Continuous feedback fluid queues.

Scheinhardt, W.; van Foreest, N.; Mandjes, M.R.H.

DOI
10.1016/j.orl.2004.11.008

Publication date
2005

Published in
Operations Research Letters

Centrum voor Wiskunde en Informatica

REPORTRAPPORT

PNA

Probability, Networks and Algorithms

Probability, Networks and Algorithms

PNA

Continuous feedback fluid queues

W.R.W. Scheinhardt, N.D. van Foreest,
M.R.H. Mandjes

REPORT PNA-E0503 FEBRUARY 2005

CWI is the National Research Institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organization for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3711

Continuous feedback fluid queues

ABSTRACT
We investigate a fluid buffer which is modulated by a stochastic background process, while the momentary behavior of the background process depends on the current buffer level in a continuous way. Loosely speaking the feedback is such that the background process behaves `as a continuous-time Markov chain' with generator Q(y) at times when the buffer level is y, where the entries of Q(y) are continuous functions of y. Moreover, the fluid-flow rates for the buffer may also depend continuously on the current buffer level. First we define the feedback behavior precisely. Then we deduce the Kolmogorov forward equations for the joint background/buffer-process under some regularity assumptions. After presenting the differential equations and boundary conditions for the stationary distributions, we find an explicit solution when the background process has two states.

2000 Mathematics Subject Classification: 60K25
Keywords and Phrases: Fluid queue, feedback, forward equations

Continuous Feedback Fluid Queues

Werner Scheinhardt[◇ ‡], Nicky van Foreest[◇], Michel Mandjes[‡ ◇]

[◇] Department of Applied Mathematics
University of Twente
P.O. Box 217
7500 AE, Enschede, The Netherlands

[‡] Center for Mathematics and Computer Science (CWI)
P.O. Box 94079
1090 GB, Amsterdam, The Netherlands

September 21, 2004

Abstract

We investigate a fluid buffer which is modulated by a stochastic background process, while the momentary behavior of the background process depends on the current buffer level in a continuous way. Loosely speaking the feedback is such that the background process behaves 'as a continuous-time Markov chain' with generator $Q(y)$ at times when the buffer level is y, where the entries of $Q(y)$ are continuous functions of y. Moreover, the fluid-flow rates for the buffer may also depend continuously on the current buffer level.

First we define the feedback behavior precisely. Then we deduce the Kolmogorov forward equations for the joint background/buffer-process under some regularity assumptions. After presenting the differential equations and boundary conditions for the stationary distributions, we find an explicit solution when the background process has two states.

Keywords: Fluid queue, feedback, forward equations.

AMS Subject Classifications (2000) — 60K25

1 Introduction

In the area of modern telecommunication systems fluid queues are often used as burst scale models for multiplexers, see e.g. [15, Ch. 17], [18]. In such models the content process $\{C(t), t \geq 0\}$ of a fluid buffer changes at a rate determined by some autonomous modulating stochastic process $\{X(t),\, t \geq 0\}$. The net fluid input rate $r(t)$ at time t is then given by r_i at times when $X(t) = i$. Of particular importance are Markov-modulated fluid models in which the background process $\{X(t)\}$ is a Markov process, see e.g. [2, 11].

More general models, known as feedback fluid queues, were introduced in [1] and [16]. Here, the behavior of the fluid buffer content is determined by the background process as before, but in turn the behavior of the background process now depends on the current buffer level. Loosely speaking, the background process behaves as a continuous-time Markov chain, but its 'generator matrix' $Q(y)$ now depends on the current buffer level y. Due to this feedback, the background process is actually no longer a Markov process.

Feedback fluid queues can be useful for studying certain production systems and for modeling modern telecommunication networks in which the network and the sources interact. For instance, the interaction between one or two TCP sources, (i.e., traffic sources that use the Transmission Control Protocol, as currently deployed in the Internet) and some buffer in the network was analyzed in [8]. In [12, 13], feedback fluid queues were used to study feedback schemes in access networks. These models have the feature that the matrix $Q(y)$ is a piecewise constant function of the buffer level y. Moreover, the fluid rates may also depend on y as piecewise constant functions $r_i(y)$. In the current paper we analyze a more general feedback model in which both $Q(y)$ and the fluid rates $r_i(y)$ depend *continuously* on y. Clearly this explains the title of our paper, although a second interpretation is also possible, namely that the background process experiences an influence from the buffer continuously over time (i.e. in each arbitrarily small interval of time).

Apart from the relation with Markov-modulated and feedback fluid queues, our model also has a relation with extensions of the classical storage process, with state-dependent output. In the classical storage model, the input is a compound Poisson process, and the release rate (i.e. the rate at which the buffer is depleted) is constant, see [14]. Early extensions of this storage process are considered in [5, 9], and references therein, in which the release rate is state-dependent; in fact it is a strictly positive piecewise continuous function of the current buffer content. Another paper worth mentioning in this context is [3], where not only the release rate, but also the arrival rate of customers is state-dependent.

