy profile $\sigma$ but with the $i^{th}$ component replaced by $\sigma'_i$. A Nash equilibrium of game $\Delta^T$ is a profile $\sigma^*$ such that for each player $i$, and any strategy $\sigma_i$ for $i$,

$$\Pi_i(\sigma^*) \geq \Pi_i(\sigma'_i, \sigma^*_{-i}).$$

The outcome path $h^T(\sigma^*)$ is an equilibrium path if $\sigma^*$ is a Nash equilibrium. We need to distinguish between the Nash equilibria of the repeated game and those of the stage game. We refer to the latter as the static equilibria. If an outcome in some period of the repeated game constitutes a static equilibrium, we say it is an equilibrium outcome.
2.2.2 Subgame perfect equilibrium

The concept of Nash equilibrium for repeated games treats the strategies as choices that are made once and for all at the outset of the first period. As such, it ignores the sequential structure of the repeated game and the fact that players can change their strategies between periods. The profile of strategies is a Nash equilibrium of the repeated game if each player’s strategy is optimal against the strategies of the other players along the equilibrium path. Off this path, that is, in case of a deviation, strategies do not need to be optimal. In particular, there is no restriction that in case of a deviation the players should be willing to continue to use their strategies.

We say that a Nash equilibrium of a repeated game is supported by non-credible threats if, in case of some deviation, some rational player does not want to follow her strategy.\(^3\) The equilibrium is supported by credible threats only if, after any history of play, it is in the self-interest of each player to continue using her original strategy. In a group of rational players the equilibrium path is sustainable if each player’s strategy is the best response not only along the equilibrium path but also in case of any deviation.

This is the spirit of the notion of subgame perfection introduced by Selten (1965) to eliminate the Nash equilibria of finite extensive form games which are sustained only as a result of non-credible threats off the equilibrium path. Selten suggested the solution concept of subgame perfection, which requires a strategy profile to be a Nash equilibrium along the equilibrium path, as well as along all other paths, that is, after any possible history of play.

Consider a finitely repeated game \(\Delta^T = \langle N, A, \pi, T \rangle\). Let \(h^t\) be a \(t\)-period history after some period \(t < T\) and let \(\Delta^{T-t} = \langle N, A, \pi, T - t \rangle\) be the game which consists of the remaining \(T - t\) periods of play. If \(\sigma_i\) is a strategy for \(i\) in \(\Delta^{T-t}\), let \(\sigma_i|_{h^t}\) be its continuation for periods \(t+1, \ldots, T\), given the history \(h^t\). Strategy \(\sigma_i|_{h^t}\) may then be seen as a strategy for player \(i\) in game \(\Delta^{T-t}\), defined iteratively by: (i) \(\sigma_i|_{h^t} = \sigma_i^{t+1}(h^t)\), and (ii) for any \(m < T - t\) and any \(m\)-period history \(h^m\),

\[
\sigma_i|_{h^t} \oplus h^m = \sigma_i^{t+m+1}(h^t \oplus h^m).
\]

The corresponding strategy profile in game \(\Delta^{T-t}\) is defined by \(\sigma|_{h^t} = (\sigma_1|_{h^t}, \ldots, \sigma_n|_{h^t})\).

A subgame of a repeated game \(\Delta^T\) is any pair \((h^t, \Delta^{T-t})\) with \(t < T\). The set of all subgames of a repeated game is generated by combining any possible history with the remaining periods of play.

**Definition 2.4** The strategy profile \(\sigma\) is a subgame perfect equilibrium of the game \(\Delta^T\) if (i) it is a Nash equilibrium of \(\Delta^T\), and (ii) for all subgames \((h^t, \Delta^{T-t})\), \(\sigma|_{h^t}\) is a Nash equilibrium of \(\Delta^{T-t}\).

