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COLLABORATIVE MATHEMATICAL INVESTIGATIONS WITH THE COMPUTER:
LEARNING MATERIALS AND TEACHER HELP

UNIVERSITEIT VAN AMSTERDAM
Graduate School of Teaching and Learning

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LEARNING MATERIALS AND TEACHER HELP

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Monique Helma José Pijls

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Monique Pijls
Amsterdam, May 2007

Chapter 1

INTRODUCTION

1. ORIGIN

1.1 Introduction

Since the introduction of reforms in upper-secondary education in the Netherlands, general skills have to be assessed in school examinations. Students are expected to regulate their own learning process. They have to be able to collaborate with peers, to carry out investigation tasks and to make use of information technology. Students must put these skills into practice in what are referred to as 'practical assignments' (*praktische opdrachten)*. Much current research in educational science focuses on teaching and assessing these skills. This study is about mathematics education. There too, investigation tasks are given attention. In our study we examine whether carrying out investigation tasks in the mathematics lesson can induce a better understanding of the mathematical topic students are learning. This is the central question of the present study. It builds on the research project: 'Information technology in the 'Studiehuis'' (Doorman, Kemme & Wijers, 1998), which consisted of pilot studies in which 16-year-old students in secondary education used a computer during investigation tasks. The aim of this project was to stimulate the use of the computer in mathematics lessons and to experiment with new developments in the field of self-regulated learning. Self-regulated learning means that students organize their own learning process, in the sense that they:

- take the initiative to learn something;
- make a plan;
- take the steps in the plan to get the desired result (Kemme & Wijers, 1996).

It appears that students in pre-university education and students in higher general secondary education (havo) with mathematics B really did carry out the investigation tasks, whereas the havo students with mathematics A did not do so. Apparently,

investigation tasks using the computer did not suit this group of students. In our project we developed investigation tasks especially for this latter group of students.
We also hope to ascertain whether executing investigation tasks may lead to these students having a better understanding of the mathematical topic. In other words, whether students transpose their informal knowledge of a domain into formal knowledge. We define this process as *mathematical level raising* (Van Hiele, 1986; Freudenthal, 1973; Elshout-Mohr & Dekker, 2000). We examine the conditions under which carrying out these investigation tasks by students in havo 4 mathematics A leads to a rise in mathematical level. Several aspects may influence the process of level raising, such as the learning materials, including the computer simulation that is used, collaboration with peers and the role of the teacher. Two conditions have our special attention: the moment in the learning process at which the tasks are done by the students, and the type of help the teacher provides. Practical assignments are usually given to students at the end of a chapter in a mathematical textbook after they have learned a certain subject. The function then is to apply the knowledge they have acquired. Although a well-founded basis of prior knowledge is a necessary condition for the success of the investigation tasks by students (see e.g. Tuovinen & Sweller, 1999), we are interested in whether students can learn from practical assignments if they do them before or during learning the mathematical theory. In the second experiment we focus on the role of the teacher. This study builds on research by Dekker and Elshout-Mohr (2004) with 17-year-old students in pre-university education. They found that students who worked with a teacher who exclusively supported the process of interaction between the students, learned more than students who worked under the guidance of a teacher who only provided mathematical hints. It is important to know whether this applies to students in havo 4 studying mathematics A.

In this section we explain what we mean by mathematical level raising, mathematical investigation tasks, collaborative learning, and the role of the computer.

1.2 Mathematical Level Raising

Mathematics A (applied mathematics) deals with the application of mathematical concepts in the field of economics, social sciences and medicine. In addition to a number of mathematical concepts, students also learn to have a critical attitude towards applying mathematics in practical situations. Students have to get used to mathematical language, to using formulae, get acquainted with several different kinds of graphical representations, learn to use mathematical models and learn to estimate their value (De Lange, 1987). It is, in fact, all about real situations in which students have to solve problems by applying mathematical concepts**.** In the different stages of their learning process, students do approach a certain problem in various different ways. Before they learn a concept they see it ‘as it is’ and do not recognize the properties of a mathematical concept in it. Once they have acquired understanding of a certain mathematical concept, students recognize aspects of that concept in a realistic problem and they can use these aspects to solve the problem. They therefore now treat the problem on a different level: level raising has taken place, from the perceptual to the conceptual level. Level raising is thus defined for a given mathematical concept. So students can be at the conceptual level for one concept and at the perceptual level for another concept. The term 'level raising' is based on, among other things, the theory of Van Hiele (1986) that was applied in the learning of geometry. He discerned the *visual level* at which students perceive structures in reality and the *descriptive level*, at which students talk about the properties of structures. In the transition from the first level to the second, students learn to approach the same problem or situation in a more abstract way. In our study, we use the general terms 'perceptual' and 'conceptual' level mentioned above.

1.3 Mathematical Investigation Tasks

Mathematics A deals with problems from reality which provide students with the opportunity to develop knowledge in the domain. Executing mathematical investigation tasks by students who study mathematics A can have several functions in the learning process. On the one hand it may facilitate the construction of knowledge, and on the other hand, it may give the students the opportunity to apply what they have learned.

The idea of investigation tasks is that students formulate a problem by themselves, make a plan to investigate this problem, execute their plan, formulate the results and finally reflect on their investigations. In the field of mathematics, problem solving and carrying out investigation tasks are closely related. This has to do with the fact that mathematics does not deal with physical experiments. In his classic work 'How to solve it' Polya (1945) discerns the following phases in the process of solving a mathematical problem: understanding the problem, making a plan, carrying out the plan and reflection. These phases resemble the different phases of the investigation tasks.

Not all the literature on investigation tasks mentions that students have to formulate the problem they will investigate themselves. In *discovery learning*, based on Bruner's ideas (Bruner, 1960), it is stressed that students get the opportunity to find out by themselves how things work (see Borich & Tombari, 1997). The teacher poses the problem or the learning goal. According to Bruner, knowledge is stored in the brain in a hierarchical way. At the top of the hierarchy are the more general, comprehensive ideas and lower down in the hierarchy are the more concrete, factual ideas. Students receive so much information that the brain must find a way to simplify it and give meaning to it. Categorizing, simplifying the ideas, and organizing are very important in this process. The brain organizes the incoming information in order to store it in the long-term memory. This information storage takes place on the basis of generalizations. The idea of discovery learning is that students discover these generalizations for themselves instead of the teacher imposing them on the students. In this way, the students better retain the concepts they learned, understand a certain topic better and are better able to apply their knowledge.