In all of the above models with state-dependent release rate, the input process is allowed to have jumps. Also in the context of fluid queues, where the input process is gradual, some models were studied where the input and/or output rates are state-dependent. The authors of [7] allow the net input rates, i.e., the difference between the input and the output rates, to be piecewise constant functions of the buffer content. In [10] the case is solved in which these rates are piecewise continuous functions. In fact these two models are special cases of the feedback models in [12] and the current paper respectively, since here also the fluid rates depend on the buffer content, while unlike [7, 10] also the behavior of the background process itself is influenced by the buffer content. Finally we mention the closely related (but independent) work in [4] where a generator approach is used to analyse a similar feedback fluid queue as here. However, the analysis is restricted to a background process with two states. Another difference is that in our case the buffer size is finite, while in [4] the buffer size is infinite.

We did not yet explain in detail what we mean by stating that the background process behaves 'as a Markov chain, with a level-dependent generator matrix $Q(y)$'. A precise description of our model is given in Section 2, where also some technical assumptions are mentioned. In Section 3 we derive the Kolmogorov forward equations for the joint Markov process $\{X(t), C(t), t \geq 0\}$. We do this by carefully following the infinitesimal approach, also employed in standard Markov-modulated fluid models. The technicalities needed for a rigorous proof are deferred to the appendix. In the final Section 4 we present the set of differential equations and boundary conditions that describe the stationary distribution, as well as a closed form solution for the case in which the background process has two states.

2 Model and Preliminaries

In this section we explain the fluid model of interest more precisely. We also state some assumptions on the functions involved and introduce some notation.

2.1 Model

Consider a fluid system consisting of a finite buffer of size B and one or more sources that transmit fluid into it. The background process, or source process, is denoted by $\{X(t), t \geq 0\}$ and has state space $\mathscr{X} = \{1, \ldots, N\}$ for some finite N. The buffer content process $\{C(t), t \geq 0\}$ takes values in the set $[0, B]$. When $X(t) = i$ and $C(t) = y$, fluid is transmitted into the buffer such that the instantaneous net input rate is given by the function $r_i(y)$. The behavior of the source process is, loosely speaking, as follows. At times when $C(t) = y$, the process $\{X(t)\}$ behaves instantaneously as a continuous-time Markov chain with generator $Q(y)$. More precisely, we mean the following:

Definition 2.1. *The behavior of the source process is such that for all $y \in [0, B]$, $i \neq j$, and for all $t \geq 0$:*

1. $\mathbb{P}\{X(t+h) = i | X(t) = i, C(t) = y\} = 1 + Q_{ii}(y)h + o(h)$,
2. $\mathbb{P}\{X(t+h) = j | X(t) = i, C(t) = y\} = Q_{ij}(y)h + o(h)$,
3. $\mathbb{P}\{X$ *makes more than one transition in* $[t, t+h] | X(t) = i, C(t) = y\} = o(h)$.

Here the function $Q_{ij}(y)$, $j \neq i$, is said to be the transition rate at which the source process jumps from state i to j when $C(t) = y$, and $Q_{ii}(y) = -\sum_{j \neq i} Q_{ij}(y)$.

Clearly, $\{X(t)\}$ and $\{C(t)\}$ are not Markov processes, as the future of each process depends on its own past through the state of the other process. In particular, the transition rates of $\{X(t)\}$ depend on the momentary content level, which, in turn, depends on the past of the process $\{X(t)\}$. However, the joint process $\{X(t), C(t)\}$

is a Markov process, with state space given by $\mathscr{X} \times [0, B]$. The joint process is characterized by the matrix $Q(y)$ and the diagonal matrix with net input rates $R(y) = \text{diag}(r_1(y), \ldots, r_N(y))$.

Finally, we define the distribution functions

$$F_i(y, t) = \mathbb{P}\{X(t) = i, C(t) \le y\}, \quad i \in \mathscr{X}, \, y \in [0, B],$$

where we do not specify the initial state in our notation for convenience, and the limiting distribution (if it exists),

$$F_i(y) = \lim_{t \to \infty} F_i(y, t), \quad i \in \mathscr{X}, \, y \in [0, B].$$

Remark 2.1. The model is such that the joint process $\{X(t), C(t), t \ge 0\}$ can in principle be viewed as a piecewise-deterministic Markov process, as described in [6] (as is indeed also the case for more traditional fluid queueing processes). For the special case of a two-state background process, [4] presents an approach as in [6], i.e. using the (extended) generator of the joint process. In the current paper we instead derive the Kolmogorov forward equations directly, as is often done in the literature on fluid queues.

2.2 Assumptions

In the sequel we need some assumptions on $Q_{ij}(y)$ and $r_i(y)$, which we collect here for ease of reference. Let $C_k(X)$ denote the space of k-times continuously differentiable functions on the set X. We assume the following.