\(^3\)See van Damme (1987), p. 4, and Fudenberg and Tirole (1991), p. 93, for some simple examples of Nash equilibria sustained by non-credible threats. See also example 4.3 below.
Take any, possibly non-equilibrium, strategy profile $\sigma \in S$. Let $h^{t-1}$ be the history in period $t$. If player $i$ follows her strategy in period $t$ she selects the action $\sigma_i^t(h^{t-1})$. We say that player $i$ deviated from $\sigma$ in period $t$ if her action at $t$ differs from $\sigma_i^t(h^{t-1})$. We say that a deviation from $\sigma$ occurred at $t$ if the outcome at $t$ differs from the actions selected by $\sigma$ given the history $h^{t-1}$.

A threat is an outcome path $h^{T-t}(|h^t)$ for the subgame $(h^t, \Delta^{T-t})$ selected by $\sigma$ if a deviation occurs at $t$. The threat selected by $\sigma$ in response to a deviation at $t$ is credible if $\sigma|_{h^t}$ is a subgame perfect equilibrium of $(h^t, \Delta^{T-t})$.

A threat keeps players from deviating if and only if, at the end of the game, the deviating players earns no more than what they would have earned had there been no deviation, assuming no other deviations occur. Suppose that player $i$ deviates at $t$, and that no more deviations occur at $t$ or in any later period. The total payoff of $i$ is then $\Pi_i(h^t \oplus h^{T-t}(\sigma|_{h^t}))$. Had she not deviated her total payoff would have been $\Pi_i(h^{t-1} \oplus h^{T-t-1}(\sigma|_{h^{t-1}}))$. The threat against her deviation is effective if

$$\Pi_i(h^t \oplus h^{T-t}(\sigma|_{h^t})) \leq \Pi_i(h^{t-1} \oplus h^{T-t-1}(\sigma|_{h^{t-1}})).$$

It can be shown that the profile $\sigma$ is a subgame perfect equilibrium if and only if all threats against one-player deviations are credible and effective. See Theorem 4.1 in Fudenberg and Tirole (1991) for the proof.

The following Lemma summarizes a few results about subgame perfect equilibria of repeated games that we use in the later chapters. Some illustrative examples are discussed in section 4.3.2. A thorough discussion is given, e.g., in Osborne and Rubinstein (1994).

**Lemma 2.1** Let $Q$ be the set of Nash equilibria of a stage game $\Delta = (N, A, \pi)$. Let $\Delta^T$ be the game $\Delta$ repeated $T$-times.

1. A strategy profile that in every period, and regardless of history, selects some static equilibrium is subgame perfect. That is, if $\sigma^{t+1}(h^t) \in Q$ for any $t$ and $h^t$ then $\sigma$ is subgame perfect.

2. If the stage game has a unique static equilibrium then the finitely repeated game has a unique subgame perfect equilibrium. It selects the static equilibrium in every period, regardless of the history. That is, if $Q = \{q\}$ then the unique subgame perfect equilibrium $\sigma$ is defined by $\sigma^{t+1}(h^t) = q$ for any $t$ and $h^t$.

3. If all static equilibria are payoff equivalent then all subgame perfect profiles are payoff equivalent. Any such profile selects in every period and regardless of the history one of the static equilibria. That is, if $\pi(q) = \pi(q')$ for any pair $q, q' \in Q$, then each subgame perfect equilibrium $\sigma$ satisfies $\sigma^{t+1}(h^t) \in Q$ for any $t$ and $h^t$. 

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$^4$This is known as the one-stage-deviation principle.
Sketch of the proof. (1) If in each period and independently of the history a static equilibrium is selected, then deviations in any period do not affect the continuation of the play. If she deviates the player therefore affects only her payoff in that period. However, since the play in any period is a static equilibrium the player cannot increase her payoff with any deviation.

(2) This can be verified by the backwards induction argument as follows. In the last period a static equilibrium will be played. If the static equilibrium is unique, it is played in the last period regardless of the history. This implies that players' decisions in the penultimate period have no influence on their decisions and payoffs in the last period. Players should thus regard the penultimate period just as the last one and play the unique static equilibrium also in the penultimate period. By backwards induction, from the last period to the first, the same argument applies to all periods. This proves that whenever the stage game has a unique Nash equilibrium, the unique subgame perfect equilibrium of the finitely repeated game is to play the static equilibrium in each of the periods.

