All normal extensions of S5-squared are finitely axiomatizable

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We prove that every normal extension of the bi-modal system $S5^2$ is finitely axiomatizable and that every proper normal extension has NP-complete satisfiability problem.

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1. Introduction

Recall that the language of $S5^2$ is the propositional language based on a fixed countably infinite set of propositional variables and equipped with the two modal operators $\Box_1$ and $\Box_2$. For a formula $\varphi$ we let $\Diamond_i \varphi$ abbreviate $\neg \Box_i \neg \varphi$ for $i = 1, 2$. We recall that $S5^2$ is the smallest set of formulas containing all substitution instances of the following axiom schemas, for $i = 1, 2$:

1) All tautologies of the classical propositional calculus;
2) $\Box_i (p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$;
3) $\Box_i p \rightarrow p$;
4) $\Box_i p \rightarrow \Box_i \Box_i p$;
5) $\Diamond_i \Box_i p \rightarrow p$;
6) $\Box_1 \Box_2 p \leftrightarrow \Box_2 \Box_1 p$;

and closed under the following rules of inference:

Modus Ponens (MP): from $\varphi$ and $\varphi \rightarrow \psi$ infer $\psi$;
Necessitation (N): from $\varphi$ infer $\Box_i \varphi$.

Recall also that a set of formulas $L$ is called a logic if it contains all tautologies of the classical propositional calculus and is closed under the rule of modus ponens. A modal logic is called normal if it contains axiom schema 2) (see above) and is closed under the rule of necessitation. A logic $L_1$ is an extension of $L_2$ if $L_2 \subseteq L_1$. 

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It is well known that $S5^2$ has the exponential size model property, and that its satisfiability problem is NEXPTIME-complete [6]. In this paper, by the complexity of a logic we will mean the complexity of its satisfiability problem. It is shown in [3] that in contrast to $S5^2$, every proper normal extension $L$ of $S5^2$ has the poly-size model property. That means that there is a polynomial $P(n)$ such that any $L$-consistent formula $\varphi$ (that is, $\neg \varphi \notin L$) has a model over a frame validating $L$ and with at most $P(|\varphi|)$ points, where $|\varphi|$ is the length of $\varphi$.

It was conjectured in [3] that every proper normal extension of $S5^2$ is finitely axiomatizable and NP-complete. In this paper we prove this conjecture. In fact, we show that for every proper normal extension $L$ of $S5^2$, there is a finite set $M_L$ of finite $S5^2$-frames such that an arbitrary finite $S5^2$-frame is a frame for $L$ iff it does not have any frame in $M_L$ as a $p$-morphic image. This condition yields a finite axiomatization of $L$. We also show that the condition is decidable in deterministic polynomial time. This, together with the poly-size model property, implies NP-completeness of (satisfiability for) $L$.

Finally, we note that general complexity results for (uni)modal logics were investigated before. Bull and Fine proved that every normal extension of $S4.3$ has the finite model property, is finitely axiomatizable and therefore is decidable (see [4, Theorems 4.96, 4.101]). Hemaspaandra strengthened the second result by showing that every normal extension of $S4.3$ is NP-complete [4, Theorem 6.41]. The proof of finite axiomatizability uses Kruskal’s theorem on well-quasi-orderings [4, Theorem 4.99]. Kracht uses the same technique for showing that every extension of the intermediate logic of leptonic strings is finitely axiomatizable [8, Theorem 14, Proposition 15]. This paper takes the same line of research beyond unimodal logics. However, as we will see below, the theory of well-quasi-orderings does not suffice for our purposes; instead, we will use better-quasi-orderings.

2. Preliminaries

Recall that a triple $F = (W, E_1, E_2)$ is an $S5^2$-frame (i.e., it validates the axioms of $S5^2$: see, e.g., [5, Corollary 5.10]) iff $W$ is a non-empty set and $E_1$ and $E_2$ are equivalence relations on $W$ such that

$$F \models (\forall w, v, u)(wE_1v \land vE_2u \rightarrow (\exists z)(wE_2z \land zE_1u)).$$

For $i = 1, 2$ we call the $E_i$-equivalence classes $E_i$-clusters. The $E_i$-cluster containing $w \in W$ is denoted by $E_i(w)$, and for $X \subseteq W$ we let $E_i(X)$ denote $\bigcup_{x \in X} E_i(x)$. 
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We identify non-negative integers with ordinals, so that for $n \geq 0$ we have $n = \{0, 1, \ldots, n - 1\}$. For positive integers $n$ and $m$, let $n \times m$ denote the $S5^2$-frame with domain $n \times m$ and with $(x_1, x_2)E_1(y_1, y_2)$ iff $x_2 = y_2$ and $(x_1, x_2)E_2(y_1, y_2)$ iff $x_1 = y_1$. Then it is well known that $S5^2$ is complete with respect to $\{n \times n : n \geq 1\}$ [11].

Given two $S5^2$-frames $\mathcal{F} = (W, E_1, E_2)$ and $\mathcal{G} = (U, S_1, S_2)$, a mapping $f : U \rightarrow W$ is called a $p$-morphism from $\mathcal{G}$ to $\mathcal{F}$ if for each $i = 1, 2$,

$$(\forall t \in U)(\forall w \in W)(f(t)E_i w \leftrightarrow (\exists u \in U)(tS_i u \land f(u) = w)).$$

It is easy to check that a map $f : U \rightarrow W$ is a $p$-morphism iff the $f$-image of every $S_i$-cluster of $\mathcal{G}$ is an $E_i$-cluster of $\mathcal{F}$, for $i = 1, 2$. We say that $\mathcal{F}$ is isomorphic to $\mathcal{G}$ if there exists a one-one $p$-morphism from $\mathcal{G}$ onto $\mathcal{F}$. We call $\mathcal{F}$ a $p$-morphic image of $\mathcal{G}$ if there is a $p$-morphism from $\mathcal{G}$ onto $\mathcal{F}$. It is well known that in this case, any formula valid in $\mathcal{G}$ is valid in $\mathcal{F}$.

