All normal extensions of S5-squared are finitely axiomatizable

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Abstract. We prove that every normal extension of the bi-modal system $S5^2$ is finitely axiomatizable and that every proper normal extension has NP-complete satisfiability problem.

Keywords: modal logic, finite axiomatization, NP-complete, better-quasi-ordering

1. Introduction

Recall that the language of $S5^2$ is the propositional language based on a fixed countably infinite set of propositional variables and equipped with the two modal operators $\Box_1$ and $\Box_2$. For a formula $\varphi$ we let $\Diamond_i \varphi$ abbreviate $\neg \Box_i \neg \varphi$ for $i = 1, 2$. We recall that $S5^2$ is the smallest set of formulas containing all substitution instances of the following axiom schemas, for $i = 1, 2$:

1) All tautologies of the classical propositional calculus;
2) $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$;
3) $\Box_i p \to p$;
4) $\Box_i p \to \Box_i \Box_i p$;
5) $\Diamond_i \Box_i p \to p$;
6) $\Box_1 \Box_2 p \leftrightarrow \Box_2 \Box_1 p$;

and closed under the following rules of inference:

Modus Ponens (MP): from $\varphi$ and $\varphi \to \psi$ infer $\psi$;
Necessitation (N)$_i$: from $\varphi$ infer $\Box_i \varphi$.

Recall also that a set of formulas $L$ is called a logic if it contains all tautologies of the classical propositional calculus and is closed under the rule of modus ponens. A modal logic is called normal if it contains axiom schema 2) (see above) and is closed under the rule of necessitation. A logic $L_1$ is an extension of $L_2$ if $L_2 \subseteq L_1$.
It is well known that $\mathbf{S5}^2$ has the exponential size model property, and that its satisfiability problem is $\text{NEXPTIME}$-complete \cite{6}. In this paper, by the complexity of a logic we will mean the complexity of its satisfiability problem. It is shown in \cite{3} that in contrast to $\mathbf{S5}^2$, every proper normal extension $L$ of $\mathbf{S5}^2$ has the poly-size model property. That means that there is a polynomial $P(n)$ such that any $L$-consistent formula $\varphi$ (that is, $\neg \varphi \notin L$) has a model over a frame validating $L$ and with at most $P(|\varphi|)$ points, where $|\varphi|$ is the length of $\varphi$.

It was conjectured in \cite{3} that every proper normal extension of $\mathbf{S5}^2$ is finitely axiomatizable and $\text{NP}$-complete. In this paper we prove this conjecture. In fact, we show that for every proper normal extension $L$ of $\mathbf{S5}^2$, there is a finite set $M_L$ of finite $\mathbf{S5}^2$-frames such that an arbitrary finite $\mathbf{S5}^2$-frame is a frame for $L$ iff it does not have any frame in $M_L$ as a $p$-morphic image. This condition yields a finite axiomatization of $L$. We also show that the condition is decidable in deterministic polynomial time. This, together with the poly-size model property, implies $\text{NP}$-completeness of (satisfiability for) $L$.

Finally, we note that general complexity results for (uni)modal logics were investigated before. Bull and Fine proved that every normal extension of $\mathbf{S4.3}$ has the finite model property, is finitely axiomatizable and therefore is decidable (see \cite[Theorems 4.96, 4.101]{4}). Hemaspaandra strengthened the second result by showing that every normal extension of $\mathbf{S4.3}$ is $\text{NP}$-complete \cite[Theorem 6.41]{4}. The proof of finite axiomatizability uses Kruskal’s theorem on well-quasi-orderings \cite[Theorem 4.99]{4}. Kracht uses the same technique for showing that every extension of the intermediate logic of leptonic strings is finitely axiomatizable \cite[Theorem 14, Proposition 15]{8}. This paper takes the same line of research beyond unimodal logics. However, as we will see below, the theory of well-quasi-orderings does not suffice for our purposes; instead, we will use better-quasi-orderings.

2. Preliminaries

Recall that a triple $F = (W, E_1, E_2)$ is an $\mathbf{S5}^2$-frame (i.e., it validates the axioms of $\mathbf{S5}^2$: see, e.g., \cite[Corollary 5.10]{5}) iff $W$ is a non-empty set and $E_1$ and $E_2$ are equivalence relations on $W$ such that

$$F \models (\forall w, v, u) (wE_1 v \land vE_2 u \rightarrow (\exists z) (wE_2 z \land zE_1 u)).$$

For $i = 1, 2$ we call the $E_i$-equivalence classes $E_i$-clusters. The $E_i$-cluster containing $w \in W$ is denoted by $E_i(w)$, and for $X \subseteq W$ we let $E_i(X)$ denote $\bigcup_{x \in X} E_i(x)$.
We identify non-negative integers with ordinals, so that for $n \geq 0$ we have $n = \{0, 1, \ldots, n-1\}$. For positive integers $n$ and $m$, let $n \times m$ denote the $\mathbf{S5}^2$-frame with domain $n \times m$ and with $(x_1, x_2)E_1(y_1, y_2)$ iff $x_2 = y_2$ and $(x_1, x_2)E_2(y_1, y_2)$ iff $x_1 = y_1$. Then it is well known that $\mathbf{S5}^2$ is complete with respect to $\{n \times n : n \geq 1\}$ [11].