(3) Consider a repeated game with different static equilibria such that each player receives the same payoff in all of them. Consider a subgame perfect equilibrium of the finitely repeated game. In the last period some static equilibrium must be selected. The selection may depend on the history, but the players are indifferent about which of them is played. Their decisions in the penultimate period do not affect their payoffs in the last period, and the players regard the penultimate period just as the last one. A static equilibrium must thus be selected also in the penultimate period, and by backwards induction, in all the periods of the repeated game. Hence, an equilibrium path of any subgame-perfect equilibrium is characterized by a sequence of static equilibrium outcomes.

2.2.3 Trigger strategies

If all static equilibria are payoff-equivalent the subgame perfect equilibrium must select an equilibrium outcome in each period of the finitely repeated game. In contrast, Friedman (1985) and Benoit and Krishna (1985) show that if static equilibria are not payoff-equivalent then a subgame perfect equilibrium may be constructed whose equilibrium path is not a sequence of static equilibrium outcomes.

Effective and credible punishment threats may be constructed by making the selection of the static equilibrium that is played in the last periods conditional on the history of play in the preceding periods. It may be possible to punish a deviating player by playing, in the last period, the equilibrium which gives her the lowest payoff. On the other hand, a player that never deviated can be rewarded if her preferred equilibrium is selected.

Intuitively, in order to construct a credible threat for each player in each period, it is necessary that the set of static Nash equilibria consists of

- a rewarding equilibrium which is played (at least in the last period) if all
players never deviated, and

- a set of punishment equilibria, one of which is played if some players deviate.

Each player must strictly prefer the rewarding equilibrium from her punishment equilibrium. Further static equilibria may exist which can be played if a group of players deviate simultaneously. For example, for each potential group of deviating players a static equilibrium may exist in which each of the deviating players earns less than in the rewarding equilibrium. However, group punishment equilibria are not necessary for subgame perfection as long as players cannot coordinate their deviations.

Effective and credible threats may be constructed as follows. If no player deviates the reward equilibrium is repeated in the last periods. This generates the reward path. A deviation, however, triggers a repetition of the punishment equilibrium. This generates the punishment path. A repetition of some static equilibrium is always subgame perfect, hence the threats are credible. The effectiveness of a punishment path grows with its length. By repeating the punishment equilibrium sufficiently many times effective and credible threats may be constructed against any deviations.

The threats against deviations in the first periods of the game consist of long punishment paths. Toward the last periods of the game only shorter and therefore less effective punishment paths are possible. Because the effectiveness of threats decreases with periods, only the outcomes with less profitable deviations may be sustained in the last periods of the game.

Example 2.1 (Trigger strategies) Let $q^*$ and $q^0$ be two Nash equilibria of the stage game $Δ = (N, A, π)$ such that $π_i(q^*) > π_i(q^0)$ for each player $i$. Consider also a non-equilibrium outcome $a ∈ A$ such that $π_i(a) > π_i(q^*)$ for each player $i$. All players prefer $a$ over any of the two equilibria, and prefer $q^*$ over $q^0$.

Even though $a$ is not a static equilibrium it can be sustained in the subgame perfect equilibrium of the repeated game $Δ^T$. Consider, for example, the following strategy profile: "Play $a$ during the early periods $1, \ldots, T - t^*$. In case of a deviation jump immediately into the punishment equilibrium $q^0$ and play it until the last period. If no deviations occur during the early periods, play the reward equilibrium $q^*$ in the remaining periods $T - t^* + 1, \ldots, T."$

The threats are credible because they consist of a repetition of a static equilibrium. Fix an early period $t ≤ T - t^*$ and assume there were no deviations before $t$. If player $i$ deviates at $t$ she earns $(T - t)π_i(q^0)$ in the remaining periods. If she does not deviate at $t$ or later she earns $(T - t - t^*)π_i(a) + t^*π_i(q^*).$ To make the threats during early periods effective the length of the reward phase $t^*$ needs to be such that for any player $i$ the smallest payoff loss from punishment, $t^*(π_i(q^*) - π_i(q^0))$, exceeds the most profitable one-period deviation from

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5For this, it is sufficient that one equilibrium is strictly preferred by all players to a particular another equilibrium.
a. Such \( t^* \) exists and the corresponding strategy profile is subgame perfect for any \( T > t^* \).