We call $\mathcal{F} = (W, E_1, E_2)$ rooted if there is a point $w \in W$ that is related to every point $v \in W$ by the reflexive transitive closure of $E_1 \cup E_2$. It is easy to check that an $S5^2$-frame $\mathcal{F}$ is rooted iff $\mathcal{F}| = (\forall w, v)(\exists u)(wE_1 u \land uE_2 v)$.

Choose a set $F_{S5^2}$ of representatives of the isomorphism types of finite rooted $S5^2$-frames. That is, for each finite rooted $S5^2$-frame, there is exactly one frame in $F_{S5^2}$ that is isomorphic to it.

Let $L$ be a normal extension of $S5^2$. An $S5^2$-frame $\mathcal{F}$ is called an $L$-frame if $\mathcal{F}$ validates all formulas in $L$. Let $F_L$ be the set of all $L$-frames in $F_{S5^2}$. Then $L$ is complete with respect to $F_L$ [1]. Thus, for our purposes it suffices to consider only finite rooted $S5^2$-frames. From now on, we will use the term “frame” to mean this.

For $\mathcal{F}, \mathcal{G} \in F_{S5^2}$, we put

$\mathcal{F} \leq \mathcal{G}$ iff $\mathcal{F}$ is a $p$-morphic image of $\mathcal{G}$.

Then it is routine to check that $\leq$ is a partial order on $F_{S5^2}$. We write $\mathcal{F} < \mathcal{G}$ if $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \not\leq \mathcal{F}$. Then $\mathcal{F} < \mathcal{G}$ implies $|\mathcal{F}| < |\mathcal{G}|$ and we see that there are no infinite descending chains in $(F_{S5^2}, <)$. Thus, for any non-empty $A \subseteq F_{S5^2}$, the set $\text{min}(A)$ of $<$-minimal elements of $A$ is non-empty, and indeed for any $\mathcal{G} \in A$ there is $\mathcal{F} \in \text{min}(A)$ such that $\mathcal{F} \leq \mathcal{G}$.
3. Finite axiomatizability

In this section we will prove the first main result of the paper — that every normal extension of $S5^2$ is finitely axiomatizable.

First we recall the Jankov-Fine formulas for $S5^2$ (see [4, §3.4] and [5, §8.4 p. 392]). Consider a frame $F = (W, E_1, E_2)$. For each point $p \in W$ we introduce a propositional variable, denoted also by $p$, and consider the formulas

$$
\alpha(F) = \Diamond_1 \Diamond_2 \left( \bigvee_{p \in W} (p \land \neg \bigvee_{p' \in W \setminus \{p\}} p') \land \bigwedge_{i=1,2} \bigvee_{p, p' \in W, p E_i p'} (p \rightarrow \Diamond_i p') \land \bigwedge_{i=1,2} \bigvee_{p, p' \in W, p \not\sim E_i p'} (p \rightarrow \neg \Diamond_i p') \right),
$$

$$
\chi(F) = \neg \alpha(F).
$$

**Lemma 3.1.** For any frames $F = (W, E_1, E_2)$ and $G = (U, S_1, S_2)$ we have that $F$ is a $p$-morphic image of $G$ iff $G \models \chi(F)$.

**Proof.** (Sketch) Suppose $F$ is a $p$-morphic image of $G$. Define a valuation $V$ on $F$ by putting $V(p) = p$ for any $p \in W$. Then $F \models V \chi(F)$ by the definition of $\chi(F)$. Now if $G \models \chi(F)$, then since $p$-morphic images preserve validity of formulas, we would also have $F \models \chi(F)$, a contradiction. Therefore, $G \models \chi(F)$.

For the converse, we use the argument of [5, Claim 8.36]. Suppose that $G \models \chi(F)$. Then there is a valuation $V'$ on $G$ and a point $u \in U$ such that $G, u \models V' \chi(F)$. Therefore, $G, u \models V' \alpha(F)$. Define a map $f : U \rightarrow W$ by putting $f(t) = p$ $\iff$ $G, t \models V', p$, for every $t \in U$ and $p \in W$. From $G$ being rooted and the truth of the first conjunct of $\alpha(F)$ it follows that $f$ is well defined. The truth of the first two conjuncts of $\alpha(F)$ together with $F$ being rooted implies that $f$ is surjective. Finally, the truth of the second and third conjuncts of $\alpha(F)$ guarantees that $f$ is a $p$-morphism. Therefore, $F$ is a $p$-morphic image of $G$. 

Let $L$ be a proper normal extension of $S5^2$. By completeness of $S5^2$ with respect to $F_{S5^2}$, the set $F_{S5^2} \setminus F_L$ is non-empty. Let $M_L = \min(F_{S5^2} \setminus F_L)$.

**Theorem 3.2.** For any proper normal extension $L$ of $S5^2$ and $\mathcal{G} \in F_{S5^2}$, $\mathcal{G} \in F_L$ iff no $F \in M_L$ is a $p$-morphic image of $\mathcal{G}$.

**Proof.** Let $\mathcal{G} \in F_L$; then since $p$-morphisms preserve validity of formulas, every $p$-morphic image of $\mathcal{G}$ belongs to $F_L$ and hence can not be in $M_L$. 