Given two $\mathbf{S5}^2$-frames $\mathcal{F} = (W, E_1, E_2)$ and $\mathcal{G} = (U, S_1, S_2)$, a mapping $f : U \to W$ is called a $p$-morphism from $\mathcal{G}$ to $\mathcal{F}$ if for each $i = 1, 2$,

$$(\forall t \in U)(\forall w \in W)(f(t)E_iw \iff (\exists u \in U)(tS_iu \land f(u) = w)).$$

It is easy to check that a map $f : U \to W$ is a $p$-morphism iff the $f$-image of every $S_i$-cluster of $\mathcal{G}$ is an $E_i$-cluster of $\mathcal{F}$, for $i = 1, 2$. We say that $\mathcal{F}$ is isomorphic to $\mathcal{G}$ if there exists a bijection $g : W \to U$ such that $wE_1w' \iff g(w)S_1g(w')$ for each $w, w' \in W$ and each $i = 1, 2$. It is easy to see that $\mathcal{F}$ is isomorphic to $\mathcal{G}$ iff there is a one-one $p$-morphism from $\mathcal{G}$ onto $\mathcal{F}$. We call $\mathcal{F}$ a $p$-morphic image of $\mathcal{G}$ if there is a $p$-morphism from $\mathcal{G}$ onto $\mathcal{F}$. It is well known that in this case, any formula valid in $\mathcal{G}$ is valid in $\mathcal{F}$.

We call $\mathcal{F} = (W, E_1, E_2)$ rooted if there is a point $w \in W$ that is related to every point $v \in W$ by the reflexive transitive closure of $E_1 \cup E_2$. It is easy to check that an $\mathbf{S5}^2$-frame $\mathcal{F}$ is rooted iff $\mathcal{F}| = (\forall w, v)(\exists u)(wE_1u \land uE_2v)$.

Choose a set $\mathbf{F}_{\mathbf{S5}^2}$ of representatives of the isomorphism types of finite rooted $\mathbf{S5}^2$-frames. That is, for each finite rooted $\mathbf{S5}^2$-frame, there is exactly one frame in $\mathbf{F}_{\mathbf{S5}^2}$ that is isomorphic to it.

Let $L$ be a normal extension of $\mathbf{S5}^2$. An $\mathbf{S5}^2$-frame $\mathcal{F}$ is called an $L$-frame if $\mathcal{F}$ validates all formulas in $L$. Let $\mathbf{F}_L$ be the set of all $L$-frames in $\mathbf{F}_{\mathbf{S5}^2}$. Then $\mathcal{L}$ is complete with respect to $\mathbf{F}_L$ [1]. Thus, for our purposes it suffices to consider only finite rooted $\mathbf{S5}^2$-frames. From now on, we will use the term “frame” to mean this.

For $\mathcal{F}, \mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$ we put

$$\mathcal{F} \leq \mathcal{G} \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of } \mathcal{G}.$$ 

Then it is routine to check that $\leq$ is a partial order on $\mathbf{F}_{\mathbf{S5}^2}$. We write $\mathcal{F} < \mathcal{G}$ if $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \not< \mathcal{F}$. Then $\mathcal{F} < \mathcal{G}$ implies $|\mathcal{F}| < |\mathcal{G}|$ and we see that there are no infinite descending chains in $(\mathbf{F}_{\mathbf{S5}^2}, <)$. Thus, for any non-empty $A \subseteq \mathbf{F}_{\mathbf{S5}^2}$, the set $\text{min}(A)$ of $<\text{-minimal elements of } A$ is non-empty, and indeed for any $\mathcal{G} \in A$ there is $\mathcal{F} \in \text{min}(A)$ such that $\mathcal{F} \leq \mathcal{G}$.
3. Finite axiomatizability

In this section we will prove the first main result of the paper — that every normal extension of $S5^2$ is finitely axiomatizable.

First we recall the Jankov-Fine formulas for $S5^2$ (see [4, §3.4] and [5, §8.4 p. 392]). Consider a frame $\mathcal{F} = (W, E_1, E_2)$. For each point $p \in W$ we introduce a propositional variable, denoted also by $p$, and consider the formulas

\[
\alpha(\mathcal{F}) = \Box_1 \Box_2 \left( \bigvee_{p \in W} (p \land \neg \bigvee_{p' \in W \setminus \{p\}} p') \right) \land \bigwedge_{i=1,2} \bigwedge_{p, p' \in W, p \neq p'} (p \rightarrow \Diamond_i p') \land \bigwedge_{i=1,2} \bigwedge_{p, p' \in W, p \neq p'} (p \rightarrow \neg \Diamond_i p') \right),
\]

\[
\chi(\mathcal{F}) = \neg \alpha(\mathcal{F}).
\]

**Lemma 3.1.** For any frames $\mathcal{F} = (W, E_1, E_2)$ and $\mathcal{G} = (U, S_1, S_2)$ we have that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$ iff $\mathcal{G} \not\models \chi(\mathcal{F})$.

**Proof.** (Sketch) Suppose $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. Define a valuation $V$ on $\mathcal{F}$ by putting $V(p) = p$ for any $p \in W$. Then $\mathcal{F} \not\models V \chi(\mathcal{F})$ by the definition of $\chi(\mathcal{F})$. Now if $\mathcal{G} \models \chi(\mathcal{F})$, then since p-morphic images preserve validity of formulas, we would also have $\mathcal{F} \models \chi(\mathcal{F})$, a contradiction. Therefore, $\mathcal{G} \not\models \chi(\mathcal{F})$.

For the converse, we use the argument of [5, Claim 8.36]. Suppose that $\mathcal{G} \not\models \chi(\mathcal{F})$. Then there is a valuation $V'$ on $\mathcal{G}$ and a point $u \in U$ such that $\mathcal{G}, u \not\models V' \chi(\mathcal{F})$. Therefore, $\mathcal{G}, u \models V' \alpha(\mathcal{F})$. Define a map $f : U \to W$ by putting $f(t) = p$ $\iff$ $\mathcal{G}, t \models V' p$, for every $t \in U$ and $p \in W$. From $\mathcal{G}$ being rooted and the truth of the first conjunct of $\alpha(\mathcal{F})$ it follows that $f$ is well defined. The truth of the first two conjuncts of $\alpha(\mathcal{F})$ together with $\mathcal{F}$ being rooted implies that $f$ is surjective. Finally, the truth of the second and third conjuncts of $\alpha(\mathcal{F})$ guarantees that $f$ is a p-morphism. Therefore, $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.