The transition from concrete, factual knowledge towards more general ideas is related to the concept of *level raising* in mathematics education. We recognize the transition from knowledge of specific situations towards more general knowledge that is applicable in more cases. According to Freudenthal (1973), it is important for level raising to give students the opportunity to explore at the visual level, the level of concrete, factual knowledge. By reflecting on what they have done, students will develop abstractions (models, formulae). This corresponds to Bruner's idea that students do have to discover generalizations for themselves. Freudenthal (1973) speaks of *reinvention*. The teacher has in mind what students will have to reinvent and by giving certain tasks or by posing certain questions, the teacher tries to get the process of invention going. This is why we speak of *guided reinvention*. Students do not have to formulate the problem by themselves. The historical development of a mathematical concept may serve as the inspiration for learning materials in accordance with the principles of guided reinvention. However, the aim is not for the students to re-experience the same process that occurred in history, but that they get the opportunity to experience a similar process as the process of invention of mathematical concepts. Learning materials for guided reinvention may also be inspired by students' solutions and strategies.

As we mentioned, mathematical investigation tasks (as the *practical assignments*) are often presented at the end of a chapter, so that the knowledge that has been constructed can be applied. Following the ideas of discovery learning as presented here, it could be instructive to have students do a investigation task about a certain subject either before they learn that concept or have the task mixed in with the learning materials. In our research we examine which leads to a greater understanding of the subject: executing the investigation tasks before, during or at the end of the chapter in the mathematical textbook.

1.4 Learning Mathematics Collaboratively

Working collaboratively in the mathematics lesson has been a point of interest for years. Especially since many students experience mathematics as a difficult subject and also since many mathematical problems can be solved in different ways that are all correct. Dekker and Elshout-Mohr (1998) developed a process model for working collaboratively on a mathematical task. They distinguished four key activities which may lead to mathematical level raising. These key activities are:

- to show one's work;
- to explain one's work;
- to justify one's work;
- to reconstruct one's work.

These key activities can be evoked by regulating activities such as asking students to show their work, asking for an explanation, and criticizing students' work. Giving explanations has a positive influence on one's learning process (Webb, 1991). Peers who ask for help or explanations do get this key activity going. Criticizing each other's work leads to justifying one's work.

Dekker (1991) mentions the following criteria for tasks which evoke interaction between students (see also Elshout-Mohr & Dekker, 2000).

1) The task is 'realistic'. This means that the task is accessible and meaningful for all students and not only for students with relevant knowledge of that particular domain. Novices must have as much access to the task as experts, although one might expect that their initial perception of relevant aspects of the problem will differ from the perception of experts.
2) The task aims at level raising. This means that the task is constructed and formulated in such a way that it will encourage students to discover aspects in which naïve knowledge is not sufficient. It needs completion or change.
3) The problem is complex. Different skills are needed to solve it and it is unlikely that one of the students will be able to solve it without consulting the others.
4) The problem asks for construction. This means that the students should make their thinking visible. In that way the differences between the students become visible and a subject for discussion.

In our research project we used these criteria to design our investigation tasks.

1.5 Learning Mathematics with the Help of the Computer

Using a computer in mathematics lessons is something that has increased over the past few decades. Using a computer only makes sense if students can use it to do something that they would not otherwise be able to do. This is the functional use of the computer. Examples include:

- Routine actions that would take a lot of time without a computer (calculations, drawing graphs) (Ruthven, 1996). The computer executes certain calculations quicker than humans. This makes it possible to do more calculations, to compare various problems and solution methods at the same time. This may lead to mathematical level raising.
- Testing hypotheses on a certain phenomenon (by using simulations) (see Doerr, 1997).

The 'Information Technology in the 'Studiehuis' Mathematics' project between 1995 and 1997 experimented with investigation tasks in which the computer was used in a functional way (Kemme & Wijers, 1996). As described in section 1.1 it appeared that investigation tasks with the computer were particularly effective with respect to the learning of mathematics for students in pre-university education and havo mathematics B. Students in havo mathematics A often executed the task only partially, so what they learned from it was questionable. Also in the PRENT continuation project, in which five schools experimented with *practical assignments* with (and without) a computer, it became clear that investigation tasks with the computer take less root (i.e. that students do really make investigations self-reliantly) in havo 4 mathematics A, than in other groups (PRINT, 1998; PRENT, 1999). Teachers gave the openness of the tasks as a possible reason for this. Students asked the teachers continuously 'what they had to do'. It is questionable whether this openness is really the most important factor, since there is an example of an open task (without a computer) that did work well for these students. Apparently, the computer program must adjust too.

What is interesting in this case is what Sweller and Chandler (1994) say about cognitive load in working with complex tasks with the help of several different kinds of learning materials (for instance a computer and a textbook at the same time). In this case the cognitive load may become too high. This applies in particular in situations in which the task is in itself already difficult for students. The cognitive load that is imposed by using different learning materials can, in many cases, be diminished. In some experiments, they make clear that the cognitive load that arises when students have to go back and forth between their textbook with instructions and the monitor can be diminished by showing a screen dump in their textbook and the text, so that students only have to consult their textbook.

2. THE PRESENT RESEARCH PROJECT

2.1 PROO Special Research Area: Math and ICT

The present study belongs to a set of five research projects which started in 1998 entitled 'Information Technology as a means for self-reliant learning of mathematics', funded by the Dutch Organization for Scientific Research (NWO). Dirk Hoek studied the use of a graphics calculator in MBO. He focused on how group work and the coaching role of the teacher led to the students being more actively involved in learning mathematics. The teacher stimulated the use of a graphics calculator in interactions and led whole class discussions (Hoek & Seegers, 2005). Paul Drijvers investigated the learning of algebra for 15-year-old students with the help of computer algebra (Drijvers, 2003). The project of Arthur Bakker was centred on 12-year-old students learning statistics with the help of Java applets in which they could visualize and manipulate various statistical concepts (median, mean, etc.) (Bakker, 2004). Michiel Doorman studied how 17-year-old students reinvented the concept of motion with the help of applets (Doorman, 2005). In this study we focus on collaborative investigation tasks with the computer for students in havo 4 mathematics A.

2.2 Aim of the Project

Our aim in this project was to investigate if and how collaborative investigation tasks using a computer can foster mathematical level raising for students in havo 4 mathematics A.

The first research question relates to the learning materials: what is the best moment in the learning process to work on the investigation tasks: before or after students work on the exercises in their regular textbook? Or should the investigation tasks be mixed in with their usual mathematics tasks? The second research question focuses on the help of the teacher. Should the teacher give mathematical hints in order to help students one step further when they are stuck? Or should the teacher stay away from the mathematical content of the tasks and create opportunities for students to think for themselves and to learn from sharing their ideas and criticizing and discussing their ideas with each other? In the next section we describe how we attempted to answer these questions with the help of two experiments in the educational field.

3. COURSE OF THE PROJECT

We now present an overview of the project and indicate the chapter in which a particular study is described.