In the above example we outlined the simplest example of a trigger strategy which can sustain a non-equilibrium outcome in the early periods of a subgame perfect equilibrium. Any deviation from the non-equilibrium outcome in the early periods triggers an immediate switch to a punishment equilibrium, hence the name of the strategy. The following definition is due to Friedman (1985).\(^6\)

**Definition 2.5** Let \( Q \) be the set of Nash equilibria of a stage game \( \Delta = (N, R, \pi) \). Take \( q^*, q^1, \ldots, q^n \in Q \), which possibly coincide, and a profile \( r \in R \). A **trigger strategy profile** for \( \Delta^T \), based upon \( (r, q^*, \{q^i\}_{i \in N}, t^*) \), \( 1 \leq t^* < T \), is denoted by \( \sigma(r, q^*, \{q^i\}_{i \in N}, T, t^*) \) and is given by

- **outcome path**: play \( r \) during the early periods \( 1, \ldots, T - t^* \); play \( q^* \) during the remaining periods \( T - t^* + 1, \ldots, T \).
- **threat**: if player \( i \) deviates early, i.e. in an early period \( t \in \{1, \ldots, T - t^*\} \), play \( q^i \) during the remaining periods \( t + 1, \ldots, T \).
- **simultaneous deviations**: if several players deviate in the same early period, exercise the threat for the deviating player with the lowest index.\(^7\)

If no player ever deviates from the trigger strategy, the (possibly non-equilibrium) outcome \( r \) is selected in periods \( 1, \ldots, T - t^* \), and the static equilibrium \( q^* \) is selected in periods \( T - t^* + 1, \ldots, T \). Any deviation, on the other hand, triggers an immediate switch to the static equilibrium \( q^i \), where \( i \) is the deviating player with the lowest index, selected ever after. A sufficient condition for the threat of punishment to be effective is that each player \( i \) strictly prefers \( q^* \) over \( q^i \). The following theorem is due to Friedman (1985).\(^8\)

**Theorem 2.1** Let \( Q \) be the set of Nash equilibria of a stage game \( \Delta = (N, R, \pi) \). Take \( q^*, q^1, \ldots, q^n \in Q \) and a profile \( r \in R \) such that \( \pi_i(r) > \pi_i(q^i) \) for each \( i \). If \( \pi_i(q^*) > \pi_i(q^i) \) for each player \( i \) then there exists a positive integer \( \gamma \) such that the trigger strategy profile \( \sigma(r, q^*, \{q^i\}_{i \in N}, T, t^*) \) is a subgame perfect equilibrium of \( \Delta^T \) for all \( T, t^* \) such that \( T > t^* \geq \gamma \).

\(^6\)Friedman (1985) makes a distinction between trigger strategies and discriminating trigger strategies. In a trigger strategy profile each deviation triggers the same collective punishment. In a discriminating trigger strategy profile the punishments can be player specific. We do not emphasize this distinction and refer to all such profiles as trigger strategy profiles.

\(^7\)The threat against simultaneous deviations is irrelevant for subgame perfection. Punishing the defector with the lowest index, however, assures that in any group of potential defectors one player will not want to participate in a simultaneous defection.