Let $L$ be a proper normal extension of $S5^2$. By completeness of $S5^2$ with respect to $F_{S5^2}$, the set $F_{S5^2} \setminus F_L$ is non-empty. Let $M^*_L = \min(F_{S5^2} \setminus F_L)$.

**Theorem 3.2.** For any proper normal extension $L$ of $S5^2$ and $\mathcal{G} \in F_{S5^2}$, $\mathcal{G} \in F_L$ iff no $F \in M^*_L$ is a p-morphic image of $\mathcal{G}$.

**Proof.** Let $\mathcal{G} \in F_L$; then since p-morphisms preserve validity of formulas, every p-morphic image of $\mathcal{G}$ belongs to $F_L$ and hence can not be in $M^*_L$. 

\[\square_1 \square_2 \left( \bigvee_{p \in W} (p \land \neg \bigvee_{p' \in W \setminus \{p\}} p') \right) \land \bigwedge_{i=1,2} \bigwedge_{p, p' \in W, p \neq p'} (p \rightarrow \Diamond_i p') \land \bigwedge_{i=1,2} \bigwedge_{p, p' \in W, p \neq p'} (p \rightarrow \neg \Diamond_i p') \right),\]
Conversely, if \( G \in F_{S5^2} \setminus F_L \) then there is \( F \in M_L \) such that \( F \leq G \) — that is, \( F \) is a \( p \)-morphic image of \( G \).

**Theorem 3.3.** Every proper normal extension \( L \) of \( S5^2 \) is axiomatizable by the axioms of \( S5^2 \) plus \( \{ \chi(F) : F \in M_L \} \).

**Proof.** Let \( G \in F_{S5^2} \). Then by Theorem 3.2, \( G \in F_L \) iff there is no \( F \in M_L \) with \( F \leq G \), iff (by Lemma 3.1) there is no \( F \in M_L \) with \( G \models \chi(F) \), iff \( G \models \chi(F) \) for all \( F \in M_L \). Thus, \( G \models \{ \chi(F) : F \in M_L \} \) iff \( G \in F_L \).

Let \( L' \) be the logic axiomatized by the axioms of \( S5^2 \) plus \( \{ \chi(F) : F \in M_L \} \). From the above it is clear that \( F_{L'} = F_L \). But \( L \) (\( L' \)) is sound and complete with respect to \( F_L \) (\( F_{L'} \)), respectively. So, \( L' = L \).

It follows that \( L \supset S5^2 \) is finitely axiomatizable whenever \( M_L \) is finite. We now proceed to show that \( M_L \) is indeed finite for every proper normal extension \( L \) of \( S5^2 \).

Suppose \( G \in F_{S5^2} \). For \( i = 1, 2 \), we say that the \( E_i \)-depth of \( G \) is \( n \), and write \( d_i(G) = n \), if the number of \( E_i \)-clusters of \( G \) is \( n \).

Fix a proper normal extension \( L \) of \( S5^2 \). Since \( S5^2 \) is complete with respect to \( \{ n \times n : n \geq 1 \} \), there is \( n \geq 1 \) such that \( n \times n \notin F_L \). Let \( n(L) \) be the least such.

**Lemma 3.4.** Let \( L \) be as above, and write \( n \) for \( n(L) \).

1. If \( G \in F_L \), then \( d_1(G) < n \) or \( d_2(G) < n \).
2. If \( G \in M_L \), then \( d_1(G) \leq n \) or \( d_2(G) \leq n \).

**Proof.**
1. If \( G \in F_L \) and \( d_1(G) \geq n \) and \( d_2(G) \geq n \), then by [3, Lemma 5], \( n \times n \) is a \( p \)-morphic image of \( G \). So, \( n \times n \notin F_L \), a contradiction.

2. If \( G \in M_L \) and both depths of \( G \) are greater than \( n \), then again \( n \times n \) is a \( p \)-morphic image of \( G \). Therefore, \( n \times n \notin G \). However, \( G \) is a minimal element of \( F_{S5^2} \setminus F_L \), implying that \( n \times n \) belongs to \( F_L \), which is false.

**Corollary 3.5.** \( M_L \) is finite iff \( \{ F \in M_L : d_i(F) = k \} \) is finite for every \( k \leq n(L) \) and \( i = 1, 2 \).

**Proof.** By Lemma 3.4, \( M_L = \bigcup_{k \leq n(L)} \{ F \in M_L : d_1(F) = k \} \cup \bigcup_{k \leq n(L)} \{ F \in M_L : d_2(F) = k \} \). Thus, \( M_L \) is finite if and only if \( \{ F \in M_L : d_i(F) = k \} \) is finite for every \( k \leq n(L) \) and \( i = 1, 2 \).
Since $\mathbf{M}_L$ is a $\leq$-antichain in $\mathbf{F}_{\mathbf{S}_5^2}$, to show that $\{ \mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k \}$ is finite for every $k \leq n(L)$ and $i = 1, 2$, it is enough to prove that for any $k$, the set $\{ \mathcal{F} \in \mathbf{F}_{\mathbf{S}_5^2} : d_i(\mathcal{F}) = k \}$ does not contain an infinite $\leq$-antichain. Without loss of generality we can consider the case when $i = 2$.

Fix $k \in \omega$. For every $n \in \omega$ let $\mathcal{M}_n$ denote the set of all $n \times k$ matrices with coefficients in $\omega (i < n, j < k)$. Let $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$. Define $\leq$ on $\mathcal{M}$ by putting $(m_{ij}) \leq (m'_{ij})$ if we have $(m_{ij}) \in \mathcal{M}_n$, $(m'_{ij}) \in \mathcal{M}_n'$, $n \leq n'$, and there is a surjection $f : n' \to n$ such that $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. It is easy to see that $(\mathcal{M}, \leq)$ is a quasi-ordered set (i.e., $\leq$ is reflexive and transitive).