Conversely, if $G \in F_{SS^2} \setminus F_L$ then there is $F \in M_L$ such that $F \leq G$ — that is, $F$ is a $p$-morphic image of $G$.

**Theorem 3.3.** Every proper normal extension $L$ of $S5^2$ is axiomatizable by the axioms of $S5^2$ plus $\{\chi(F) : F \in M_L\}$.

**Proof.** Let $G \in F_{SS^2}$. Then by Theorem 3.2, $G \in F_L$ iff there is no $F \in M_L$ with $F \leq G$, iff (by Lemma 3.1) there is no $F \in M_L$ with $G \not\models \chi(F)$, iff $G \models \chi(F)$ for all $F \in M_L$. Thus, $G \models \{\chi(F) : F \in M_L\}$ iff $G \in F_L$.

Let $L'$ be the logic axiomatized by the axioms of $S5^2$ plus $\{\chi(F) : F \in M_L\}$. From the above it is clear that $F_{L'} = F_L$. But $L$ ($L'$) is sound and complete with respect to $F_L$ ($F_{L'}$, respectively). So, $L' = L$.

It follows that $L \supset S5^2$ is finitely axiomatizable whenever $M_L$ is finite. We now proceed to show that $M_L$ is indeed finite for every proper normal extension $L$ of $S5^2$.

Suppose $G \in F_{SS^2}$. For $i = 1, 2$, we say that the $E_i$-depth of $G$ is $n$, and write $d_i(G) = n$, if the number of $E_i$-clusters of $G$ is $n$.

Fix a proper normal extension $L$ of $S5^2$. Since $S5^2$ is complete with respect to $\{n \times n : n \geq 1\}$, there is $n \geq 1$ such that $n \times n \not\in F_L$. Let $n(L)$ be the least such.

**Lemma 3.4.** Let $L$ be as above, and write $n$ for $n(L)$.

1. If $G \in F_L$, then $d_1(G) < n$ or $d_2(G) < n$.
2. If $G \in M_L$, then $d_1(G) \leq n$ or $d_2(G) \leq n$.

**Proof.**

1. If $G \in F_L$ and $d_1(G) \geq n$ and $d_2(G) \geq n$, then by [3, Lemma 5], $n \times n$ is a $p$-morphic image of $G$. So, $n \times n \in F_L$, a contradiction.

2. If $G \in M_L$ and both depths of $G$ are greater than $n$, then again $n \times n$ is a $p$-morphic image of $G$. Therefore, $n \times n \not\in F_L$. However, $G$ is a minimal element of $F_{SS^2} \setminus F_L$, implying that $n \times n$ belongs to $F_L$, which is false.

**Corollary 3.5.** $M_L$ is finite iff $\{F \in M_L : d_i(F) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$.

**Proof.** By Lemma 3.4, $M_L = \bigcup_{k \leq n(L)} \{F \in M_L : d_1(F) = k\} \cup \bigcup_{k \leq n(L)} \{F \in M_L : d_2(F) = k\}$. Thus, $M_L$ is finite if and only if $\{F \in M_L : d_i(F) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. ■
Since \( M_L \) is a \( \leq \)-antichain in \( F_{SS^2} \), to show that \( \{ F \in M_L : d_i(F) = k \} \) is finite for every \( k \leq n(L) \) and \( i = 1, 2 \), it is enough to prove that for any \( k \), the set \( \{ F \in F_{SS^2} : d_i(F) = k \} \) does not contain an infinite \( \leq \)-antichain. Without loss of generality we can consider the case when \( i = 2 \).

Fix \( k \in \omega \). For every \( n \in \omega \) let \( M_n \) denote the set of all \( n \times k \) matrices \((m_{ij})\) with coefficients in \( \omega (i < n, j < k) \). Let \( \mathcal{M} = \bigcup_{n \in \omega} M_n \). Define \( \leq \) on \( \mathcal{M} \) by putting \( (m_{ij}) \leq (m'_{ij}) \) if we have \( (m_{ij}) \in M_n \), \( (m'_{ij}) \in M_n' \), \( n \leq n' \), and there is a surjection \( f : n' \to n \) such that \( m_{f(i)j} \leq m'_{ij} \) for all \( i < n' \) and \( j < k \). It is easy to see that \( (\mathcal{M}, \leq) \) is a quasi-ordered set (i.e., \( \leq \) is reflexive and transitive).

Let \( F_{SS^2}^k = \{ F \in F_{SS^2} : d_2(F) = k \} \). For each \( F \in F_{SS^2}^k \) we fix enumerations \( F_0, \ldots, F_{n-1} \) of the \( E_1 \)-clusters of \( F \) (where \( n = d_1(F) \)) and \( F_0', \ldots, F_{k-1} \) of the \( E_2 \)-clusters of \( F \). Define a map \( H : F_{SS^2}^k \to \mathcal{M} \) by putting \( H(F) = (m_{ij}) \) if \( |F_i \cap F_j| = m_{ij} \) for \( i < d_1(F) \) and \( j < k \). As \( F \in F_{SS^2}^k \), it follows that \( m_{ij} > 0 \) for each such \( i, j \). Recall that a map \( f : P \to P' \) between ordered sets \( (P, \leq) \) and \( (P', \leq') \) is called order reflecting if \( f(w) \leq' f(v) \) implies \( w \leq v \) for any \( w, v \in P \).