\(^8\)The original theorem in Friedman (1985) allows for player-specific discount rates. Throughout this chapter we assume, for simplicity, that players do not discount future. We thus state a simplified version of the original theorem.
2.2.4 Prisoner's dilemma

Prisoner's dilemma games stylize situations in which there is a conflict between individual and common interests. Consider a pair of crime suspects being investigated in separate locations and with no possibility of communication. Each is given an opportunity to witness against the other. The court considers any witness favorably. If both confess each receives a sentence of 3 years in prison. If only one confesses against the other, he is set free and the other receives a sentence of 4 years. If neither confesses each receives a sentence of 1 year for a minor offense.

The suspects achieve the mutually optimal outcome if neither confesses. However, each has an incentive to confess regardless of the behavior of the other. Hence, rational suspects always witness against each other and serve 3 years each.

Let \( v \), the payoff function of the prisoner's dilemma game \( \langle \{1, 2\}, \{C, D\}^2, v \rangle \), be described by the following payoff matrix,

\[
\begin{array}{cc}
\text{Player } j & C & D \\
\text{Player } i & C & c, c & e, f \\
& D & f, e & d, d \\
\end{array}
\]

where \( f > c > d > e \), and \( 2c > f + e \). We interpret action \( D \) as defection and action \( C \) as cooperation. An instance of a prisoner's dilemma game is shown in Figure 2.1 on page 9. For each player action \( D \) is the best response to any action of the other player. Hence, \( (D, D) \) is the unique Nash equilibrium of a prisoner’s dilemma game.

Several generalizations of the prisoner's dilemma game can be made. An \( n \)-player prisoner's dilemma is the game \( \langle N, \{C, D\}^n, \pi \rangle \), where

\[
\pi_i(a) = \sum_{j \in N \setminus \{i\}} v_i(a_i, a_j).
\]

In the \( n \)-player prisoner's dilemma each player chooses either cooperation or defection and plays a 2-player game with each other player in turn.\(^9\)

In each of the prisoner's dilemma games above defection is the unique best response of each player. Hence, each of the games above has a unique Nash equilibrium in which all players defect. In contrast, the collectively optimal outcome in each of these games is attained when each player cooperates. The individual incentives thus conflict with common interest. This conflict is cumulated if any of the prisoner’s dilemma games above is finitely repeated: there is no cooperation in the unique subgame perfect equilibrium (see Lemma 2.1(2)).

A common feature of all these games is the fixed group structure: players do not choose with whom they play the game. In this thesis we explore the

\(^9\)This game can also be described as a linear public goods game with discrete binary choice, see Palfrey and Rosenthal (1984).
possibility to attain cooperation if players are free to choose their partners and establish the group structure endogenously. The corresponding stage game will be introduced in chapter 3. We outline in the following section the basic concepts of the mathematical theory of networks.

2.3 Elements of network theory

A network consists of a set of agents and a set of links. Each link connects a pair of agents. Formally, a network can be described by the mathematical concept of a graph. The elements of the network theory, introduced in this section, coincide with several basic concepts of graph theory. For advanced reading in the mathematical theory of graphs see e.g. Berge (1973).

We consider networks in which interaction is undirected. Whenever two agents have a mutual link they both interact with each other. We do not discuss directed networks and thus omit the classifier "undirected".

Definition 2.6 A network is characterized by

- a finite set of agents $N = \{1, \ldots, n\}$, and
- a set of links $L = \{ij \mid \{i,j\} \in N \times N$ and $i \neq j\}$.

Following the mathematical convention we denote the link between agents $i$ and $j$ simply by $ij$. In the definition of a network we implicitly assume that links are binary: two agents are either linked or not. Furthermore, a pair of agents can have at most one link and no agent can have a link with herself.

A network $(N, L)$ can be represented by an $n \times n$ binary adjacency matrix $g$, where

$$g_{ij} = 1 \text{ if } ij \in L \text{ and } g_{ij} = 0 \text{ otherwise.}$$

Matrix $g$ is symmetric: if $ij \in L$ then $ji \in L$. We refer to $g$ simply as network.