Let $\mathbf{F}_{\mathbf{S}_5^2}^k = \{ \mathcal{F} \in \mathbf{F}_{\mathbf{S}_5^2} : d_2(\mathcal{F}) = k \}$. For each $\mathcal{F} \in \mathbf{F}_{\mathbf{S}_5^2}^k$ we fix enumerations $F_0, \ldots, F_{n-1}$ of the $E_1$-clusters of $\mathcal{F}$ (where $n = d_1(\mathcal{F})$) and $F^0, \ldots, F^{k-1}$ of the $E_2$-clusters of $\mathcal{F}$. Define a map $H : \mathbf{F}_{\mathbf{S}_5^2}^k \to \mathcal{M}$ by putting $H(\mathcal{F}) = (m_{ij})$ if $|F_i \cap F^j| = m_{ij}$ for $i < d_1(\mathcal{F})$ and $j < k$. As $\mathcal{F} \in \mathbf{F}_{\mathbf{S}_5^2}^k$, it follows that $m_{ij} > 0$ for each such $i, j$. Recall that a map $f : P \to P'$ between ordered sets ($P, \leq$) and ($P', \leq'$) is called order reflecting if $f(w) \leq' f(v)$ implies $w \leq v$ for any $w, v \in P$.

**Lemma 3.6.** $H : (\mathbf{F}_{\mathbf{S}_5^2}^k, \leq) \to (\mathcal{M}, \leq)$ is an order-reflecting injection.

**Proof.** Since $\mathbf{F}_{\mathbf{S}_5^2}$ consists of non-isomorphic frames, $H$ is one-one. Now let $\mathcal{F} = (W, E_1, E_2)$, $\mathcal{G} = (U, S_1, S_2)$, $\mathcal{F}, \mathcal{G} \in \mathbf{F}_{\mathbf{S}_5^2}^k$, and $(m_{ij}), (m'_{ij}) \in \mathcal{M}$ be such that $H(\mathcal{F}) = (m_{ij}), H(\mathcal{G}) = (m'_{ij})$, and $(m_{ij}) \leq (m'_{ij})$. We need to show that $\mathcal{F} \leq \mathcal{G}$. Suppose $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. Then there is a surjection $f : n' \to n$ such that $m_{f(i)j} \leq m'_{ij}$ for $i < n'$ and $j < k$. Then $|G_i \cap G^j| \geq |F_{f(i)} \cap F^j| > 0$ for any $i < n'$ and $j < k$. Hence there exists a surjection $h_i^j : G_i \cap G^j \to F_{f(i)} \cap F^j$. Define $h : U \to W$ by putting $h(u) = h_i^j(u)$, where $i < n'$, $j < k$, and $u \in G_i \cap G^j$. It is obvious that $h$ is well defined and onto.

Now we show that $h$ is a p-morphism. If $uS_1v$, then $u, v \in G_i$ for some $i < n'$. Therefore, $h(u), h(v) \in F_{f(i)}$, and so $h(u)E_1h(v)$. Analogously, if $uS_2v$, then $u, v \in G^j$ for some $j < k$, $h(u), h(v) \in F^j$, and so $h(u)E_2h(v)$. Now suppose $u \in G_i \cap G^j$ for some $i < n'$ and $j < k$. If $h(u)E_2h(v)$, then $h(u), h(v) \in F^j$ and $v \in G^j$. As both $u$ and $v$ belong to $G^j$ it follows that $uS_2v$. Finally, if $h(u)E_1h(v)$, then $h(u) \in F_{f(i)} \cap F^j$ and $h(v) \in F_{f(i)} \cap F^j'$, for some $j' < k$. Therefore, there exists $z \in G_i \cap G^{j'}$ (since $z \in G_i$ we have $uS_1z$) such that $h(z) = h(v)$. Thus, $h$ is an onto p-morphism, implying that $\mathcal{F} \leq \mathcal{G}$. Thus, $H$ is order reflecting. \[\blacksquare\]

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1 By an $n \times k$ matrix we mean a matrix with $n$ rows and $k$ columns.
Corollary 3.7. If $\Delta \subseteq F^{k}_{s5^2}$ is a $\preceq$-antichain, then $H(\Delta) \subseteq M$ is a $\preceq$-antichain.

Proof. Immediate.

Now we will show that there are no infinite $\preceq$-antichains in $M$. For this we define a quasi-order $\preceq$ on $M$ included in $\preceq$ and show that there are no infinite $\preceq$-antichains in $M$. To do so we first introduce two quasi-orders $\preceq_1$ and $\preceq_2$ on $M$ and then define $\preceq$ as the intersection of these quasi-orders. For $(m_{ij}) \in M_n$ and $(m'_{ij}) \in M'_n$, we say that:

i. $(m_{ij}) \subseteq_1 (m'_{ij})$ if there is a one-one order-preserving map $\varphi : n \to n'$ (i.e., $i < i' < n$ implies $\varphi(i) < \varphi(i')$) such that $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$;

ii. $(m_{ij}) \subseteq_2 (m'_{ij})$ if there is a map $\psi : n' \to n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$.

Let $\preceq$ be the intersection of $\preceq_1$ and $\preceq_2$.

Lemma 3.8. For any $(m_{ij}), (m'_{ij}) \in M$, if $(m_{ij}) \subseteq (m'_{ij})$, then $(m_{ij}) \preceq (m'_{ij})$.