**Lemma 3.6.** \( H : (F_{SS^2}^k, \leq) \to (\mathcal{M}, \leq) \) is an order-reflecting injection.

**Proof.** Since \( F_{SS^2} \) consists of non-isomorphic frames, \( H \) is one-one. Now let \( F = (W, E_1, E_2), G = (U, S_1, S_2), \) \( F, G \in F_{SS^2}^k \), and \( (m_{ij}), (m'_{ij}) \in \mathcal{M} \) be such that \( H(F) = (m_{ij}) \), \( H(G) = (m'_{ij}) \), and \( (m_{ij}) \leq (m'_{ij}) \). We need to show that \( F \leq G \). Suppose \( (m_{ij}) \in M_n \) and \( (m'_{ij}) \in M_n' \). Then there is a surjective \( f : n' \to n \) such that \( m_{f(i)j} \leq m'_{ij} \) for \( i < n' \) and \( j < k \). Then \( |G_i \cap G'_j| \geq |F_{f(i)} \cap F_j| > 0 \) for any \( i < n' \) and \( j < k \). Hence there exists a surjection \( h' : G_i \cap G'_j \to F_{f(i)} \cap F_j \). Define \( h : U \to W \) by putting \( h(u) = h'(u) \), where \( i < n', j < k \), and \( u \in G_i \cap G'_j \). It is obvious that \( h \) is well defined and onto.

Now we show that \( h \) is a \( p \)-morphism. If \( uS_1v \), then \( u, v \in G_i \) for some \( i < n' \). Therefore, \( h(u), h(v) \in F_{f(i)} \), and so \( h(u)E_1h(v) \). Analogously, if \( uS_2v \), then \( u, v \in G_j \) for some \( j < k \). Now suppose \( u \in G_i \cap G'_j \) for some \( i < n' \) and \( j < k \). If \( h(u)E_2h(v) \), then \( h(u), h(v) \in F_{f(j)} \) and \( v \in G'_j \). As both \( u \) and \( v \) belong to \( G_j \) it follows that \( uS_2v \). Finally, if \( h(u)E_1h(v) \), then \( h(u) \in F_{f(i)} \cap F_j \) and \( h(v) \in F_{f(j)} \cap F'_j \), for some \( j' < k \). Therefore, there exists \( z \in G_i \cap G'_j \) (since \( z \in G_i \) we have \( uS_1z \) such that \( h(z) = h(v) \). Thus, \( h \) is an onto \( p \)-morphism, implying that \( F \leq G \). Thus, \( H \) is order reflecting.

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1 By an \( n \times k \) matrix we mean a matrix with \( n \) rows and \( k \) columns.
Corollary 3.7. If $\Delta \subseteq \mathbb{F}_s^k$ is a $\preceq$-antichain, then $H(\Delta) \subseteq \mathcal{M}$ is a $\preceq$-antichain.

Proof. Immediate.

Now we will show that there are no infinite $\preceq$-antichains in $\mathcal{M}$. For this we define a quasi-order $\sqsubseteq$ on $\mathcal{M}$ included in $\preceq$ and show that there are no infinite $\sqsubseteq$-antichains in $\mathcal{M}$. To do so we first introduce two quasi-orders $\sqsubseteq_1$ and $\sqsubseteq_2$ on $\mathcal{M}$ and then define $\sqsubseteq$ as the intersection of these quasi-orders. For $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$, we say that:

- $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ if there is a one-one order-preserving map $\varphi : n \to n'$ (i.e., $i < i' < n$ implies $\varphi(i) < \varphi(i')$) such that $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$;
- $(m_{ij}) \sqsubseteq_2 (m'_{ij})$ if there is a map $\psi : n' \to n$ such that $m'_{\psi(i)j} \leq m_{ij}$ for all $i < n'$ and $j < k$.

Let $\sqsubseteq$ be the intersection of $\sqsubseteq_1$ and $\sqsubseteq_2$.

Lemma 3.8. For any $(m_{ij}), (m'_{ij}) \in \mathcal{M}$, if $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \preceq (m'_{ij})$.

Proof. Suppose $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. If $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ and $(m_{ij}) \sqsubseteq_2 (m'_{ij})$. By $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ there is a one-one order-preserving map $\varphi : n \to n'$ with $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$; and by $(m_{ij}) \sqsubseteq_2 (m'_{ij})$ there is a map $\psi : n' \to n$ such that $m'_{\psi(i)j} \leq m_{ij}$ for all $i < n'$ and $j < k$. Let $\text{rng}(\varphi) = \{\varphi(i) : i < n\}$. Define $f : n' \to n$ by putting

$$f(i) = \begin{cases} \varphi^{-1}(i), & \text{if } i \in \text{rng}(\varphi), \\ \psi(i), & \text{otherwise}. \end{cases}$$

Then $f$ is a surjection. Moreover, for $i < n'$ and $j < k$, if $i \in \text{rng}(\varphi)$, then $m_{f(i)j} = m'_{\varphi^{-1}(i)j} \leq m'_{ij}$ by the definition of $\sqsubseteq_1$; and if $i \notin \text{rng}(\varphi)$, then $m_{f(i)j} = m'_{\psi(i)j} \leq m'_{ij}$ by the definition of $\sqsubseteq_2$. Therefore, $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Thus, $(m_{ij}) \preceq (m'_{ij})$.

Thus, it is left to show that there are no infinite $\sqsubseteq$-antichains in $\mathcal{M}$. For this we use the theory of better-quasi-orderings (bqos). Our main source of reference is Laver [9].