1. $Q_{ij}(y) \in C(0, B)$ and finite on $[0, B]$ for all $i, j \in \mathscr{X}$.
2. $r_i(y) \in C_1(0, B)$ and finite on $[0, B]$ for all $i \in \mathscr{X}$.
3. $\min_i \inf_{y \in (0,B)} |r_i(y)| > 0$, i.e., the functions r_i are strictly bounded away from 0 on $(0, B)$.
4. There exists at least one i and one j such that $r_i(y) < 0 < r_j(y)$, for some (and hence all) $y \in (0, B)$.

The continuity assumptions in 1 and 2 may be weakened; we refer to Remark 3.1 to see how models with discontinuous $Q(y)$ and/or $R(y)$ can be analyzed. Due to the assumptions on $r_i(y)$ we can unambiguously define two disjoint subsets of $\mathscr{X}$: the set of *up-states* $\mathscr{X}_+ = \{i \in \mathscr{X} | r_i(y) > 0\}$, and the set of *down-states* $\mathscr{X}_- = \{i \in \mathscr{X} | r_i(y) < 0\}$, when $y \in (0, B)$. Let $N_- = |\mathscr{X}_-|$ and $N_+ = |\mathscr{X}_+|$. Clearly, by Assumption 3 we have that $\mathscr{X} = \mathscr{X}_+ \cup \mathscr{X}_-$, while Assumption 4 ensures that both subsets are non-empty, thereby avoiding trivial models. The reason why we assume the r_i to be strictly bounded away from zero will soon become clear.

Because the boundaries 0 and B act as impenetrable barriers for the content process we have to assume

5. $r_i(0) = 0$ if $i \in \mathscr{X}_-$ and $r_j(B) = 0$ if $j \in \mathscr{X}_+$.
6. $r_i(0) = r_i(0+) > 0$ if $i \in \mathscr{X}_+$ and $r_j(B) = r_j(B-) < 0$ if $j \in \mathscr{X}_-$.

Here, and in the sequel, we use shorthands like $r_i(0+) = \lim_{y \downarrow 0} r_i(y)$ and $r_i(B-) = \lim_{y \uparrow B} r_i(y)$.

Our next concern is the irreducibility of the process. Because it seems difficult to formally characterize when the process is irreducible, we will only give a sufficient condition. Notice that by Assumptions 2 and 3 the integrals

$$\int_x^y \frac{du}{r_j(u)}, \quad j \in \mathscr{X}_+ \qquad \text{and} \qquad \int_y^x \frac{du}{-r_j(u)}, \quad j \in \mathscr{X}_-$$

are finite for all $x, y \in [0, B]$. These integrals represent the time it takes for the joint process to move from (j, x) to (j, y) without making jumps in between. The boundedness of $Q(y)$ then ensures that in both cases there is a positive probability that, starting from (j, x), the process will indeed move to (j, y) without jumps. As a consequence, the following condition is sufficient (but not necessary) for irreducibility.

7a. For all $j \in \mathscr{X}_-$ ($j \in \mathscr{X}_+$) there is some $i \in \mathscr{X}_+$ ($i \in \mathscr{X}_-$) such that $Q_{ij}(B) > 0$ ($Q_{ij}(0) > 0$);

b. The matrix $\tilde{Q}$ whose entries are defined as $\tilde{Q}_{ij} = 1\{i \in \mathscr{X}_+\} Q_{ij}(B) + 1\{i \in \mathscr{X}_-\} Q_{ij}(0)$ is irreducible,

where $1\{i \in \mathscr{X}_+\} = 1$ when $i \in \mathscr{X}_+$ and 0 otherwise. To illustrate the line of reasoning, suppose that we would like to show that it is possible to reach state (j, y) from (i, x) with $i, j \in \mathscr{X}_-$. Then by Assumption 7a, a state (k, B) exists with $k \in \mathscr{X}_+$ from which the process can jump to (j, B) (followed by a drift from (j, B) to (j, y)), while assumption 7b ensures that it is possible to reach state (k, B) from $(i, 0)$ (and hence from (i, x)). Similar arguments for the cases where i and/or j are not in $\mathscr{X}_-$ then establish that 7a and 7b indeed imply that any state (j, y) can be reached with positive probability from any starting state. For models that do no satisfy Assumption 7, we note that the results in this paper remain valid as long as the process under study is irreducible or has a single recurrent class. In many instances this is not difficult to verify.

Representing $F_i(y,t)$ in the form $F_i(y,t) = A_i(y,t) + D_i(y,t)$, where $A_i(y,t)$ is an absolutely continuous function of y for all t and $D_i(y,t)$ is a jump function of y, it is clear that when $D_i(y,0) = 0,\ y \in [0,B]$, we also have $D_i(y,t) = 0,\ y \in (0,B)$ for all $t \geq 0$. Hence the densities $f_i(y,t) = \partial_y F_i(y,t)$ in that case exist for $y \in (0,B)$. In Section 3 and the Appendix we will actually need the following stronger assumption.