Proof. Suppose $(m_{ij}) \in M_n$ and $(m'_{ij}) \in M'_n$. If $(m_{ij}) \subseteq (m'_{ij})$, then $(m_{ij}) \subseteq_1 (m'_{ij})$ and $(m_{ij}) \subseteq_2 (m'_{ij})$. By $(m_{ij}) \subseteq_1 (m'_{ij})$ there is a one-one order-preserving map $\varphi : n \to n'$ with $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$; and by $(m_{ij}) \subseteq_2 (m'_{ij})$ there is a map $\psi : n' \to n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Let $\text{rng}(\varphi) = \{\varphi(i) : i < n\}$. Define $f : n' \to n$ by putting

\[ f(i) = \begin{cases} \varphi^{-1}(i), & \text{if } i \in \text{rng}(\varphi), \\ \psi(i), & \text{otherwise}. \end{cases} \]

Then $f$ is a surjection. Moreover, for $i < n'$ and $j < k$, if $i \in \text{rng}(\varphi)$, then $m_{f(i)j} = m_{\varphi^{-1}(i)j} \leq m'_{ij}$ by the definition of $\preceq_1$; and if $i \notin \text{rng}(\varphi)$, then $m_{f(i)j} = m_{\psi(i)j} \leq m'_{ij}$ by the definition of $\preceq_2$. Therefore, $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Thus, $(m_{ij}) \preceq (m'_{ij})$.

Thus, it is left to show that there are no infinite $\preceq$-antichains in $M$. For this we use the theory of better-quasi-orderings (bqos). Our main source of reference is Laver [9].

For any set $X \subseteq \omega$ let $[X]^{<\omega} = \{Y \subseteq X : |Y| < \omega\}$, and for $n < \omega$ let $[X]^n = \{Y \subseteq X : |Y| = n\}$. We say that $Y$ is an initial segment of $X$ if there is $n \in \omega$ such that $Y = \{x \in X : x \leq n\}$.
**Definition 3.9.** Let $X$ be an infinite subset of $\omega$. We say that $\mathcal{B} \subseteq [X]^{<\omega}$ is a barrier on $X$ if $\emptyset \notin \mathcal{B}$ and:

- for every infinite $Y \subseteq X$, there is an initial segment of $Y$ in $\mathcal{B}$;
- $\mathcal{B}$ is an antichain with respect to $\subseteq$.

A barrier is a barrier on some infinite $X \subseteq \omega$.

Note that for any $n \geq 1$, $[\omega]^n$ is a barrier on $\omega$.

**Definition 3.10.**

1. If $s, t$ are finite subsets of $\omega$, we write $s \triangleleft t$ to mean that there are $i_1 < \ldots < i_k$ and $j$ ($1 \leq j < k$) such that $s = \{i_1, \ldots, i_j\}$ and $t = \{i_2, \ldots, i_k\}$.

2. Given a barrier $\mathcal{B}$ and a quasi-ordered set $(Q, \leq)$, we say that a map $f : \mathcal{B} \to Q$ is good if there are $s, t \in \mathcal{B}$ such that $s \triangleleft t$ and $f(s) \leq f(t)$.

3. Let $(Q, \leq)$ be a quasi-order. We call $\leq$ a better-quasi-ordering (bqo) if for every barrier $\mathcal{B}$, every map $f : \mathcal{B} \to Q$ is good.

Now we recall basic constructions and properties of bqos.

**Proposition 3.11.** If $(Q, \leq)$ is a bqo, there are no infinite $\leq$-antichains in $Q$.

**Proof.** Let $(\xi_n)_{n \in \omega}$ be an infinite sequence of distinct elements of $Q$. As we pointed out, $\mathcal{B} = [\omega]^1 = \{\{n\} : n < \omega\}$ is a barrier. Define a map $\theta : \mathcal{B} \to Q$ by putting $\theta(\{n\}) = \xi_n$. Since $(Q, \leq)$ is a bqo, $\theta$ is good. Therefore, there are $\{n\}, \{m\} \in \mathcal{B}$ such that $\{n\} \triangleleft \{m\}$ (i.e., $n < m$) and $\xi_n \leq \xi_m$. So, no infinite subset of $Q$ forms a $\leq$-antichain. \qed

We write $\text{On}$ for the class of all ordinals. Let $(Q, \leq)$ be a quasi-order. Define $\leq^*$ on the class $\bigcup_{\alpha \in \text{On}} Q^\alpha$, and on any set contained in it, by putting $(x_i)_{i < \alpha} \leq^* (y_i)_{i < \beta}$ if there is a one-one order-preserving map $\varphi : \alpha \to \beta$ such that $x_i \leq y_{\varphi(i)}$ for all $i < \alpha$.

Let $\varphi(Q)$ be the power set of $Q$. The order $\leq$ can be extended to $\varphi(Q)$ as follows: For $\Gamma, \Delta \in \varphi(Q)$, we say that $\Gamma \leq \Delta$ if for all $\delta \in \Delta$ there is $\gamma \in \Gamma$ with $\gamma \leq \delta$.

Recall that $(P, \leq')$ is called a suborder of $(Q, \leq)$ if $P \subseteq Q$ and $\leq' = \leq \cap P^2$.

**Theorem 3.12.**

1. $(\omega, \leq)$ is a bqo.
2. Any suborder of a bqo is a bqo.

3. If $\leq$ and $\leq'$ are bqos on $Q$, then $\leq \cap \leq'$ is also a bqo on $Q$.

4. If $(Q, \leq)$ is a bqo, then $(\bigcup_{\alpha \in \mathcal{O}_n} Q^\alpha, \leq^*)$ is also a (proper class) bqo.

   Hence, by (2), its suborders $(Q^\delta, \leq^*)$ and $(\bigcup_{\alpha<\omega} Q^\alpha, \leq^*)$ are bqos.

5. If $(Q, \leq)$ is a bqo, then $(\varphi(Q), \leq)$ is a bqo.

**Proof.** (1) follows from Lemma 1.2 of [9]. (2) is trivial.

(3): By [9, Lemma 1.8], $(Q \times Q, \leq \otimes \leq')$ is a bqo, where we define

$(x, x') \leq \otimes \leq' (y, y')$ iff $x \leq y$ and $x' \leq' y'$.

By (2), its suborder $(\{(q, q) : q \in Q\}, \leq \otimes \leq')$ is also a bqo, and this is isomorphic to $(Q, \leq \cap \leq')$.

(4) — see [9, Theorem 1.10].