For any set $X \subseteq \omega$ let $[X]^{<\omega} = \{Y \subseteq X : |Y| < \omega\}$, and for $n < \omega$ let $[X]^n = \{Y \subseteq X : |Y| = n\}$. We say that $Y$ is an initial segment of $X$ if there is $n \in \omega$ such that $Y = \{x \in X : x \leq n\}$.
Definition 3.9. Let $X$ be an infinite subset of $\omega$. We say that $B \subseteq [X]<\omega$ is a barrier on $X$ if $\emptyset \notin B$ and:

- for every infinite $Y \subseteq X$, there is an initial segment of $Y$ in $B$;
- $B$ is an antichain with respect to $\subseteq$.

A barrier is a barrier on some infinite $X \subseteq \omega$.

Note that for any $n \geq 1$, $[\omega]^n$ is a barrier on $\omega$.

Definition 3.10.

1. If $s, t$ are finite subsets of $\omega$, we write $s \triangleleft t$ to mean that there are $i_1 < \ldots < i_k$ and $j$ ($1 \leq j < k$) such that $s = \{i_1, \ldots, i_j\}$ and $t = \{i_2, \ldots, i_k\}$.

2. Given a barrier $B$ and a quasi-ordered set $(Q, \leq)$, we say that a map $f : B \to Q$ is good if there are $s, t \in B$ such that $s \triangleleft t$ and $f(s) \leq f(t)$.

3. Let $(Q, \leq)$ be a quasi-order. We call $\leq$ a better-quasi-ordering (bqo) if for every barrier $B$, every map $f : B \to Q$ is good.

Now we recall basic constructions and properties of bqos.

Proposition 3.11. If $(Q, \leq)$ is a bqo, there are no infinite $\leq$-antichains in $Q$.

Proof. Let $(\xi_n)_{n \in \omega}$ be an infinite sequence of distinct elements of $Q$. As we pointed out, $B = [\omega]^1 = \{\{n\} : n < \omega\}$ is a barrier. Define a map $\theta : B \to Q$ by putting $\theta(\{n\}) = \xi_n$. Since $(Q, \leq)$ is a bqo, $\theta$ is good. Therefore, there are $\{n\}, \{m\} \in B$ such that $\{n\} \triangleleft \{m\}$ (i.e., $n < m$) and $\xi_n \leq \xi_m$. So, no infinite subset of $Q$ forms a $\leq$-antichain.

We write $On$ for the class of all ordinals. Let $(Q, \leq)$ be a quasi-order. Define $\leq^*$ on the class $\bigcup_{\alpha \in On} Q^\alpha$, and on any set contained in it, by putting $(x_i)_{i < \alpha} \leq^* (y_i)_{i < \beta}$ if there is a one-one order-preserving map $\varphi : \alpha \to \beta$ such that $x_i \leq y_{\varphi(i)}$ for all $i < \alpha$.

Let $\varphi(Q)$ be the power set of $Q$. The order $\leq$ can be extended to $\varphi(Q)$ as follows: For $\Gamma, \Delta \in \varphi(Q)$, we say that $\Gamma \leq \Delta$ if for all $\delta \in \Delta$ there is $\gamma \in \Gamma$ with $\gamma \leq \delta$.

Recall that $(P, \leq')$ is called a suborder of $(Q, \leq)$ if $P \subseteq Q$ and $\leq' = \leq \cap P^2$.

Theorem 3.12.

1. $(\omega, \leq)$ is a bqo.
2. Any suborder of a bqo is a bqo.

3. If $\leq$ and $\preceq$ are bqos on $Q$, then $\leq \cap \preceq$ is also a bqo on $Q$.

4. If $(Q, \leq)$ is a bqo, then $(\bigcup_{\alpha \in \Omega} Q^\alpha, \preceq^*)$ is also a (proper class) bqo.

Hence, by (2), its suborders $(Q^k, \preceq^*)$ and $(\bigcup_{n<\omega} Q^n, \preceq^*)$ are bqos.

5. If $(Q, \leq)$ is a bqo, then $(\wp(Q), \leq)$ is a bqo.

**Proof.** (1) follows from Lemma 1.2 of [9]. (2) is trivial.

(3): By [9, Lemma 1.8], $(Q \times Q, \leq \otimes \leq')$ is a bqo, where we define $(x, x') \leq \otimes \leq' (y, y')$ iff $x \leq y$ and $x' \preceq y'$. By (2), its suborder $(\{(g, q) : q \in Q \}, \leq \otimes \leq')$ is also a bqo, and this is isomorphic to $(Q, \leq \cap \leq')$.

(4) — see [9, Theorem 1.10].

(5) Finally to show $(\wp(Q), \leq)$ is a bqo we adapt the proof of Lemma 1.3 of [9]. Let $B$ be a barrier and consider $f : B \rightarrow \wp(Q)$. Suppose $f$ is not good. Then for each $s, t \in B$ with $s < t$ we have $f(s) \not\subseteq f(t)$. Let $B(2) = \{ s \cup t : s, t \in B \text{ and } s < t \}$. Thus for every element $s \cup t \in B(2)$ there is an element $\delta_{st} \in f(t)$ such that for every $\gamma \in f(s)$ we have $\gamma \not\subseteq \delta_{st}$.

Define a map $h : B(2) \rightarrow Q$ by putting $h(s \cup t) = \delta_{st}$ for every $s \cup t \in B(2)$. It can be checked that $h$ is well defined. It is known (see, e.g., [9, p. 35]) that $B(2)$ is a barrier. Since $(Q, \leq)$ is a bqo, $h$ is good, so there exist $s \cup t, s' \cup t' \in B(2)$ with $s \cup t < s' \cup t'$ and $h(s \cup t) \leq h(s' \cup t')$. It is easy to check (see [9, p. 35]) that $t = s'$. But now $\delta_{s't'} = h(s' \cup t') \geq h(s \cup t) \in f(t) = f(s')$. This contradicts the definition of $\delta_{s't'}$, hence $f$ is good and therefore $(\wp(Q), \leq)$ is a bqo.