8. $F_i(y,t) \in C_2((0,B) \times [0,\infty)), \quad i \in \mathscr{X}$,

because it implies that $\partial_t f_i(y,t)$ and $\partial_y f_i(y,t)$ are continuous for $t \geq 0$ and that $\partial_y \partial_t F_i = \partial_t \partial_y F_i$. Note that Assumption 8 is of a different nature then the previous ones; in fact it is unclear what precise conditions on $R(y)$ and $Q(y)$ make sure that it is satisfied. Similar assumptions are common practice in the literature on fluid queues, see e.g. [2].

Finally, we introduce some extra notation for the atoms of $F_i(y,t)$ at $y = 0$ and $y = B$, given by

$$D_i(0,t) = F_i(0,t) \qquad \text{and} \qquad D_i(B,t) = F_i(B,t) - F_i(B-,t), \quad i \in \mathscr{X};$$

note that $F_i(B,t) = \mathbb{P}\{X(t) = i\}, \quad i \in \mathscr{X}$.

3 Derivation of the Kolmogorov Forward Equations

In this section we derive the Kolmogorov forward equations for the joint process $\{X(t), C(t)\}$. First we summarize the derivation from [10] for the situation in which Q, but not necessarily R, is constant as a function of the buffer content. This derivation is completely analogous to that of the standard case, in which R and Q are fixed. Then we focus on the case in which $Q(y)$ is a non-constant function of y. Although at first sight it may seem obvious that again a similar system of forward equations is found, this is in fact not the case. By a more careful analysis we obtain the correct result.

3.1 Constant Q

The usual approach to derive the forward equations when Q is a constant matrix is to express the distribution function $F_i(y, t+h)$ in terms of $F_i(y,t)$ for sufficiently small h and $y \in (0,B)$, i.e.,

$$F_i(y, t+h) = (1 + hQ_{ii}) F_i(y - r_i(y)h, t) + \sum_{j \neq i} hQ_{ji} F_j(y,t) + o(h). \tag{1}$$

By Assumption 8 we may write $F_i(y - r_i(y)h, t) = F_i(y,t) - hr_i(y)\partial_y F_i(y,t) + o(h)$. Rearranging terms, dividing by h, and letting h approach to zero, yields

$$\frac{\partial}{\partial t} F_i(y,t) = \lim_{h \to 0} \frac{F_i(y,t+h) - F_i(y,t)}{h} = -r_i(y) \frac{\partial}{\partial y} F_i(y,t) + \sum_j F_j(y,t) Q_{ji}. \tag{2}$$

This is precisely the result obtained in [10]. Notice that (2) has the same form as the corresponding result for traditional Markov-modulated fluid queues, where both R and Q are constant matrices, only with the constants r_i replaced by $r_i(y)$. To complete the analysis we should also provide boundary conditions. However, as they are not relevant for the sequel, we will not consider these, but rather turn to the case of primary interest.

3.2 Variable Q

Focusing on the interval $(0,B)$, it may be tempting to assume that in case the coefficients of Q are non-constant functions of y, the partial differential equation (2) can be adapted simply by replacing Q_{ji} by $Q_{ji}(y)$, just as (2) is found from the equations for the standard Markov-modulated fluid model by replacing the constants r_i by $r_i(y)$. This is *not* the case, however, since the $Q_{ji}(y)$ are transition rate functions for the source process, *provided* $C(t) = y$, while the event $\{X(t) = j, C(t) \leq y\}$ — the probability of which is given by $F_j(y,t)$ — does not at all imply that $C(t) = y$. This problem can be circumvented by considering densities rather than distribution functions. Thus, we write $d_yF_j(y,t) \equiv \mathbb{P}\{X(t)=j, C(t)\in dy\} = f_j(y,t)dy$, the last equality being valid if $y\in(0,B)$, and find the following extension of (1) for $y\in(0,B)$:

$$F_i(y,t+h) = \int_0^{y-r_i(y)h} (1+hQ_{ii}(x))d_xF_i(x,t) + h\sum_{j\neq i}\int_0^y Q_{ji}(x)d_xF_j(x,t) + o(h). \tag{3}$$

Again using $F_i(y-r_i(y)h,t) = F_i(y,t) - hr_i(y)\partial_y F_i(y,t) + o(h)$ and the fact that $\int_{y-r_i(y)h}^{y} hQ_{ii}(x)d_xF_i(x,t) = o(h)$, we find

$$F_i(y,t+h) = F_i(y,t) - hr_i(y)\frac{\partial}{\partial y}F_i(y,t) + h\sum_j\int_0^y Q_{ji}(x)d_xF_j(x,t) + o(h). \tag{4}$$

By subtracting $F_i(y,t)$ from both sides, dividing by h and taking the limit $h\to 0$, we find

$$\frac{\partial}{\partial t}F_i(y,t) = -r_i(y)\frac{\partial}{\partial y}F_i(y,t) + \int_0^y\sum_j Q_{ji}(x)d_xF_j(x,t). \tag{5}$$

This is the correct generalization of (2) at the interior of $[0,B]$. Notice that we were somewhat careless in the derivation above; in particular we did not prove assertion (3). To take care of this omission, a precise derivation of (4) is presented in the Appendix, with proper attention to $o(h)$ details.