3.1 Pilot Study

In the spring of 1999 a pilot study was carried out in which investigation tasks on the subject of 'Routes and Probabilities' was tested with a number of students. The aim was to investigate whether these tasks did encourage these students to learn certain concepts in probability theory. Chapter 2 presents this pilot study and provides the answers to the questions we posed about the learning materials.

3.2 The First Experiment

Based on the results of the pilot study, learning materials and a computer simulation were developed. In January-February 2001, a field study was undertaken in a school in which three versions of the learning materials were compared in which investigation tasks were presented before, after and mixed in with the learning materials. Students written materials were collected and audio recordings made. A pre- and post-test was given to the students. The results were analysed and are reported on in chapter 3. The audiotapes of two students were transcribed and the learning process is analyzed, as described in chapter 4.

3.3 The Second Experiment

Based on the results of the first experiment and a workshop with experts at the International Conference for the Psychology of Mathematics Education (see Doorman & Pijls, 2001), the learning materials were further developed. A field experiment with the new materials was conducted in January 2004 in which two different types of teacher help were compared (process help and product help). Students' written materials were collected, pre- and post-tests were carried out and audio recordings made. The results are analysed and described in chapter 5. The audiotapes of two students were transcribed and analyzed, as reported in chapter 6.

3.4 Discussion of the Project

The results of the project are described and evaluated in the chapter 7. The theoretical and educational implications are discussed and attention is given to the role of group size, types of teacher help, the learning materials and the key question whether students in havo 4 mathematics A can truly benefit from making investigations in their learning process.

Chapter 2

MATHEMATICAL INVESTIGATIONS USING A COMPUTER[1]

We conducted a pilot study to test whether two investigation tasks based on the gambling game 'Plinko' (with the underlying structure of a grid) could be used to stimulate mathematical level raising in the subject of 'Routes and Probabilities'. The tasks were conducted by two pairs of students at a secondary school. It appeared that one of the tasks was approached on different levels: one pair approached the task at the perceptual level and the other pair at the conceptual level. However, the task did not stimulate mathematical level raising. This is why we decided, in the continuation of the research, to design a computer simulation in which students could experiment with the different aspects of a grid. In this way students at the perceptual level could make a start with mathematical level raising.

1. INTRODUCTION

The curriculum at the upper secondary school level in the Netherlands is currently undergoing considerable reform. The main focus in this reform is on independent and self-regulated learning. The new curriculum includes mathematical investigation tasks using a computer, and the tasks have been developed for students to work together in small groups. They have to explore a real world or a theoretical problem and find their own solution. The computer is used for calculation and computation purposes and therefore serves as a tool for solving mathematical problems. The question is whether these tasks really do improve mathematics learning, and whether they work for all students. What we mean here by work is that they lead to *level raising* (Elshout-Mohr & Dekker, 2000). This phrase is used to indicate the transition from thinking processes on an intuitive level towards reasoning on a more abstract level. For example, students have an intuitive notion of 'chance' even before they have learned anything about probability theory in the classroom. They think and talk about chances at a *perceptual level* (cf. the visual Van Hiele level in geometry (van Hiele, 1986; Dekker, Elshout-Mohr & Pijls, 1998)). In mathematics lessons students learn to reason about chances and to calculate them, often with the help of

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[1] This chapter is based on:
Pijls, M. (2000). Mathematical investigations with the computer. In A. Ahmed, H. Williams, & J.M. Kraemer (Eds.), *Cultural diversity in mathematics (education): CIEAEM 51*. (pp. 361-367). Chichester, Great Britain: Horwood Publishing.

combinatorics. They learn to deal with chances at a *conceptual level* (cf. the descriptive Van Hiele level (van Hiele, 1986; Dekker, Elshout-Mohr & Pijls, 1998)). The didactical means that we use to achieve this level raising are the computer, working together in twos, and tasks for mathematical investigation.

Using computers in mathematics is something that has become more popular over the past few years. Using computers makes sense only if students can use a computer to do something they could not otherwise do. This is what we refer to as the functional use of computers (Doorman, Kemme & Wijers, 1998). Examples:

- To support routine calculus (calculations, graphing) (Ruthven, 1996). The computer performs specific calculations much more quickly than humans. This makes it possible to make more calculations and to compare different solution methods, which could lead to level raising.
- To test hypotheses on a phenomenon by using simulations (Doerr, 1997; de Jong & van Joolingen, 1998).

Cooperative learning has been a focus in the mathematics classroom over the past few decades (Webb, 1991). Dekker and Elshout-Mohr (1998) developed a process model for students working together on a mathematical task. It involves four key activities that lead to level raising. These key activities are: to show one's work, to explain one's work, to justify one's work, and to reconstruct one's work.

Dekker (1994) presented four criteria for tasks that elicit interaction in group work. Firstly, tasks must be realistic and meaningful for students. Secondly, they must be complex so that students with different abilities and qualities have a need for each other. Thirdly, the task must be to construct something so that the thinking processes can become visible. Finally, the aim of the tasks must be to raise levels. They must be difficult enough for students to need each other's help, criticism and explanations.

The aim of our investigation tasks is firstly for students to get the opportunity to explore a subject at the perceptual level, and secondly for them to develop concepts to deal with the problem at the conceptual level. When learning mathematics, problem solving and conducting investigations are very closely related. The reason for this is that no physical experiments are actually carried out in mathematics. In his classic work 'How to solve it' (1945), Polya distinguishes several phases in solving a mathematical problem: understanding the problem, making a plan, carrying out the plan, and looking back. These phases are akin to the phases of conducting investigations.

2. THE INVESTIGATION TASK 'PLINKO'

We developed an investigation task based on the gambling game 'Plinko', which is popular in the United States and can be found on the Internet to be played on the computer. The player clicks on one of the five 'go-boxes' and then a 'ball drops down'. The ball has the same chance of dropping to the left or to the right from every box. At the bottom it arrives in a box where a number of points is indicated. The aim of the game is to gain as many points as possible. To make this more exciting, we changed the ‘40 point box’ to a ‘1000 point box’. The task is to find out whether there is a strategy to win this game. In order to do this, probability theory and combinatorics are needed. Counting the possible paths that the ball can take is an application of ‘counting in a grid’. The second task is to develop a new, exciting form of Plinko. Students are then allowed to change the point boxes, the number of points and the shape of the grid. We hope that students will create games with other grids, so that they explore the phenomena ‘counting in a grid’ in different grids, which may lead to an understanding of ‘counting in a grid’ at the descriptive level. We conducted a try-out with the investigation tasks on Plinko in April 1999 in order to answer the following questions:

- Do the students carry out the investigation tasks?
- Is the aim of the tasks clear to the students?
- Do students cooperate when they work on these tasks?
- Do students make functional use of the computer?
- Do these tasks elicit exploration at the perceptual level?
- Do these tasks produce answers at the conceptual level?