**Remark 3.13.** A quasi-order $\leq$ on a set $Q$ is called a well-quasi-ordering (wqo) if for any sequence $(x_i)_{i<\omega}$ in $Q$ there exist $i < j < \omega$ with $x_i \leq x_j$. As we said in the introduction, wqos have been used to prove finite axiomatizability results in modal logic on many previous occasions. The following facts are known about them (cf. Theorem 3.12):

1. Any bqo is a wqo.

2. If $\leq$ and $\preceq$ are wqos on $Q$, then $\leq \cap \preceq$ is also a wqo on $Q$.

3. (Higman’s Lemma, proved in [7]) If $(Q, \leq)$ is a wqo then $(\bigcup_{n<\omega} Q^n, \preceq^*)$ is also a wqo.

An example of a wqo $(Q, \leq)$ with $(\bigcup_{\alpha \in \Omega} Q^\alpha, \preceq^*)$ not a wqo was constructed by Rado [10]: let $Q = \{(i, j) : i < j < \omega\}$, ordered by $(i, j) \leq (k, l)$ iff either $i = k$ and $j \leq l$, or else $i, j < k$. This is a wqo on $Q$. Now for $i < \omega$ let $\xi_i$ be the sequence $((i, i + 1), (i, i + 2), \ldots)$. Then $\xi_i \not\subseteq^* \xi_j$ for all $i < j < \omega$. This example can be used to show that for a wqo $(Q, \leq)$, in general $(\wp(Q), \leq)$ fails
to be a wqo, even if we restrict to finite subsets of $Q$ (see also the discussion on p. 33 of [9]). This failure is why we use bqos and not wqos here.

By Proposition 3.11, to show that there are no $\sqsubseteq$-antichains in $\mathcal{M}$ it suffices to show that $(\mathcal{M}, \sqsubseteq)$ is a bqo. It follows from Theorem 3.12(3) that the intersection of two bqos is again a bqo. Hence, it is enough to prove that $(\mathcal{M}, \sqsubseteq_1)$ and $(\mathcal{M}, \sqsubseteq_2)$ are bqos.

**Lemma 3.14.** $(\mathcal{M}, \sqsubseteq_1)$ is a bqo.

**Proof.** By Theorem 3.12(1), $(\omega, \leq)$ is a bqo. By Theorem 3.12(4), $(\omega^k, \leq^*)$ is also a bqo. By Theorem 3.12(4) again, $(\mathcal{M}, \sqsubseteq_1) \cong (\bigcup_{n<\omega}(\omega^k)^n, \leq^{**})$ is a bqo as well.

It remains to show that $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

**Lemma 3.15.** $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

**Proof.** For a matrix $(m_{ij}) \in \mathcal{M}_n$ let $m_i = (m_{i0}, \ldots, m_{ik-1})$ denote the $i$-th row of $(m_{ij})$. Note that each row of $(m_{ij})$ is a $1 \times k$ matrix, and so $m_i \in \mathcal{M}_1$ for any $i < n$. We write $\text{row}(m_{ij})$ for the set $\{m_i : i < n\}$. Obviously, $\text{row}(m_{ij}) \in \wp(\mathcal{M}_1) \subseteq \wp(\mathcal{M})$. Consider an arbitrary barrier $B$ and a map $f : B \to \mathcal{M}$. We need to show that $f$ is good with respect to $\sqsubseteq_2$. Define $g : B \to \wp(\mathcal{M})$ by $g(s) = \text{row}(f(s))$. Since $(\mathcal{M}, \sqsubseteq_1)$ is a bqo, by Theorem 3.12(5), $(\wp(\mathcal{M}), \sqsubseteq_1)$ is also a bqo. Hence, there are $s, t \in B$ such that $s \sqsubseteq t$ and $g(s) \sqsubseteq_1 g(t)$. Therefore, for each $\delta \in g(t)$ there is $\gamma \in g(s)$ with $\gamma \sqsubseteq_1 \delta$.

Now we show that $f(s) \sqsubseteq_2 f(t)$. Write $(m_{ij})$ for $f(s)$ and $(m'_{ij})$ for $f(t)$. Suppose that $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. We define $\psi : n' \to n$ as follows. Let $i < n'$. Then $m'_{ij} \in g(t)$. By the above, we may choose $\psi(i) < n$ such that $m_{\psi(i)j} \in m'_i$. This defines $\psi$, and we have $m_{\psi(i)j} \leq m'_{ij}$ for any $i < n'$ and $j < k$. Thus, $f(s) \sqsubseteq_2 f(t)$, $f$ is a good map, and so $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

It follows that $(\mathcal{M}, \sqsubseteq)$ is a bqo. Therefore, there are no infinite $\sqsubseteq$-antichains in $\mathcal{M}$. Thus, by Lemma 3.8 there are no infinite $\preceq$-antichains in $\mathcal{M}$.

Now we are in a position to prove the first main theorem of this paper.