We now provide the forward equations at the boundaries $y=0$ and $y=B$. The equation for $y=0$ follows easily by letting $y\downarrow 0$ in (5). Taking this limit yields

$$\frac{\partial}{\partial t}D_i(0,t) = -f_i(0+,t)r_i(0+) + \sum_j D_j(0,t)Q_{ji}(0). \tag{6}$$

To obtain the equation at $y=B$ we first write down the forward equations for the process $\{X(t)\}$:

$$\frac{\partial}{\partial t}F_i(B,t) = \int_0^B\sum_j Q_{ji}(x)d_xF_j(x,t), \tag{7}$$

which can be obtained by similar methods as those used in the Appendix to derive (5). Next, we take the limit $y\uparrow B$ in (5),

$$\frac{\partial}{\partial t}F_i(B-,t) = -f_i(B-,t)r_i(B-) + \int_0^{B-}\sum_j Q_{ji}(x)d_xF_j(x,t).$$

and subtract this from (7) to find

$$\frac{\partial}{\partial t}D_i(B,t) = f_i(B-,t)r_i(B-) + \sum_j D_j(B,t)Q_{ji}(B). \tag{8}$$

Finally, with respect to the boundary conditions it is clear, on physical grounds, that for $t>0$ we must have

$$\mathbb{P}\{X(t)\in\mathscr{X}_-, C(t)=B\} = \mathbb{P}\{X(t)\in\mathscr{X}_+, C(t)=0\} = 0.$$

With these boundary conditions, equations (5), (6) and (8) fully specify the stochastic behaviour of the process (together with initial conditions). However, we prefer to state our main result as a partial differential equation for densities instead of the integro-differential equation (5). Thus, we differentiate with respect to y, using Assumptions 3 and 8 from Section 2.2 (so far we only needed $r_i(y)\in C(0,B)$ and $F_i(y,t)\in C_1((0,B)\times[0,\infty))$). The result is straightforward:

$$\frac{\partial}{\partial t}f_i(y,t) = -\frac{\partial}{\partial y}(f_i(y,t)r_i(y)) + \sum_j f_j(y,t)Q_{ji}(y). \tag{9}$$

Notice that this equation is exactly the same as the corresponding equation from Section 3.1 (which can be found by differentiating (2) with respect to y), only with Q_{ji} replaced by $Q_{ji}(y)$. Thus, although in equation (2) for distribution functions we could not simply replace Q_{ji} with $Q_{ji}(y)$, we apparently can in the equations for densities. This may not come as a surprise, in view of the discussion leading to equation (3).

The following theorem summarizes the results in vector form, where the row vector $f(y,t)$ (respectively $D(y,t)$) has components $f_i(y,t)$ (respectively $D_i(y,t)$), $i=1,\ldots,N$.

Theorem 3.1. *Under the assumptions of Section 2.2, the Kolmogorov forward equations for the joint process* $\{X(t), C(t)\}$ *are, in row vector form,*

$$\frac{\partial}{\partial t}f(y,t) = -\frac{\partial}{\partial y}(f(y,t)R(y)) + f(y,t)Q(y) \tag{10a}$$

$$\frac{d}{dt}D(0,t) = -f(0+,t)R(0+) + D(0,t)Q(0) \tag{10b}$$

$$\frac{d}{dt}D(B,t) = f(B-,t)R(B-) + D(B,t)Q(B). \tag{10c}$$

The boundary conditions to be satisfied for $t > 0$ *are*

$$D_i(0,t) = D_j(B,t) \equiv 0, \quad \textit{if } i \in \mathscr{X}_+, j \in \mathscr{X}_-. \tag{11}$$

Remark 3.1. Our analysis extends easily to the case in which $Q(y)$ and $R(y)$ depend piecewise continuously on the buffer content y by combining the ideas presented here with those of [12]. Considering thresholds $0 = B_0 < B_1 < \ldots < B_N = B$ in the buffer as in [12], we obtain a system of differential equations as in (10a) for each interval (B_k, B_{k+1}), while the equations (10b) and (10c) will be supplemented with similar equations for $D(B_k, t)$.

4 Stationary Behavior

In this section we present the differential equations for the stationary distribution and give the closed form solution for the case in which the source process has two states.

Theorem 4.1. *Assuming it exists, the stationary distribution for the process* $\{X(t), C(t)\}$ *satisfies the following system of (ordinary) differential and algebraic equations,*

$$\frac{d}{dy}(f(y)R(y)) = f(y)Q(y) \tag{12a}$$

$$f(0+)R(0+) = D(0)Q(0) \tag{12b}$$

$$-f(B-)R(B-) = D(B)Q(B), \tag{12c}$$

with boundary conditions

$$D_i(0) = D_j(B) = 0, \quad \textit{if } i \in \mathscr{X}_+, j \in \mathscr{X}_-, \tag{12d}$$

and normalization condition

$$\sum_j F_j(B) = \sum_j D_j(0) + \sum_j \int_0^B f_j(x)dx + \sum_j D_j(B) = 1. \tag{12e}$$