The subjects were four students from a senior general secondary education school in Amsterdam. The students worked in twos: Anouk & Michel, and Joost & Brigitte. Earlier in the school year they had followed lessons on probability theory, which they worked on for two 45 minute lessons.

During the first lesson the students worked on the task about the winning strategy for Plinko. The couples showed striking similarities and differences in their working on the task. The similarities are as follows:

- The task was meaningful for the students: we concluded from their comments that they often played computer games; they started to work straightaway, and the aim of the task was clear.
- The students played the game several times and stated the number of points they win; the task requires an element of experimentation.
- They soon assumed that the balls from the boxes on the left have more chance of dropping into the 1000 slot than balls from the boxes on the right.
- Both couples found a winning strategy.

The differences between the couples were:

- Anouk and Michel compared two strategies by playing them several times and writing down the number of points they won. One of the strategies was to click alternating on one of the two boxes on the left. Each time, ten goes resulted in an average of 2448 points. The other strategy (clicking alternating on one of the three boxes on the right) resulted in 694 points on average.
- Anouk and Michel compared the chances of falling into the 1000 slot from the left hand and the right hand boxes. Anouk tried to draw all possible routes from the left box. She drew several of them, and it became rather cluttered (figure 1). Michel stated that there were only two routes from the box on the right that lead to the 1000 slot (figure 2). They wrote: ‘Between the left go and the right go, the left go has more possibilities to drop into the 1000 slot because the box on

the right has two possibilities and the box on the left more than six. The second go has more possibilities, but not more possibilities to go to the right or the left.'

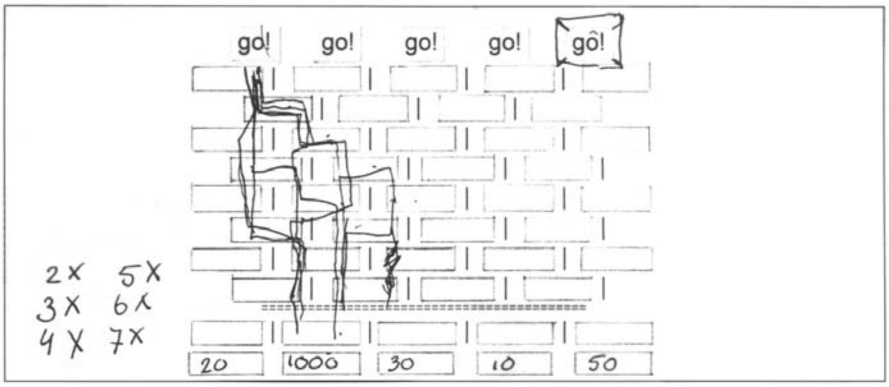


Figure 1. Anouk's answer.

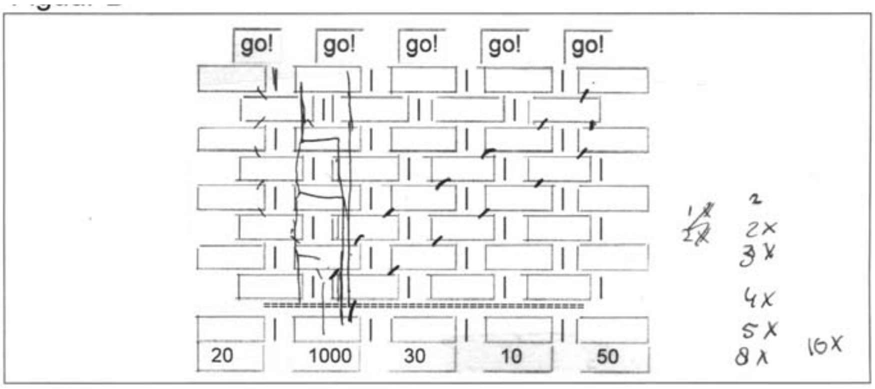


Figure 2. Michel's answer.

For the three boxes on the left, Joost and Brigitte calculated the chance of reaching the 1000 slot. They wrote: 'The left 'go' boxes have the greatest chance of reaching 1000, i.e. to get the highest score. The two 'go' boxes on the right have the least chance. Because of the chance to go to the left every time is smaller than to go to the left or to the right. And for the box that is furthest left, you must go to the right at the beginning, so the chance of dropping into the 1000 slot is greater. The box in the middle has a chance but not so great as the first two. We have calculated how much chance the first three 'go' boxes have to reach the 1000 slot. First box: 56/128; second box: 49/128 and third box: 32/128 chance.' In figure 3 we see one of the three calculations made by Joost and Brigitte.

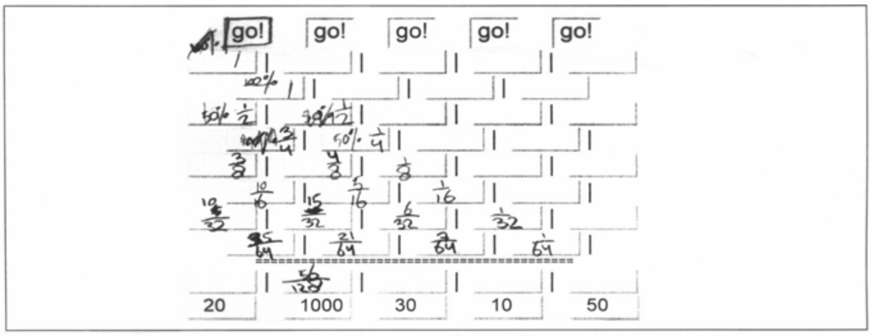


Figure 3. Answer given by Joost and Brigitte.

The second task was performed without using a computer. The students worked together in the same couples as on the first task. Here we list the main results of the second lesson:

- The task was meaningful for both couples. They were familiar with the idea of several versions of one game.
- They worked on a design on their own, and not together on one design. When their design was finished, they showed each other their results, explained their games, defended them and reconstructed them.
- This task was done without a computer, but it referred to a game on the computer. Joost mentioned that he would like to simulate his game on the computer to test it.
- Both couples introduced ‘special effects’: Anouk & Michel introduced a *loser* (a box where you lose all your points), a *bonus* (a box where you get another ball) and a *bounce* (the ball goes back to its starting position and then takes a different route). Joost & Brigitte introduced *black boxes* (into which the ball cannot drop) that change position with every game, and they give *additional points* at the starting position (the further to the right you start, the more points you get). The students probably introduced these effects to make the game more exciting. However, the consequence was that they could escape the original grid structure and hence they did not gain experience with the subject of ‘counting in a grid’.

3. ANALYSIS OF THE RESULTS

We analyze the results of the try-out in the light of the questions we formulated above.