(5) Finally to show $(\varphi(Q), \leq)$ is a bqo we adapt the proof of Lemma 1.3 of [9]. Let $B$ be a barrier and consider $f : B \to \varphi(Q)$. Suppose $f$ is not good. Then for each $s, t \in B$ with $s \prec t$ we have $f(s) \not\leq f(t)$. Let $B(2) = \{s \cup t : s, t \in B$ and $s \prec t\}$. Thus for every element $s \cup t \in B(2)$ there is an element $\delta_{st} \in f(t)$ such that for every $\gamma \in f(s)$ we have $\gamma \not\leq \delta_{st}$.

Define a map $h : B(2) \to Q$ by putting $h(s \cup t) = \delta_{st}$ for every $s \cup t \in B(2)$. It can be checked that $h$ is well defined. It is known (see, e.g., [9, p. 35]) that $B(2)$ is a barrier. Since $(Q, \leq)$ is a bqo, $h$ is good, so there exist $s \cup t, s' \cup t' \in B(2)$ with $s \cup t \prec s' \cup t'$ and $h(s \cup t) \leq h(s' \cup t')$. It is easy to check (see [9, p. 35]) that $t = s'$. But now $\delta_{st'} = h(s' \cup t') \geq h(s \cup t) \in f(t) = f(s')$. This contradicts the definition of $\delta_{st'}$, hence $f$ is good and therefore $(\varphi(Q), \leq)$ is a bqo. 

**Remark 3.13.** A quasi-order $\leq$ on a set $Q$ is called a well-quasi-ordering (wqo) if for any sequence $(x_i)_{i<\omega}$ in $Q$ there exist $i < j < \omega$ with $x_i \leq x_j$. As we said in the introduction, wqos have been used to prove finite axiomatizability results in modal logic on many previous occasions. The following facts are known about them (cf. Theorem 3.12):

1. Any bqo is a wqo.

2. If $\leq$ and $\leq'$ are wqos on $Q$, then $\leq \cap \leq'$ is also a wqo on $Q$.

3. (Higman’s Lemma, proved in [7]) If $(Q, \leq)$ is a wqo then $(\bigcup_{\alpha<\omega} Q^\alpha, \leq^*)$ is also a wqo.

An example of a wqo $(Q, \leq)$ with $(\bigcup_{\alpha<\omega} Q^\alpha, \leq^*)$ not a wqo was constructed by Rado [10]: let $Q = \{(i, j) : i < j < \omega\}$, ordered by $(i, j) \leq (k, l)$ iff either $i = k$ and $j \leq l$, or else $i, j < k$. This is a wqo on $Q$. Now for $i < \omega$ let $\xi_i$ be the sequence $(i, i+1, (i, i+2)), \ldots$. Then $\xi_i \not\leq^* \xi_j$ for all $i < j < \omega$. This example can be used to show that for a wqo $(Q, \leq)$, in general $(\varphi(Q), \leq)$ fails
to be a wqo, even if we restrict to finite subsets of $Q$ (see also the discussion on p. 33 of [9]). This failure is why we use bqos and not wqos here.

By Proposition 3.11, to show that there are no $\sqsubseteq$-antichains in $\mathcal{M}$ it suffices to show that $(\mathcal{M}, \sqsubseteq)$ is a bqo. It follows from Theorem 3.12(3) that the intersection of two bqos is again a bqo. Hence, it is enough to prove that $(\mathcal{M}, \sqsubseteq_1)$ and $(\mathcal{M}, \sqsubseteq_2)$ are bqos.

**Lemma 3.14.** $(\mathcal{M}, \sqsubseteq_1)$ is a bqo.

**Proof.** By Theorem 3.12(1), $(\omega, \leq)$ is a bqo. By Theorem 3.12(4), $(\omega^k, \leq^*)$ is also a bqo. By Theorem 3.12(4) again, $(\mathcal{M}, \sqsubseteq_1) \cong (\bigcup_{n<\omega}(\omega^k)^n, \leq^{**})$ is a bqo as well. $\square$

It remains to show that $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

**Lemma 3.15.** $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

**Proof.** For a matrix $(m_{ij}) \in \mathcal{M}_n$ let $m_i = (m_{i0}, \ldots, m_{ik-1})$ denote the $i$-th row of $(m_{ij})$. Note that each row of $(m_{ij})$ is a $1 \times k$ matrix, and so $m_i \in \mathcal{M}_1$ for any $i < n$. We write row$(m_{ij})$ for the set $\{m_i : i < n\}$. Obviously, row$(m_{ij}) \in \wp(\mathcal{M}_1) \subseteq \wp(\mathcal{M})$. Consider an arbitrary barrier $\mathcal{B}$ and a map $f : \mathcal{B} \to \mathcal{M}$. We need to show that $f$ is good with respect to $\sqsubseteq_2$. Define $g : \mathcal{B} \to \wp(\mathcal{M})$ by $g(s) = \text{row}(f(s))$. Since $(\mathcal{M}, \sqsubseteq_1)$ is a bqo, by Theorem 3.12(5), $(\wp(\mathcal{M}), \sqsubseteq_1)$ is also a bqo. Hence, there are $s, t \in \mathcal{B}$ such that $s \triangleleft t$ and $g(s) \sqsubseteq_1 g(t)$. Therefore, for each $\delta \in g(t)$ there is $\gamma \in g(s)$ with $\gamma \sqsubseteq_1 \delta$.

Now we show that $f(s) \sqsubseteq_2 f(t)$. Write $(m_{ij})$ for $f(s)$ and $(m'_{ij})$ for $f(t)$. Suppose that $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. We define $\psi : n' \to n$ as follows. Let $i < n'$. Then $m'_i \in g(t)$. By the above, we may choose $\psi(i) < n$ such that $m_{\psi(i)} \sqsubseteq_1 m'_i$. This defines $\psi$, and we have $m_{\psi(i)j} \leq m'_{ij}$ for any $i < n'$ and $j < k$. Thus, $f(s) \sqsubseteq_2 f(t)$, $f$ is a good map, and so $(\mathcal{M}, \sqsubseteq_2)$ is a bqo. $\square$

It follows that $(\mathcal{M}, \sqsubseteq)$ is a bqo. Therefore, there are no infinite $\sqsubseteq$-antichains in $\mathcal{M}$. Thus, by Lemma 3.8 there are no infinite $\preceq$-antichains in $\mathcal{M}$.