Agents $i$ and $j$ are neighbors in network $g$ if they have a mutual link, that is, if $g_{ij} = 1$. Set $L_i = \{j \mid ij \in L\}$ is the neighborhood of agent $i$. The size of $i$'s neighborhood is the number of her links $l_i = |L_i|$.\(^{10}\) Agent $i$ is isolated if it has no neighbors, i.e. if $l_i = 0$.

A path of length $k$ between agents $i$ and $j$ is the sequence of agents $(i, i_1, \ldots, i_{k-1}, j)$ such that $g_{ii_1} = g_{i_1i_2} = \ldots = g_{i_{k-1}j} = 1$. The (geodesic) distance between $i$ and $j$ is the length of the shortest path between them. The distance between $i$ and $j$ is infinite if there is no such path, and 1 if $i$ and $j$ are neighbors. If $i$ and $j$ are not neighbors but there is a path between them we say that they are indirectly linked.

A graph is connected if every pair of agents is (indirectly) linked. If a graph is not connected it has several connected components. A component of $g$ is a subset of agents $N' \subseteq N$ such that (i) all agents in $N'$ are (indirectly) linked and (ii) agents in $N' \setminus N'$ have no links with agents in $N \setminus N'$. An isolated agent,

\(^{10}\)In graph theory this is the degree of node $i$. 

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for example, constitutes a component herself. A subset of agents \( N' \subseteq N \) is a clique if all agents in \( N' \) are neighbors, that is, if each agent in \( N' \) is linked to every other agent in \( N' \).

A large body of literature on networks deals with network measures. A network cannot be captured by one unidimensional measure. Analysis of social, economic or technological networks usually relies on several such measures, each designed to capture one particular aspect of the network structure of topic-specific significance. Social scientists, for example, study network centrality, power distribution and cohesion, and have developed several measures of each (see e.g. Freeman, 1979, or Wasserman and Faust, 1994). Economists are interested in efficiency and equality of networks, and engineers care for network reliability and throughput. We evoke one simple measure that is used in most network analyzes. A density of a network is the number of established links divided by the number of all possible links,

\[
\text{density}(g) = \frac{2|L|}{n(n-1)}.
\]

**Example 2.2** For demonstration of several concepts we discuss the network illustrated in Figure 2.2. Circles correspond to agents and lines correspond to links. For example, there is a link between agents 1 and 2. The network has 6 agents and 5 links. The neighborhood of agent 1 consists of agents 2 and 4. Agent 1 is also indirectly linked to agents 3 and 5, but not to agent 6. Two different paths connect agent 1 with agent 5: \((1,4,5)\) and \((1,2,4,5)\). The shortest one is of length two, which is then the distance between agents 1 and 5. The agents \(\{1,2,3,4,5\}\) constitute a connected component, but the network is not connected. Player 6 is isolated. Players 1, 2 and 4 constitute the largest clique in the network.

This network can be captured by a \(6 \times 6\) binary matrix \(g\), defined by \(g_{12} = g_{21} = g_{14} = g_{41} = g_{24} = g_{42} = g_{34} = g_{43} = g_{45} = g_{54} = 1\) and otherwise \(g_{ij} = 0\). Its density is \(\frac{1}{3}\).

Several classes of networks, called network architectures, are prominent in the network literature. Figure 2.3 illustrates the following four architectures.
Figure 2.3: Common network architectures.

Empty network  No links are established: \( g_{ij} = 0 \) for all \( i \) and \( j \).

Complete network  All links are established: \( g_{ij} = 1 \) for all \( i \) and \( j \).

Wheel network  Links constitute a cycle spanning all agents. There is an ordering of all agents \((i_1, i_2, ..., i_N)\) such that \( g_{i_1i_n} = g_{i_ni_1} = 1 \) and \( g_{i_ji_{j+1}} = g_{i_{j+1}i_j} = 1 \) for all \( j \in N\{n\} \) and \( g_{k,k'} = 0 \) otherwise.

Star network  There is one central agent, linked to all other agents. There are no other links. If \( i \) is the central agent then \( g_{ik} = 1 \) for all \( k \in N\{i\} \) and \( g_{jk} = 0 \) for all \( j, k \in N\{i\} \).