Do the students carry out the investigation tasks? Is the aim of the tasks clear for the students?
Yes! They carry them out immediately. The tasks are meaningful for them and they are linked to their reality. The aim of the first task, to find a winning strategy, is clear. The students perform mathematical investigations. They explore the problem, formulate hypotheses which they try to prove by calculations. The methods of proving differ, but it is clear for the students that this is the aim. The aim in the second task is less clear. The students have invented creative games, but they did not follow the rules exactly. This was possible because there was no medium (like the computer) that forced them to follow the construction rules. Moreover, there seems to be less opportunity to test hypotheses because there was no computer.

Do students cooperate when they work on these tasks?
Each couple shows a high degree of cooperation on the first task in particular. The following key activities were observed:

- *To show one's work*: In the first task they try several strategies and demonstrate them to each other. This is a very natural activity when two people work on one computer. They also show each other possible routes.
- *To explain one's work:* In the first task Anouk & Michel explain the possible routes they found to each other.
- *To justify one's work:* In the first task, when Joost & Brigitte make calculations, they criticize each other, and justify their work.
- *To reconstruct one's work*: In the first task, when Joost & Brigitte make calculations, they criticize each other, and they reconstruct their work.

We also see that the criteria for tasks that evoke interaction from Dekker (1994) do indeed evoke interaction: the task is real and meaningful for the students, it is a complex task requiring different skills (to play the game, to count the points, to analyze the strategy, to reason about the possibilities), the task is to construct something (a strategy), the task aims at level raising (gaining an insight into the game).

Do students make functional use of the computer?
This only involves the first task. Both couples use the computer for their explorations at the perceptual level. This is a functional use of the computer: the computer simulates the game of Plinko and calculates the points. In addition, the computer would also seem to be particularly motivating because it is something new for the students: it is exciting for them to play a game on the Internet. They wanted to use a computer for the second task, but it was not allowed.

Does the task evoke exploration at the perceptual level?
Concerning the first task: both couples first play the game several times. They explore the game on the perceptual level by trying out different strategies. Secondly, they then compare the strategies. This is the start of level raising. They formulate conjectures on the basis of the differences in outcomes ('the boxes on the left result in more points than the ones on the right'). Anouk and Michel give solutions at the perceptual level. They have drawn some possible routes to 1000. They did not draw all possible routes. If they had systematically drawn all the possible routes, then they would have come up with a solution at the conceptual level. In the second task all the answers were at the perceptual level because they involved specific situations, and not general strategies.

Does the task evoke answers at the conceptual level?
Concerning the first task: Brigitte and Joost give a solution at the conceptual level. They calculate from all the top-boxes the chance of reaching the different boxes at the bottom. In this way they describe all possibilities that could happen in this game, although they do not calculate this in the most efficient way. They calculate with chances; it is easier to first count possibilities and then to divide to obtain the chances. This could be a refining of their answer at the conceptual level. In the second task there were no answers at the conceptual level. Each couple gave an example of a specific game. Their creativity made them invent such complex games that none of them had an airtight strategy for their own game.

4. DISCUSSION

Generally speaking, we have seen that the investigation tasks using a computer on Plinko worked well. It is therefore possible to continue with these tasks. However, the second task (designing a new version) could be slightly changed by doing it with the Plinko game on the computer. The task could be to design a new version of Plinko within the boundaries of the old version (it is possible to change the number of points). This may make this task more explicit from a mathematical point of view and give students the opportunity to test their designs and to improve on them. What students learned from this task with respect to mathematics is not measured. We do not know which couple learned the most: the couple that gave an answer at the perceptual level or the couple that gave an answer at the conceptual level.

In our try-out the students conducted the investigation task *after* they had learned about probability theory. It would be interesting to see how students do this task *before* they have learned about probability theory. We may expect that when students do these tasks before they have learned anything about probability theory, that their reasoning will tend to be intuitive and that they give answers at the perceptual level. The task may lead to questions such as: 'We have the feeling that our strategy works, but is this always true, how can we decide that?'. In this case the task would contribute to the understanding of probability theory. A third possibility is to make students learn about probability theory with the help of investigation tasks *during* their learning from the regular textbook chapter. If students work with mathematical investigation tasks with the computer alternated with regular tasks from their mathematical textbook, they may fully benefit from their experiences with the computer simulation for building up the concept of counting routes. We hope to develop these ideas in further research.

Chapter 3

MATHEMATICAL LEVEL RAISING THROUGH COLLABORATIVE INVESTIGATIONS WITH THE COMPUTER[1]

Investigations with the computer can have different functions in the mathematical learning process, such as to let students explore a subject domain, to guide the process of reinvention, or to give them the opportunity to apply what they have learned. Which function has most effect on mathematical level raising? We investigated that question in the context of developing learning materials for 16-year-old students in the domain of probability theory, consisting of computer simulations based on a gambling game and investigation tasks about these games. We compared the difference in level raising between three versions of the learning materials: investigations with the computer *before*, *during* or *after* the learning of a mathematical concept. It was shown that there was no significant difference in the final mathematical level that students attained in the three conditions (the product). However, there were differences in the level on which students approached the investigation tasks (the process). Furthermore, we found evidence of new categories in the students' answers, lying between the perceptual and conceptual levels, which may give important insight into the process of level raising.

1. INTRODUCTION

In the past decade, technological developments have lead to some important new tools for mathematics education. The Dutch research program 'Mathematics and ICT' has investigated possibilities to use these tools in the learning of mathematics by means of guided reinvention, with the aim of mathematical level raising. In one of the projects a computer game was developed (Pijls, 2000, 2001; Pijls, Dekker et al., 2000, 2001) which laid the foundation for several simulations that have been used to teach the subject of binomially- distributed probabilities and counting routes (see description below) to 16-year-old students of higher prevocational education. Earlier research made clear that the students in this type of education were not al-

[1] Pijls, M., Dekker, R., & Van Hout–Wolters, B. (2003). Mathematical level raising through collaborative investigations with the computer. *International Journal of Computers for Mathematical Learning, 8,* 191–213.

ways motivated to perform investigations with the help of the computer (PRINT, 1998; PRENT, 1999). We ascribed this to the fact that many computer simulations on probability theory were rather abstract. That is why we tried to develop simulations in a way that is very accessible for students, offering them many opportunities to explore the subject on a concrete level. In a field experiment, we wanted to investigate how students can learn most from working with the simulations, that is: how can they attain level raising in the domain of binomially distributed probabilities? In this paper we compare three versions of the learning materials and find out which version enhanced level raising the most.

We will look at both the product (have students attained the conceptual level at the end of the experimental lessons?) and the process (how do students attain level raising during the lessons?). This study is part of a developmental project and the results are being used to improve the learning materials for future research and development.