Now we are in a position to prove the first main theorem of this paper.

**Theorem 3.16.** Every normal extension of $S5^2$ is finitely axiomatizable.

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2 To apply this theorem, we needed to require in the definition of $\sqsubseteq_1$ on $\mathcal{M}$ that $\varphi$ is order preserving. This is the only time this assumption is used.
Proof. Clearly, $S5^2$ is finitely axiomatizable. Suppose $L$ is a proper normal extension of $S5^2$. Then by Theorem 3.3 $L$ is axiomatizable by the $S5^2$ axioms plus $\{\chi(F) : F \in M_L\}$. Since there are no infinite $\approx$-antichains in $M$, by Corollary 3.7 there are no infinite antichains in $F^k_{S5^2}$, for each $k \in \omega$. Therefore, $\{F \in M_L : d_i(F) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. Thus, $M_L$ is finite by Corollary 3.5. It follows that $L$ is finitely axiomatizable. □

Corollary 3.17. The lattice of normal extensions of $S5^2$ is countable.

Proof. Immediately follows from Theorem 3.16 since there are only countably many finitely axiomatizable normal extensions of $S5^2$. □

Remark 3.18. In algebraic terminology, Corollary 3.17 says that the lattice of subvarieties of the variety $Df_2$ of two-dimensional diagonal-free cylindric algebras is countable. This is in contrast with the variety $CA_2$ of two-dimensional cylindric algebras (with diagonals), since, as was shown in [2], the cardinality of the lattice of subvarieties of $CA_2$ is that of continuum.

4. Complexity

Note that Theorem 3.16, and the fact that every normal extension $L$ of $S5^2$ is complete with respect to a class of finite frames $(F_L)$ for which (up to isomorphism) membership is decidable, imply that $L$ is decidable. This section will be devoted to showing that if $L$ is a proper normal extension, then its satisfiability problem is NP-complete. Fix such an $L$. We will see in Corollary 4.3 below that NP-completeness follows from the poly-size model property if we can decide in time polynomial in $|W|$ whether a finite structure $A = (W, R_1, R_2)$ is in $F_L$ (up to isomorphism). It suffices to decide in polynomial time (1) whether $A$ is a (rooted $S5^2$-) frame; (2) whether a given frame is in $F_L$. The first is easy. We concentrate on the second.

By Lemma 3.4(1), there is $n(L) \in \omega$ such that for each frame $G = (U, S_1, S_2)$ in $F_L$ we have $d_1(G) < n(L)$ or $d_2(G) < n(L)$. So, if both depths of a given frame $G$ are greater than or equal to $n(L)$ (which obviously can be checked in polynomial time in the size of $G$), then $G \notin F_L$. So, without loss of generality we can assume that $d_1(G) < n(L)$.

By Theorem 3.2, $G$ is in $F_L$ iff it has no p-morphic image in $M_L$. Because $M_L$ is a fixed finite set, it suffices to provide, for an arbitrary fixed frame $F = (W, E_1, E_2)$, an algorithm that decides in time polynomial in the size of $G$ whether there is a p-morphism from $G$ onto $F$. If we considered every map $f : U \rightarrow W$ and checked whether it is a p-morphism, it would take
exponential time in the size of $G$ (since there are $|W|^{|U|}$ different maps from $U$ to $W$). Now we will give a different algorithm to check in polynomial time in $|U|$ whether the fixed frame $F$ is a $p$-morphic image of a given frame $G = (U, S_1, S_2)$ with $d_1(G) < n(L)$.

**Lemma 4.1.** $F$ is a $p$-morphic image of $G$ iff there is a partial surjective map $g : U \to W$ with the following properties:

1. For each $u \in U$, there is $v \in \text{dom}(g)$ such that $u S_1 v$.

2. For each $v \in \text{dom}(g)$, the restriction $g \upharpoonright (\text{dom}(g) \cap S_1(v))$ is one-one and has range $E_1(g(v))$.

3. For each $u \in U$ there is $w \in W$ such that

   (a) $g(v) E_2 w$ for all $v \in \text{dom}(g) \cap S_2(u)$,

   (b) for each $w' \in W$, writing

   $$X_{w'} = S_1(g^{-1}(E_1(w'))) \cap S_2(u),$$
   $$Y_{w'} = E_1(w') \cap E_2(w),$$

   we have $|Y_{w'} \setminus \text{rng}(g \upharpoonright [\text{dom}(g) \cap X_{w'}])| \leq |X_{w'} \setminus \text{dom}(g)|$.

**Proof.** Recall that a map $f : U \to W$ is a $p$-morphism iff the $f$-image of every $S_i$-cluster of $G$ is an $E_i$-cluster of $F$, for $i = 1, 2$.

Suppose there is a surjective $p$-morphism $f : U \to W$. Then for each $S_1$-cluster $C \subseteq U$, the map $f \upharpoonright C$ is a surjection from $C$ onto $E_1(f(u))$ for any $u \in C$, so we may choose $C' \subseteq C$ such that $f \upharpoonright C'$ is a bijection from $C'$ onto $E_1(f(u))$. Let $U' = \bigcup\{C' : C$ is an $S_1$-cluster of $G\}$. Then it is easy to check that $g = f \upharpoonright U'$ satisfies conditions 1–2 of the lemma. To check condition 3, take any $u \in U$, and put $w = f(u)$. Condition 3a is clearly true. For 3b, fix any $w' \in W$. Pick any $x \in S_2(u)$. Note that $f(x) \in E_2(w)$. Define $X_{w'}, Y_{w'}$ as in the lemma. Then $x \in X_{w'}$ iff $x \in S_1(g^{-1}(E_1(w'))) \cap S_2(u)$, iff there is $y \in U'$ such that $x S_1 y$ and $g(y) E_1 w'$, iff $f(x) E_1 w'$, iff $f(x) \in Y_{w'}$. Now $f$ maps $S_2(u)$ onto $E_2(w)$, so $f(S_2(u)) \supseteq Y_{w'}$. It now follows that $f$ maps $X_{w'}$ onto $Y_{w'}$. Plainly, $f$ must therefore map a subset of $X_{w'} \setminus U'$ onto $Y_{w'} \setminus g(X_{w'} \cap U')$, so we must have $|X_{w'} \setminus U'| \geq |Y_{w'} \setminus g(X_{w'} \cap U')|$ as required.