For convenience we introduce the following notation. Given a network \( g \) and a pair of distinct agents \( ij \) let
\[
g \oplus ij
\]
be the network established by adding link \( ij \) to network \( g \), and let
\[
g \ominus ij
\]
be the network established when link \( ij \) is removed from network \( g \). If \( ij \) is already present in \( g \) then \( g \oplus ij = g \). Similarly, if \( ij \) is not present in \( g \) then \( g \ominus ij = g \).

2.4 Markov chains

In the following we shortly describe the elements of the theory of stationary discrete-time Markov chains. The theory is used in the analysis of dynamic
processes in chapter 6. An accessible introduction into Markov chain theory may be found in Tijms (1994). Further details and proofs may be found in advanced textbooks on the theory of stochastic processes, e.g. Bhattacharya and Waymire (1990).

Markov chains are among the simplest stochastic processes. A stochastic process is, roughly, an infinite sequence of random events. The sequence often represents the ordering of events in time, in which case we can speak of past and future events, of a current state, and of a history. Such a stochastic process is described by a set of possible states, and by a function that assigns a probability distribution over the future events to any history.

A stochastic process is discrete-time if events occur only at given, discrete points in time, usually denoted by \(t = 0, 1, 2, \ldots\). In such a process an event describes a transition from one possible state to another. At each discrete time point \(t\), the history of the process describes the sequence of states that the process went through at times \(0, \ldots, t\). A discrete-time stochastic process is described by a set of possible states and, for each possible history, a probability distribution over possible transitions between the states. The probability distribution over possible transitions between times \(t\) and \(t + 1\) may, in general, depend on the whole history up to \(t\). If, however, at each time point it depends only on the current state then the process is a Markov chain.

A discrete-time Markov chain is therefore a random sequence of transitions between states, such that at each time \(t\) the probability distribution over transitions between times \(t\) and \(t + 1\) is independent of the past states of the system during times \(0, \ldots, t - 1\), but may depend on the current state at \(t\). A Markov chain is stationary if the distribution over future transitions is independent of the time index \(t\). For simplicity we omit general definitions and define only the Markov chains used in this thesis.

**Definition 2.7** A finite stationary discrete-time Markov chain is characterized by

- a finite state space \(S = \{s_1, \ldots, s_k\}\) and
- a \(k \times k\) stochastic transition matrix \(M\).

The Markov chain starts at time 0 in some initial state \(s^0 \in S\). The choice of the initial state can be random. The evolution of the process continues as follows. Let \(s[t]\) be the state of Markov chain at time \(t\). If at time \(t\) the Markov chain is in state \(s\) then the probability that it will be in state \(s'\) at time \(t + 1\) is

\[M_{ss'} = \text{prob}[s[t + 1] = s' \mid s[t] = s].\]

The value of \(M_{ss'}\) therefore gives the probability of the transition from state \(s\) into state \(s'\). The value of \(M_{ss}\) is the probability that the process will not make any transition from state \(s\).

\(^{11}\) A \(k \times k\) matrix \(M\) is stochastic if each row constitutes a probability vector, that is, if \(M_{ij} \geq 0\) for each \(i, j\) and \(\sum_{j=1}^{k} M_{ij} = 1\) for each row \(i\).
For any non-negative integer $\tau$ let $M^\tau$ be the $\tau$-fold product of the matrix $M$, that is

$$M^\tau = M \cdot M \cdot \ldots \cdot M.$$ 

For each $t, \tau \in \mathbb{N}$ the value of $M^\tau_{ss'}$ gives the probability that the process is in state $s'$ at time $t + \tau$, conditional on being in state $s$ at time $t$.\(^\text{12}\) A state $s'$ is accessible from state $s$ if there exists a finite $\tau > 0$ such that $M^\tau_{ss'} > 0$. The process is aperiodic if for every pair of states $s$ and $s'$ such that $M^\tau_{ss'} > 0$ there exists a finite $t > 0$ such that $M^\tau_{ss'} > 0$ for all $\tau \geq t$.