2. MATHEMATICAL LEVEL RAISING

When using the term 'level raising' we refer to the level theory of Van Hiele (1986). Van Hiele distinguishes several levels in the learning of a mathematical concept and gives examples of it in the field of geometry. On the first level students deal with mathematical objects in their physical appearance, such as a rhombus. On the second level students deal with properties of the objects such as the fact that it has four equal sides. In our research we define level raising as follows. In the different stages of their learning, students approach a problem in various ways. Before they learn a concept, they approach it at a perceptual level. They see it 'as it is' and do not recognize properties of a mathematical concept in it. After they have acquired a certain mathematical concept, students will recognize aspects of that concept in a realistic problem and they can use these to solve the problem. Thus, they now treat the problem at another level: level raising has taken place, from the perceptual to the conceptual level. Level raising is thus defined for a given concept.

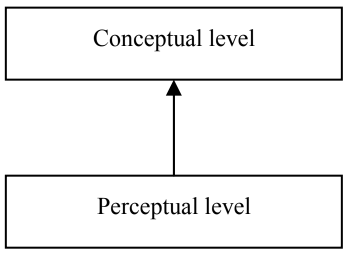


Figure 1. Mathematical level raising.

Level raising, as interpreted above, can be linked to the idea of *conceptual change* (Pressley and McCormick, 1995). Both level raising and conceptual change deal with the acquisition of a certain concept. The difference, however, is that with conceptual change it is supposed that students start with misconceptions, which have to 'be transformed' into correct scientific concepts, whereas with level raising there is not necessarily a misconception at the perceptual level. For instance, regarding the concept 'probability' (defined as 'favorable amount of probabilities/total amount of probabilities'), at the perceptual level students can correctly estimate probabilities without using numbers, so this estimation is not necessarily a 'misconception'. On the other hand, in some cases mathematical level raising does deal with misconceptions at the perceptual level, so level raising may involve conceptual change.

3. COLLABORATIVE INVESTIGATION TASKS WITH THE COMPUTER

A number of studies have shown that the interplay between building knowledge and carrying out investigations by students is twofold. Studies in the field of discovery learning show that discovery activity only makes sense if students have a sufficiently extensive knowledge base at their disposal (see for instance Tuovinen and Sweller, 1998). On the other hand, other studies have shown that carrying out investigations can be useful for students to build up their knowledge base. For example, Freudenthal speaks of explorations at the visual (which we call perceptual) level as an essential aspect of the process of reinvention; and reflection on one's own activities at the perceptual level may evoke level raising (Freudenthal, 1973). The assumption that carrying out investigations contributes to the building of knowledge is in favor of using investigation tasks *before* or *during* the learning of a mathematical concept, whereas the idea that students have to know enough about a certain subject before they can profit from investigation tasks argues for performing such tasks only *after* learning a concept. In the first two cases, the use of a computer simulation could allow students to make explorations at the perceptual level, in the third case it would enable them to apply and restructure what they learned at the conceptual level. This last case was the most commonly used in an earlier research project, on which the current project is based (PRINT, 1998; PRENT, 1999).

Not only the use of investigation tasks with the computer may lead students to reflection, working together with peers may contribute to this as well (see for example, Dekker, 1991; Webb, 1991). It has been shown that giving explanations about one's own work has a positive influence on learning. The same holds for showing, justifying and reconstructing one's own work. All those activities are evidently related to reflection, either because they give rise to reflection (showing one's work) or because they are the result of it (justifying and reconstructing). Dekker and Elshout- Mohr (1998) developed a process model in which they describe these key activities for two students who are working on the same mathematical task but on different levels. Regulating activities (activities that evoke key activities, such as to ask to show one's work, to ask for explanation, to criticize), and mental activities

(activities that are assumed to take place 'in the heads of the students', such as to become conscious of one's own work, to become conscious of the work of the other) are also mentioned in the model (Dekker and Elshout-Mohr, 1998).

4. RESEARCH QUESTIONS

In the previous section, we mentioned that in current classroom practice, students frequently carry out investigation tasks with the computer *after* they have learned a certain concept. However, educational theories suggest that carrying out investigations with the computer *before* or *during* the learning of a mathematical concept could be instructive too. Therefore, we carried out a field experiment to find out which of the three approaches enhances mathematical level raising the most. The three functions of collaborative investigation tasks with the computer are presented in the three conditions chosen for the experiment. In the condition BEFORE, students perform investigation tasks with the computer before studying a chapter from a regular textbook. In the condition DURING, they perform the same investigation tasks with the computer, and a set of paper-andpencil tasks connected with the computer simulation instead of studying the regular textbook. Furthermore, in the DURING condition the students have to 'reinvent' the theory, that is, there are no 'rules' given in the teaching materials. Instead, there are exercises in which students have to formulate and summarize what they have discovered. In order to reduce the cognitive load while students are working with both the computer simulation and written materials (see Sweller and Chandler, 1994), the exercises are formulated as much as possible in the context of the games of the simulation, using screenshots of the software. In the condition AFTER students perform the investigation tasks with the computer after having studied the tasks using their regular textbook. Figure 2 gives an overview of the lessons in the three conditions.

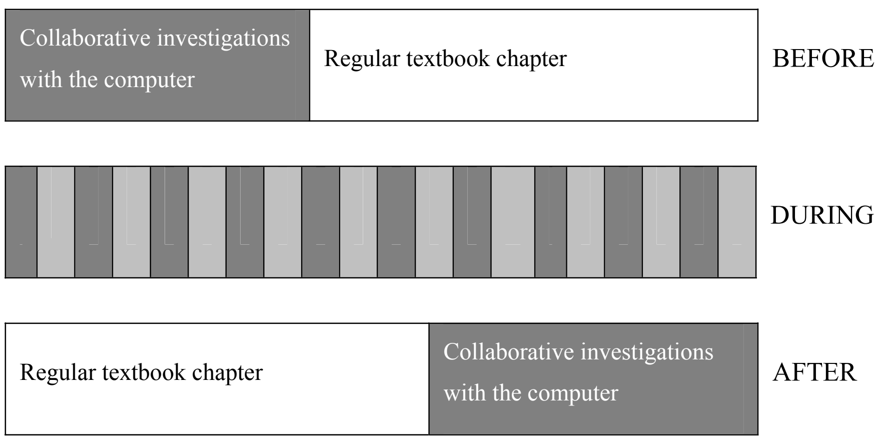


Figure 2. The lessons in the three conditions.

The research questions concern firstly the mathematical levels attained by the students, and secondly the process of mathematical level raising in the three conditions:

(1) Are there differences in attained mathematical level between the following three conditions?

a. BEFORE students work on collaborative investigation tasks with the computer before their regular textbook chapter;
b. DURING students work on collaborative investigation tasks with the computer without a textbook, but with textbook-like questions in the context of the computer simulations;
c. AFTER students work on collaborative investigation tasks with the computer after their regular textbook chapter.