In [17] it is proven that a stationary distribution exists under the same Assumptions as in Section 2.2, but with Assumption 1 replaced by $Q_{ij}(y) \in C[0,B]$ (and hence finite) on $[0,B]$ for all $i,j \in \mathscr{X}$. On physical grounds it seems clear that the same should hold when $Q(y)$ is discontinuous in 0 or B, but unfortunately the proof given in [17] does not go through for that case. The statement in Theorem 4.1 follows immediately from (10) by removing the dependence on t and equating the left hand sides to 0. It remains to check that the number of equations suffices to make the system complete. The number of unknowns is, formally, $3N$: the N coefficients appearing in the general solution of (12a), and $2N$ constants in the form of the atoms $D_i(0)$ and $D_i(B)$. The required number of conditions should then also equal $3N$. It is evident that (12d) and (12e) together contain $N+1$ conditions. However, the number of conditions provided by (12b) and (12c) is not immediately obvious. Note that in the presence of (12d) we may replace both $Q(0)$ and $Q(B)$ in these equations by the matrix $\tilde{Q}$, which we introduced in Assumption 7b. Since we assumed there that this matrix is irreducible, its rank is $N-1$. So, formally speaking, it might seem that (12b) and (12c) provide only $2N-2$ conditions. The lacking condition is provided by the fact that any solution of (12) should satisfy

$$\sum_i f_i(y)r_i(y) = 0, \quad y \in (0,B). \tag{13}$$

To see this, note that the row sums of $Q(y)$ equal 0 for all $y \in [0, B]$. Hence, taking this sum in (12a) and then integrating yields that $\sum_i f_i(y) r_i(y) = C$ for all $y \in (0, B)$ and some constant C. From (12b) and (12c) it immediately follows that $C = 0$. As an aside we note that (13) can be interpreted as a level crossing identity, as is also employed in e.g. [3].

For a two-state model the above equations can be solved explicitly. The solution is presented here; for proofs consult [17]. Similar results are presented in [4] for the case of an infinite buffer. Let

$$R(y) = \text{diag}(r_1(y), r_2(y)),$$

where, without loss of generality, $r_1(y) < 0 < r_2(y)$, $y \in (0, B)$, and

$$Q(y) = \begin{pmatrix} -\lambda(y) & \lambda(y) \\ \mu(y) & -\mu(y) \end{pmatrix}.$$

Hence, the system of differential equations (12a) becomes

$$f_1'(y) r_1(y) + f_1(y) r_1'(y) = -\lambda(y) f_1(y) + \mu(y) f_2(y) \tag{14a}$$

$$f_2'(y) r_2(y) + f_2(y) r_2'(y) = \lambda(y) f_1(y) - \mu(y) f_2(y). \tag{14b}$$

Theorem 4.2. *The unique solution of (14) satisfying (12b), (12c), (12d) and (12e) is given by*

$$f(y) = \lambda(0) D_1(0) e^{-g(y)} \left(-\frac{1}{r_1(y)}, \frac{1}{r_2(y)} \right), \tag{15}$$

where

$$g(y) = \int_0^y \left(\frac{\lambda(x)}{r_1(x)} + \frac{\mu(x)}{r_2(x)} \right) dx, \tag{16}$$

and

$$\begin{aligned} D_1(0) &= \left[1 + \lambda(0) \int_0^B e^{-g(x)} \left(\frac{-1}{r_1(x)} + \frac{1}{r_2(x)} \right) dx + e^{-g(B)} \frac{\lambda(0)}{\mu(B)} \right]^{-1}, \\ D_2(B) &= e^{-g(B)} \frac{\lambda(0)}{\mu(B)} D_1(0). \end{aligned} \tag{17}$$

In terms of the distribution function we have

$$F(y) = \Big(D_1(0)\,,\, D_2(B) 1_{\{y=B\}} \Big) + \lambda(0) D_1(0) \int_0^y e^{-g(x)} \left(\frac{-1}{r_1(x)}, \frac{1}{r_2(x)} \right) dx, \qquad 0 \le y \le B, \tag{18}$$

where $1_{\{y=B\}} = 1$ when $y = B$ and 0 else.

We also refer to [17] for examples and a convenient numerical method to solve the two-point boundary value problem in the general case where the background process has more than two states.

A Appendix

In Section 3 we expressed the dynamics of $F_i(y, t+h)$ for $h > 0$ sufficiently small and $y \in (0, B)$ in terms of the expansion

$$F_i(y, t+h) = \int_0^{y - r_i(y)h} (1 + hQ_{ii}(x)) d_x F_i(x, t) + h \sum_{j \neq i} \int_0^y Q_{ji}(x) d_x F_j(x, t) + o(h).$$

It is not immediately obvious that this expansion is indeed correct. For instance, the upper limit of integration y in the second term suggests that we assume that the content level remains constant during $[t, t+h]$, while this is certainly *not* the case. Compensating for this effect is difficult since the knowledge that $X(t) = j, X(t+h) = i$, and $C(t) = y$ is not sufficient to determine $C(t+h)$. To do that, we also need the epoch $\tau \in [t, t+h]$ at which the source makes its transition from state j to i, which is random. Similarly, we notice that the upper limits of both integrals should incorporate some $o(h)$ term. In the following we prove that, when the assumptions of Section 2.2 are satisfied, the influence of such subtleties can be absorbed in the term $o(h)$. In fact we immediately prove (4), from which (5) then easily follows.