Conversely, let $g$ be as stated. We will extend $g$ to a surjective $p$-morphism $f : U \to W$. Since $U$ is a disjoint union of $S_2$-clusters, it is enough to define $f$ on an arbitrary $S_2$-cluster of $G$. Pick $u \in U$. We will extend $g \upharpoonright S_2(u)$ to the whole of $S_2(u)$. Pick $w \in W$ according to condition 3 of the lemma. By condition 3a, $\text{rng}(g \upharpoonright S_2(u)) \subseteq E_2(w)$. Now we extend $g$ to
f such that \( \text{rng}(f \upharpoonright S_2(u)) = E_2(w) \) and \( f(x)E_1g(v) \) whenever \( v \in \text{dom}(g) \) and \( x \in S_2(u) \cap S_1(v) \).

For each \( w' \in W \), define \( X_{w'}, Y_{w'} \) as in the lemma. By conditions 1 and 2, \( S_2(u) = \bigcup \{ X_{w'} : w' \in W \} \), and \( X_{w'} \cap X_{w''} = \emptyset \) whenever \( \neg (w'E_1w'') \). For each \( w' \in W \), we take the restriction of \( g \) to \( X_{w'} \) (this restriction may be empty), observe that its range is a subset of \( Y_{w'} \), and extend it to a surjection from \( X_{w'} \) onto \( Y_{w'} \). By condition 3, \( |X_{w'} \setminus \text{dom}(g)| \geq |Y_{w'} \setminus \text{rng}(g \upharpoonright X_{w'})| \). So, there exists a surjection \( f_{X_{w'}} : X_{w'} \to Y_{w'} \) extending \( g \). Repeating this for a representative \( w' \) of each \( E_1 \)-cluster in turn yields an extension of \( g \) to \( S_2(u) \). Repeating for a representative \( u \) of each \( S_2 \)-cluster in turn yields an extension of \( g \) to \( U \) as required.

It is left to show that \( f \) is a \( p \)-morphism. But it follows immediately from the construction of \( f \) that \( f \upharpoonright S_i(u) : S_i(u) \to E_i(f(u)) \) is surjective for each \( u \in U \) and each \( i = 1, 2 \). As we pointed out above this implies that \( f \) is a \( p \)-morphism.

\begin{corollary}
It is decidable in polynomial time in the size of \( \mathcal{G} \), whether \( \mathcal{F} \) is a \( p \)-morphic image of \( \mathcal{G} \).
\end{corollary}

\begin{proof}
By Lemma 4.1 it is enough to check whether there exists a partial map \( g : U \to W \) satisfying conditions 1–3 of the lemma. There are at most \( n(L) \) \( S_1 \)-clusters in \( \mathcal{G} \), and the restriction of \( g \) to each \( S_1 \)-cluster is one-one; hence, \( d = |\text{dom}(g)| \leq n(L) \cdot |W| \), and this is independent of \( \mathcal{G} \). There are at most \( d^{|W|} \) maps from a set of size at most \( d \) into \( W \). Obviously, there are \( \binom{|U|^d}{|W|^d} \) partial maps which may satisfy conditions 1 and 2 of the lemma. Our algorithm enumerates all partial maps from \( U \) to \( W \) with domain of size at most \( d \), and for each one, checks whether it satisfies conditions 1–3 or not. It is not hard to see that this check can be done in \( p \)-time; indeed, it is clear that conditions 1 and 2 can be checked in time polynomial in \( |U| \) and there is a first-order sentence \( \sigma_F \) such that \( \mathcal{G} \models \sigma_F \) iff \( \mathcal{G} \) satisfies condition 3. The algorithm states that \( \mathcal{F} \) is a \( p \)-morphic image of \( \mathcal{G} \) if and only if it finds a map satisfying the conditions. Therefore, this is a \( p \)-time algorithm checking whether \( \mathcal{F} \) is a \( p \)-morphic image of \( \mathcal{G} \).
\end{proof}

\begin{corollary}
Let \( L \) be a proper normal extension of \( \mathbf{S5}^2 \).
\end{corollary}

\begin{enumerate}
\item It can be checked in polynomial time in \(|U| \) whether a finite \( \mathbf{S5}^2 \)-frame \( \mathcal{G} = (U, S_1, S_2) \) is an \( L \)-frame.
\item The satisfiability problem for \( L \) is \( \text{NP-complete}. \)
\item The validity problem for \( L \) is \( \text{co-NP-complete}. \)
\end{enumerate}
Proof. 1. Follows directly from Theorem 3.2, Corollary 4.2, and the fact (shown in the proof of Theorem 3.16) that $M_L$ is finite.

2. It is a well known result of modal logic (see, e.g., [4, Lemma 6.35]) that if $L$ is a consistent normal modal logic having the poly-size model property, and the problem of whether a finite structure $A$ is an $L$-frame is decidable in time polynomial in the size of $A$, then the satisfiability problem of $L$ is NP-complete. The poly-size model property of every $L \supset S5^2$ is proven in [3, Corollary 9]. (1) implies that the problem $G \in F_L$ can be decided in polynomial time in the size of $G$. The result follows.

3. Follows directly from (2).

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