(2) Do students in the different conditions approach the collaborative investigation tasks with the computer in the following ways?

a. do students in BEFORE approach the tasks at the perceptual level?
b. do students in DURING approach the tasks initially at the perceptual level, and at the end at the conceptual level?
c. do students in AFTER approach the tasks at the conceptual level?

Concerning question (1) we had the following expectations. The BEFORE students will be able to explore the domain at the perceptual level with the computer simulations. This may lead to level raising if they are able to link these experiences to the theory they learn in the textbook chapter. We expect the DURING students to profit the most from the possibilities of the learning materials since the investigation tasks are highly embedded in the learning process. We expect the AFTER students to apply what they have learned in the investigation tasks. If they did not yet reach the conceptual level, working with the investigation tasks might help them to achieve level raising. Question (1) will be answered by analyzing the results of a posttest on the topic of binomially distributed probabilities.

For the second question we expect the students to approach the tasks as expressed in (2a), (2b) and (2c). This question will be answered by analyzing the students' answers in the investigation tasks.

5. METHOD

The experiment took place at a Montessori school in Amsterdam, with three classes (67 students) of 16-year-old students of higher pre-vocational education. The students in this school were used to working self-reliantly, while the teachers had a coaching role. The teachers attended the experimental lessons too, giving minimal help when students asked for it. The researcher carried out a major part of the teacher's role. This was because the learning materials were still in a developmental stage and the researcher knew most about them. The experiment consisted of 10 lessons, each lasting 45 minutes.

One month before the lessons started, the students completed in 90 minutes two pre-tests: pre-test 1 on prior knowledge of probability theory, and pretest 2 on the topic of binomially distributed probabilities and counting routes. Pretest 1 was a test of topics covered in preceding lessons on combinatorics and probability theory, and it consisted of 10 items. The researcher and the teacher constructed it and the reliability proved to be sufficient (Cronbach's alpha = 0.65). Pretest 2 consisted of 11 items on binomially distributed probabilities and counting routes. Those items were comparable to the items of the post-test. The reliability of this pre-test was not sufficient (Cronbach's alpha = 0.57), probably because the students could not yet answer the questions of this test. That is why we have left it out of our analysis.

After the 10 lessons of the instructional sequence the students completed a 45 minute post-test. It consisted of 13 items, 11 of which were very much comparable with pre-test 2. The items were constructed in such a way that they could not be answered correctly unless students had reached the conceptual level. The reliability was sufficient (Cronbach's alpha = 0.70) and the inter coder reliability was 95%.

Each class was divided into three condition groups that were equal in results on pre-test 1, by listing the students from low to high results and assigning them one by one to a different group. In each condition group the students could choose semi-heterogeneous pairs. This was organized as follows: each condition group was divided into two groups, A and B, on the basis of pre-test 1 (on prior knowledge). Group A consisted of students with a high or low score and group B of students with an average score. Then students could choose pairs themselves, so long as an A-student worked together with a B-student.

The three condition groups of each class worked together in a large room with computers. Each group had an own tester that carried out the teacher role, i.e., starting the lessons with a short introduction, handing out the learning materials and giving minimal help if the students asked for it. The regular class teacher also moved among the three groups to give minimal help. The students worked in pairs on the investigation tasks with the computer: they shared one computer and one task booklet and produced one answer. The tasks from the traditional textbook were done individually, that is, each student produced his or her own answer although they discussed their answers with each other. The BEFORE students first had four lessons on investigation tasks and then six lessons on their regular textbook chapter.The DURING students had ten lessons with the experimental learning materials. The AFTER students started with six lessons on their textbook chapter and then four lessons on investigation tasks. We collected from all students their (written) answers from the learning materials, both answers from the investigation tasks (made in pairs) and answers to the textbook questions (made individually). Furthermore, we collected for all students log files created while they were working with the computer. Audio recordings of nine pairs (three per condition, one from each class) were made during all ten lessons. For these recordings, we chose pairs that contained at least one student that had made extensive elaborations (drawings, diagrams) in pre-test 1, because we expected that their written products might give rise to discussion by the pair. The log files and audiotapes were used as background information in case the answers of the students were unclear, and to get more information about the level of answer of the students.

6. THE LEARNING MATERIALS

6.1 The Topic of Routes and Probabilities

In this section, we give an outline of the topic 'Routes and Probabilities', for which we developed the learning materials. This is part of the Dutch pre-examination curriculum for 16 year-old students of higher prevocational education, which mainly involves study of applied mathematics.

The central topic is the calculation of binomially distributed probabilities, as in example 1. The students are taught to do this with the help of counting routes in a grid, according to a given procedure.

Example 1

In the grid of Figure 3 you have a probability of 45% to go downwards and a probability of 55% to go upwards. You start in point A. What is the probability that you end up in point B?

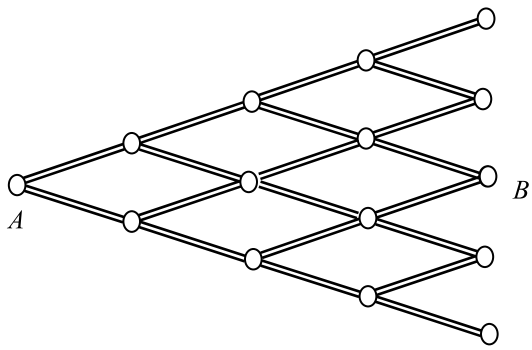


Figure 3. Grid.

One can answer this question by starting in point A with 100% and dividing this at each point. Another possibility is to count all possible routes from point A to point B and multiply this by the probability of a route from A to B. We will call the second procedure 'counting routes in a grid'. Students are taught to do this with help of the procedure 'adding in end points', as shown in Figure 4.

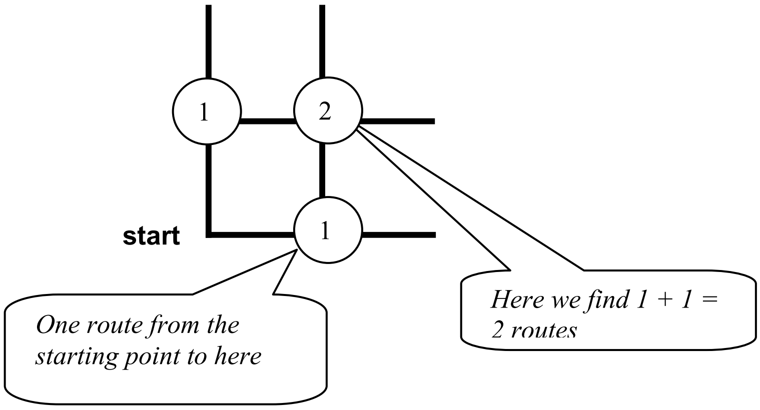


Figure 4. The procedure 'adding in end points'.