Lemma A.1. *Under the assumptions of Theorem 3.1, the expansion* (5) *in Section 3.2, i.e.,*

$$F_i(y, t+h) = F_i(y,t) - hr_i(y)\frac{\partial}{\partial y}F_i(y,t) + h\sum_j \int_0^y Q_{ji}(x)d_xF_j(x,t) + o(h)$$

is valid for any $i \in \mathscr{X}$.

Proof. The proof consists of four steps. First we state three differential equations that bound all possible paths of $\{C(s), s \in [t, t+h]\}$. Then we define a family of transition functions that allow us in the third step to express the functions $F_i(y, t+h)$ in terms of some integrals, each involving the functions $F_j(x,t)$. In the last step we rewrite these and thereby prove the lemma. In the sequel consider $i \in \mathscr{X}$ fixed.

First we wish to introduce three differential equations with the aim to relate events at time $t+h$, e.g., $\{C(t+h) \le y\}$, to events at time t. To that end we define for $y \in (0, B)$,

$$\underline{r}(y) = \min(r_1(y), \dots, r_N(y)), \qquad \overline{r}(y) = \max(r_1(y), \dots, r_N(y)).$$

Clearly, $\underline{r}(y)$ and $\overline{r}(y)$ are continuous and finite functions on $(0, B)$ by the continuity and finiteness of the functions $r_i(y), 1 \le i \le N$. The three terminal value problems of interest are now given as follows,

$$\begin{aligned}
\dot{y}(s) &= r_i(y(s)), & y(t+h) &= y,\\
\underline{\dot{y}}(s) &= \underline{r}(\underline{y}(s)), & \underline{y}(t+h) &= y,\\
\dot{\overline{y}}(s) &= \overline{r}(\overline{y}(s)), & \overline{y}(t+h) &= y,
\end{aligned}$$

where $y \in (0, B)$ is some fixed terminal value. Notice that these are well defined on $s \in [t, t+h]$, provided that h is so small that $\underline{y}(t), \overline{y}(t) \in (0, B)$ if $y \in (0, B)$. The solutions to these terminal value problems are unique; consequently, trajectories with different terminal conditions do not cross. In the remainder we will be particularly interested in these solutions evaluated at t, for which we have

$$\overline{y}(t) \le y(t) \le \underline{y}(t)$$

Now, let J_0, J_1 and J_2 denote the events that the source process makes respectively $0, 1$, or more than 1 transitions in the interval $[t, t+h]$. Then we can define the following transition functions:

$$P_n(i, y, t+h; j, x, t) = \mathbb{P}\{X(t+h) = i, C(t+h) \le y, J_n \mid X(t) = j, C(t) = x\}.$$

From our definitions of $\overline{y}(t)$, $y(t)$ and $\underline{y}(t)$ and the definition in Section 2.1 it can be seen that for h sufficiently small we actually have

$$\begin{aligned}
P_0(i, y, t+h; j, x, t) &= \begin{cases} 1 + hQ_{ii}(x) + o(h) & \text{if } j = i \text{ and } x \le y(t)\\ 0 & \text{else}\end{cases}\\
P_1(i, y, t+h; j, x, t) &= \begin{cases} hQ_{ji}(x) + o(h) & \text{if } j \ne i \text{ and } x \le \overline{y}(t)\\ 0 & \text{if } j \ne i \text{ and } x \ge \underline{y}(t), \text{ or if } j = i\end{cases}\\
P_1(i, y, t+h; j, x, t) &\le \quad hQ_{ji}(x) + o(h) \qquad \text{if } j \ne i \text{ and } \overline{y}(t) \le x \le \underline{y}(t)\\
P_2(i, y, t+h; j, x, t) &= \quad o(h),
\end{aligned}$$

where the inequality in the third line is due to the fact that $r_i(\cdot) \ge \underline{r}(\cdot)$ and $r_j(\cdot) \ge \underline{r}(\cdot)$; hence, when the process starts in (j, x) at time t with $\overline{y}(t) \le x \le \underline{y}(t)$, it may end up above y at time $t+h$.

Moreover, by conditioning we know that

$$F_i(y, t+h) = \mathbb{P}\{X(t+h) = i, C(t+h) \le y\} = I_0 + I_1 + I_2, \tag{19}$$

where

$$I_n = \sum_{j \in \mathscr{X}} \int_0^B P_n(i, y, t+h; j, x, t)d_xF_j(x,t).$$

We consider the individual integrals I_0, I_1, and I_2, consecutively.