A set of states $Z \subseteq S$ is recurrent, if

1. every state $s' \in Z$ is accessible from every other state $s \in Z$, and
2. the states outside $Z$ are not accessible from the states in $Z$: if $s \in Z$ and $s'' \in S \setminus Z$ then $s''$ is not accessible from $s$.

It can be shown that every Markov chain with a finite state space has at least one recurrent set. A state $s \in S$ is an absorbing state if $\{s\}$ constitutes a singleton recurrent set. If the process reaches an absorbing state it remains in that state forever. A state that is not absorbing and does not belong to a recurrent set is called transient. Every Markov chain eventually leaves transient states and converges to a recurrent set with probability one. For the analysis of the long run behavior of a Markov chain it is therefore necessary to characterize the recurrent sets of the process.

Suppose an initial state $s^0 \in S$. For each $t > 0$, let $\mu^t(s | s^0)$ be the relative frequency with which state $s$ is visited during the first $t$ periods. As $t$ goes to infinity, $\mu^t(s | s^0)$ converges almost surely to a probability distribution $\mu^\infty(s | s^0)$, called the asymptotic frequency distribution of the process conditional on $s^0$. Distribution $\mu^\infty$ can be interpreted as a selection criterion: over the long run, the process selects only those states on which $\mu^\infty(s | s^0)$ puts positive probability. Which states are selected in this sense may depend on the initial state, $s^0$. If it does not, that is, if the asymptotic distribution $\mu^\infty(s | s^0)$ is independent of $s^0$, we say that the process is ergodic.

Consequently, for every initial state $s^0$ the states on which $\mu^\infty(s | s^0)$ puts positive probability constitute a union of recurrent sets. Hence, the process is ergodic if and only if it has a unique recurrent set.

The asymptotic properties of finite Markov chains can be studied as follows. Let $\mu = (\mu(s))_{s \in S}$ be a probability distribution on $S$. Consider the system of linear equations

$$\mu M = \mu, \text{ where } \mu \geq 0 \text{ and } \sum_{s \in S} \mu(s) = 1. \quad (2.2)$$

It can be shown that this system always has at least one solution $\mu$, which is called a stationary distribution of the Markov chain with transition matrix $M$.

\(^{12}\)In matrix terms, if $\rho$ is the probability vector over the states at time $t$, the probability vector at time $t + \tau$ is given by $\rho M^\tau$. 22
Equation (2.2) states that, if the probability distribution over the initial states is given by $\mu$, the probability distribution over the states of the process at any time $t > 0$ is also given by $\mu$. The dynamics of the process is in this sense stationary, which motivates the use of the term. In general a Markov chain may have many stationary distributions. If, however, the process is ergodic the stationary distribution is unique and, furthermore, coincides with the asymptotic frequency distribution $\mu^\infty(s \mid s^0)$ for any $s^0$. More precisely, for an ergodic process,

$$\lim_{t \to \infty} \mu^t(s \mid s^0) = \mu^\infty(s \mid s^0) = \mu(s) \text{ for all } s, s^0.$$  \hfill (2.3)

The following proposition illustrates these ideas for a Markov chain in which transition between any pair of states occurs with positive probability. These results are used in chapter 6. Proofs may be found in Tijms (1994).

**Proposition 2.1** Consider a transition matrix $M^*$ with only positive entries: $M^*_{ss'} > 0$ for all $s, s' \in S$. The corresponding Markov chain has a number of properties:

1. the unique recurrent set consists of the whole state space $S$,
2. the process is aperiodic and ergodic,
3. the stationary distribution $\mu(s)$ is unique,
4. regardless of the initial state, for large $t$ the probability that the process is in state $s$ is approximately $\mu(s)$, and
5. regardless of the initial state, the proportion of time that the process spends in state $s$ is approximately $\mu(s)$.

\footnote{Above we also refer to stationary Markov chains, which should not be confused with a stationary distribution of a Markov chain.}