**Theorem 3.16.** Every normal extension of $S5^2$ is finitely axiomatizable.

---

\(^2\) To apply this theorem, we needed to require in the definition of $\sqsubseteq_1$ on $\mathcal{M}$ that $\varphi$ is order preserving. This is the only time this assumption is used.
PROOF. Clearly, $S5^2$ is finitely axiomatizable. Suppose $L$ is a proper normal extension of $S5^2$. Then by Theorem 3.3 $L$ is axiomatizable by the $S5^2$ axioms plus $\{\chi(F) : F \in M_L\}$. Since there are no infinite $\mathbin{\preceq}$-antichains in $\mathcal{M}$, by Corollary 3.7 there are no infinite antichains in $F^k_{S5^2}$, for each $k \in \omega$. Therefore, $\{F \in M_L : d_i(F) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. Thus, $M_L$ is finite by Corollary 3.5. It follows that $L$ is finitely axiomatizable. 

COROLLARY 3.17. The lattice of normal extensions of $S5^2$ is countable.

PROOF. Immediately follows from Theorem 3.16 since there are only countably many finitely axiomatizable normal extensions of $S5^2$.

REMARK 3.18. In algebraic terminology, Corollary 3.17 says that the lattice of subvarieties of the variety $Df_2$ of two-dimensional diagonal-free cylindric algebras is countable. This is in contrast with the variety $CA_2$ of two-dimensional cylindric algebras (with diagonals), since, as was shown in [2], the cardinality of the lattice of subvarieties of $CA_2$ is that of continuum.

4. Complexity

Note that Theorem 3.16, and the fact that every normal extension $L$ of $S5^2$ is complete with respect to a class of finite frames $(F_L)$ for which (up to isomorphism) membership is decidable, imply that $L$ is decidable. This section will be devoted to showing that if $L$ is a proper normal extension, then its satisfiability problem is NP-complete. Fix such an $L$. We will see in Corollary 4.3 below that NP-completeness follows from the poly-size model property if we can decide in time polynomial in $|W|$ whether a finite structure $A = (W, R_1, R_2)$ is in $F_L$ (up to isomorphism). It suffices to decide in polynomial time (1) whether $A$ is a (rooted $S5^2$-) frame; (2) whether a given frame is in $F_L$. The first is easy. We concentrate on the second.

By Lemma 3.4(1), there is $n(L) \in \omega$ such that for each frame $G = (U, S_1, S_2)$ in $F_L$ we have $d_1(G) < n(L)$ or $d_2(G) < n(L)$. So, if both depths of a given frame $G$ are greater than or equal to $n(L)$ (which obviously can be checked in polynomial time in the size of $G$), then $G \not\in F_L$. So, without loss of generality we can assume that $d_1(G) < n(L)$.

By Theorem 3.2, $G$ is in $F_L$ iff it has no $p$-morphic image in $M_L$. Because $M_L$ is a fixed finite set, it suffices to provide, for an arbitrary fixed frame $F = (W, E_1, E_2)$, an algorithm that decides in time polynomial in the size of $G$ whether there is a $p$-morphism from $G$ onto $F$. If we considered every map $f : U \to W$ and checked whether it is a $p$-morphism, it would take
exponential time in the size of $\mathcal{G}$ (since there are $|W|^{|U|}$ different maps from $U$ to $W$). Now we will give a different algorithm to check in polynomial time in $|U|$ whether the fixed frame $\mathcal{F}$ is a $p$-morphic image of a given frame $\mathcal{G} = (U, S_1, S_2)$ with $d_1(\mathcal{G}) < n(L)$.

**Lemma 4.1.** $\mathcal{F}$ is a $p$-morphic image of $\mathcal{G}$ iff there is a partial surjective map $g : U \to W$ with the following properties:

1. For each $u \in U$, there is $v \in \text{dom}(g)$ such that $u S_1 v$.
2. For each $v \in \text{dom}(g)$, the restriction $g \upharpoonright (\text{dom}(g) \cap S_1(v))$ is one-one and has range $E_1(g(v))$.
3. For each $u \in U$ there is $w \in W$ such that
   
   (a) $g(v) E_2 w$ for all $v \in \text{dom}(g) \cap S_2(u)$,
   (b) for each $w' \in W$, writing
   
   \[
   X_{w'} = S_1(g^{-1}(E_1(w')) \cap S_2(u),
   Y_{w'} = E_1(w') \cap E_2(w),
   \]
   
   we have $|Y_{w'} \setminus \text{rng}(g \upharpoonright [\text{dom}(g) \cap X_{w'}])| \leq |X_{w'} \setminus \text{dom}(g)|$.

**Proof.** Recall that a map $f : U \to W$ is a $p$-morphism iff the $f$-image of every $S_i$-cluster of $\mathcal{G}$ is an $E_i$-cluster of $\mathcal{F}$, for $i = 1, 2$.

Suppose there is a surjective $p$-morphism $f : U \to W$. Then for each $S_1$-cluster $C \subseteq U$, the map $f \upharpoonright C$ is a surjection from $C$ onto $E_1(f(u))$ for any $u \in C$, so we may choose $C' \subseteq C$ such that $f \upharpoonright C'$ is a bijection from $C'$ onto $E_1(f(u))$. Let $U' = \bigcup\{C' : C \text{ is an } S_1\text{-cluster of } \mathcal{G}\}$. Then it is easy to check that $g = f \upharpoonright U'$ satisfies conditions 1–2 of the lemma. To check condition 3, take any $u \in U$, and put $w = f(u)$. Condition 3a is clearly true. For 3b, fix any $w' \in W$. Pick any $x \in S_2(u)$. Note that $f(x) \in E_2(w)$. Define $X_{w'}, Y_{w'}$ as in the lemma. Then $x \in X_{w'}$ iff $x \in S_1(g^{-1}(E_1(w')))$, iff there is $y \in U'$ such that $x S_1 y$ and $g(y) E_1 w'$, iff $g(x) E_1 w'$, iff $f(x) \in Y_{w'}$. Now $f$ maps $S_2(u)$ onto $E_2(w)$, so $f(S_2(u)) \supseteq Y_{w'}$. It now follows that $f$ maps $X_{w'}$ onto $Y_{w'}$. Plainly, $f$ must therefore map a subset of $X_{w'} \setminus U'$ onto $Y_{w'} \cap g(X_{w'} \cap U')$, so we must have $|X_{w'} \setminus U'| \geq |Y_{w'} \setminus g(X_{w'} \cap U')|$ as required.