With the expression above for $P_0(i,y,t+h;j,x,t)$, the integral I_0 becomes

$$I_0 = \int_0^{y(t)} (1 + hQ_{ii}(x)) d_x F_i(x,t) + o(h),$$

where we used dominated convergence to establish that $\int_0^{y(t)} o(h) d_x F_i(x,t) = o(h)$. We rewrite this expression further by using Assumption 8 and the Taylor expansion of $y(t)$ around $t+h$, i.e.,

$$y(t) = y(t+h) - h\dot{y}(t+h) + o(h) = y - hr_i(y) + o(h).$$

The result becomes

$$I_0 = F_i(y,t) - hr_i(y)\frac{\partial}{\partial y}F_i(y,t) + \int_0^y hQ_{ii}(x)d_xF_i(x,t) + \int_y^{y(t)} hQ_{ii}(x)d_xF_i(x,t) + o(h). \qquad (20)$$

Notice that the second integral in (20) may be omitted since it is $o(h)$.

For the second integral I_1 we can derive that

$$-\sum_{j\neq i}\int_{\overline{y}(t)}^{y} hQ_{ji}(x)d_xF_j(x,t)+o(h) \leq I_1 - \sum_{j\neq i}\int_0^y hQ_{ji}(x)d_xF_j(x,t) \leq \sum_{j\neq i}\int_y^{\underline{y}(t)} hQ_{ji}(x)d_xF_j(x,t)+o(h),$$

and since the left-hand side and right-hand side are both $o(h)$, we find

$$I_1 = \sum_{j\neq i}\int_0^y hQ_{ji}(x)d_xF_j(x,t) + o(h). \qquad (21)$$

Finally, it is clear that $I_2 = o(h)$. Combining this with (19), (20) and (21) yields the desired result. □

References

[1] I.J.B.F. Adan, E.A. van Doorn, J.A.C. Resing, and W.R.W. Scheinhardt. Analysis of a single-server queue interacting with a fluid reservoir. *Queueing Systems*, 29:313–336, 1998.

[2] D. Anick, D. Mitra, and M.M. Sondhi. Stochastic theory of a data-handling system with multiple sources. *Bell Sys. Tech. J.*, 61(8):1871–1894, 1982.

[3] R. Bekker, S.C. Borst, O.J. Boxma, and O. Kella. Queues with workload-dependent arrival and service rates. *Queueing Systems*, 46:537–556, 2004.

[4] O. Boxma, H. Kaspi, O. Kella, and D. Perry. On/off storage systems with state dependent input, output and switching rates. To appear in *Probability in the Engineering and Informational Sciences*, 2004.

[5] E. Çinlar and M. Pinsky. A stochastic integral in storage theory. *Z. Wahr. verw. Geb.*, 17:227–240, 1971.

[6] M.H.A. Davis. *Markov Models and Optimization*. Chapman and Hall, 1993.

[7] A.I. Elwalid and D. Mitra. Statistical multiplexing with loss priorities in rate-based congestion control of high-speed networks. *IEEE Transactions on Communications*, 42(11):2989–3002, 1994.

[8] N.D. van Foreest, M.R.H. Mandjes, and W.R.W. Scheinhardt. Analysis of a feedback fluid model for heterogeneous TCP sources. *Stochastic Models*, 19(3):299–324, 2003.

[9] J. M. Harrison and S. I. Resnick. The stationary distribution and first exit probabilities of a storage process with general release rule. *Mathematics of Operations Research*, 1(4):347–358, 1976.

[10] O. Kella and W. Stadje. Exact results for a fluid model with state-dependent flow rates. *Probability in the Engineering and Informational Sciences*, 16(4):389–402, 2002.

[11] L. Kosten. Stochastic theory of a multi-entry buffer, part 1. *Delft Progress Report, Series F*, 1:10–18, 1974.

[12] M.R.H. Mandjes, D. Mitra, and W.R.W Scheinhardt. Models of network access using feedback fluid queues. *Queueing Systems*, 44:365–398, 2003.

[13] M.R.H. Mandjes, D. Mitra, and W.R.W Scheinhardt. A simple model of network access: feedback adaptation of rates and admission control. *Computer Networks*, 41:489–504, 2003.

[14] N. U. Prahbu. *Stochastic Storage Processes*. Springer Verlag, 1980.

[15] J. Roberts, U. Mocci, and J. Virtamo, editors. *Broadband Network Traffic*, volume 1155 of *Lecture Notes in Computer Science*. Springer Verlag, 1996.

[16] W.R.W. Scheinhardt. *Markov-Modulated and Feedback Fluid Queues*. PhD thesis, Faculty of Mathematical Sciences, University of Twente, Enschede, The Netherlands, 1998.

[17] W.R.W. Scheinhardt, N.D. van Foreest, and M.R.H. Mandjes. Continuous feedback fluid queues. Memorandum 1686, Department of Applied Mathematics, Enschede, The Netherlands, 2003. Available at http://www.math.utwente.nl/publications/

[18] M. Schwartz. *Broadband Integrated Networks*. Prentice Hall, 1996.