Conversely, let $g$ be as stated. We will extend $g$ to a surjective $p$-morphism $f : U \to W$. Since $U$ is a disjoint union of $S_2$-clusters, it is enough to define $f$ on an arbitrary $S_2$-cluster of $\mathcal{G}$. Pick $u \in U$. We will extend $g \upharpoonright S_2(u)$ to the whole of $S_2(u)$. Pick $w \in W$ according to condition 3 of the lemma. By condition 3a, $\text{rng}(g \upharpoonright S_2(u)) \subseteq E_2(w)$. Now we extend $g$ to
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Let $f$ such that $\text{rng}(f \upharpoonright S_2(u)) = E_2(w)$ and $f(x)E_1g(v)$ whenever $v \in \text{dom}(g)$ and $x \in S_2(u) \cap S_1(v)$.

For each $w' \in W$, define $X_{w'}, Y_{w'}$ as in the lemma. By conditions 1 and 2, $S_2(u) = \bigcup\{X_{w'} : w' \in W\}$, and $X_{w'} \cap X_{w''} = \emptyset$ whenever $-(w' E_1 w'')$. For each $w' \in W$, we take the restriction of $g$ to $X_{w'}$ (this restriction may be empty), observe that its range is a subset of $Y_{w'}$, and extend it to a surjection from $X_{w'}$ onto $Y_{w'}$. By condition 3, $|X_{w'} \setminus \text{dom}(g)| \geq |Y_{w'} \setminus \text{rng}(g \upharpoonright X_{w'})|$. So, there exists a surjection $f_{X_{w'}} : X_{w'} \to Y_{w'}$ extending $g$. Repeating this for a representative $w'$ of each $E_1$-cluster in turn yields an extension of $g$ to $S_2(u)$. Repeating for a representative $u$ of each $S_2$-cluster in turn yields an extension of $g$ to $U$ as required.

It is left to show that $f$ is a $p$-morphism. But it follows immediately from the construction of $f$ that $f \upharpoonright S_i(u) : S_i(u) \to E_i(f(u))$ is surjective for each $u \in U$ and each $i = 1, 2$. As we pointed out above this implies that $f$ is a $p$-morphism.

**Corollary 4.2.** It is decidable in polynomial time in the size of $\mathcal{G}$, whether $\mathcal{F}$ is a $p$-morphic image of $\mathcal{G}$.

**Proof.** By Lemma 4.1 it is enough to check whether there exists a partial map $g : U \to W$ satisfying conditions 1–3 of the lemma. There are at most $n(L)$ $S_1$-clusters in $\mathcal{G}$, and the restriction of $g$ to each $S_1$-cluster is one-one; hence, $d = |\text{dom}(g)| \leq n(L) \cdot |W|$, and this is independent of $\mathcal{G}$. There are at most $d^{|W|}$ maps from a set of size at most $d$ into $W$. Obviously, there are $\binom{|U|}{d} \leq |U|^d$ subsets of $U$ of size $d$. Hence there are at most $d^{|W|}|U|^d$ partial maps which may satisfy conditions 1 and 2 of the lemma. Our algorithm enumerates all partial maps from $U$ to $W$ with domain of size at most $d$, and for each one, checks whether it satisfies conditions 1–3 or not. It is not hard to see that this check can be done in $p$-time; indeed, it is clear that conditions 1 and 2 can be checked in time polynomial in $|U|$ and there is a first-order sentence $\sigma_{\mathcal{F}}$ such that $\mathcal{G} \models \sigma_{\mathcal{F}}$ iff $\mathcal{G}$ satisfies condition 3. The algorithm states that $\mathcal{F}$ is a $p$-morphic image of $\mathcal{G}$ if and only if it finds a map satisfying the conditions. Therefore, this is a $p$-time algorithm checking whether $\mathcal{F}$ is a $p$-morphic image of $\mathcal{G}$.

**Corollary 4.3.** Let $L$ be a proper normal extension of S5².

1. It can be checked in polynomial time in $|U|$ whether a finite S5²-frame $\mathcal{G} = (U, S_1, S_2)$ is an L-frame.
2. The satisfiability problem for $L$ is NP-complete.
3. The validity problem for $L$ is co-NP-complete.
Proof. 1. Follows directly from Theorem 3.2, Corollary 4.2, and the fact (shown in the proof of Theorem 3.16) that $\mathbb{M}_L$ is finite.

2. It is a well known result of modal logic (see, e.g., [4, Lemma 6.35]) that if $L$ is a consistent normal modal logic having the poly-size model property, and the problem of whether a finite structure $\mathcal{A}$ is an $L$-frame is decidable in time polynomial in the size of $\mathcal{A}$, then the satisfiability problem of $L$ is NP-complete. The poly-size model property of every $L \supseteq \text{S5}^2$ is proven in [3, Corollary 9]. (1) implies that the problem $\mathcal{G} \in \text{F}_L$ can be decided in polynomial time in the size of $\mathcal{G}$. The result follows.

3. Follows directly from (2).

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