Repeating this procedure leads to the structure of Pascal's triangle. However, students are not taught the formula for binomial coefficients, so they have to keep in mind the counting procedure and/or the numbers in Pascal's triangle. Moreover, they have to be able to apply the counting of routes and the calculation of binomially distributed probabilities for solving word problems as in example 2.

Example 2

We have a box with three red and four white balls. How many patterns of colors can you get by putting these balls in a row?

This is an example of a 'counting word problem'. Using their prior knowledge, students may try to draw a 'probability tree' to solve this problem, or try to systematically write down all strings having three R's and four W's, like RRRWWWW, RRWRWWW. However, this is a rather complex problem and the two ways of solving it just described are difficult to carry out correctly. That is why students are taught to solve this problem another way, by drawing a three-by-four grid and counting all possible routes from one angle to the opposite angle as in Figure 5, or to solve it by using Pascal's triangle.

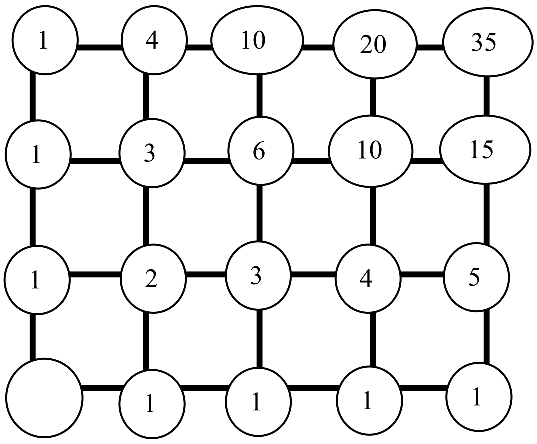


Figure 5. Counting routes from one angle to the opposite angle in a three-by-four grid.

The regular Dutch textbooks deal with this subject in the following order. Students have prior knowledge about systematic counting and probability trees. The chapter 'Routes and Probabilities' of 'Moderne wiskunde' [Modern mathematics] (Boer et al., 1998), the mathematics textbook that was used in our experiment, starts with the subject of counting routes in a grid, followed by the use of counting routes to solve 'counting word problems' and Pascal's triangle. After that, follows the calculation of probabilities in a grid, like in example 1, and the use of calculating probabilities in a grid to solve 'probability word problems'. As we can see, the subjects covered by this chapter become increasingly complex.

For students, difficulties arise at several points. First, it can be difficult for them to calculate routes in a grid with a special shape, for instance a grid with a 'hole' in it. Second, they find it difficult to see that Pascal's triangle and the grid are in fact the same, in other words, that it is the structure of the numbers that counts and not the orientation. Third, the calculation of probabilities in a grid is problematic for them.

6.2 The Software

The software 'WhooPY Trainer' (Pijls, 2001) consists of several games that have a game board with a grid structure. Games were chosen as the subject of the software because students of this age are used to playing many computer games. We expected that counting routes in a grid will become more meaningful for them when they play games with this same structure. One of the games, TIC-TAC, is shown in Figure 6.

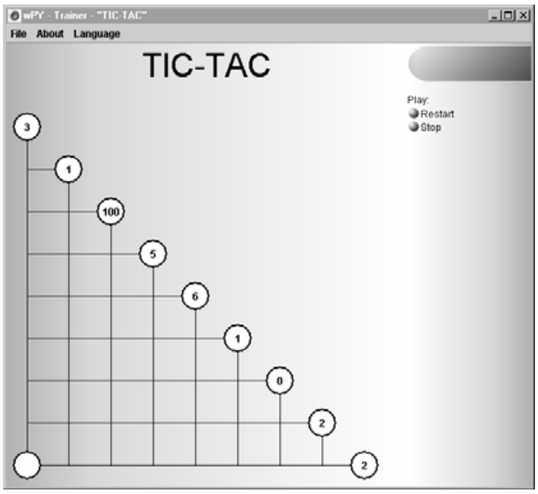


Figure 6. Computer screen of the game TIC-TAC.

The game is played as follows: at the bottom left of the game board one can start a little ball moving. This ball 'chooses' a route to one of the nine boxes by going upwards or to the right. One then earns the number of points indicated on the box the ball gets into. The probability to go to the right and the probability to go upwards are both 50%. The route of a certain ball is shown step by step. In every round, a player can play five balls.

The essential ideas behind this game are

1) to focus on the relationship between the *routes* to a certain box and the *probability* of getting there;
2) to focus on the fact that every route towards a certain box consists of the same number of steps upwards and the same number of steps to the right.

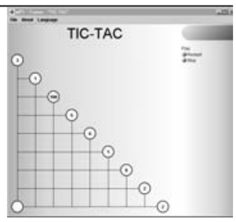
We explicitly did not add the possibility to let the computer count the balls ending up in a certain box in this part of the learning material. The reason for this is that we want to give students the opportunity to problematize the counting of routes and to develop their own strategies. The software provides opportunities for teachers and researchers to design other games, that vary in orientation (i.e., does a ball go from left to right or from the top downwards), lay-out (a grid with lines or boxes at end points), different shapes (so called 'grids with holes') and probabilities (in the game TIC-TAC the probability is 50–50, but it can be asymmetric too). Furthermore one can change the number of points in the boxes, the number of balls per round and whether play proceed by rounds, or just one or 100 balls at a time. Teachers can also use the software to design an environment for students, in which they can determine which variables of the game the students can manipulate. Thus, students can design their own games as well as playing the games provided.

6.3 Investigation Tasks

In Figure 7 we give an overview of the collaborative investigation tasks that we developed to be used with the software. There are five tasks that concern the analysis and playing of a game, and in the final investigation task students have to design a new game themselves. The aim of the first task is to raise the question of counting routes. The aim of the second task is to make clear that this game has the same underlying structure; it is made smaller so that students have the opportunity to develop a counting strategy. The third task lets students explore so-called 'grids with a hole'. The fourth task lets them experiment with probabilities. The idea is that in order to answer the question students should calculate all probabilities. The aim of the fifth task is to let students experience an asymmetric distribution. In the last investigation task students can apply what they have learned about grid structures, routes, and probabilities.

6.4 The Experimental DURING Tasks

In the conditions BEFORE and AFTER students use the mathematics textbook and the collaborative investigation tasks with the computer. In the experimental DURING condition, the investigation tasks take about ten lessons. Between the investigation tasks, students work on paper and pencil tasks based on the context of the computer simulation. The idea of these tasks is to give students the opportunity to make a start with level raising. Here is an example of that. After working on task 1 (see Figure 7) the two students of a pair both get a picture of the game board and they are asked to draw several routes from the starting point to the '100 box'. The second step is to describe the routes they drew in terms of 'Right' and 'Above'. Then they are asked to compare the routes they have described. The next step is to ask them whether they have found all possible routes from the start to the 100 box. The aim of these tasks is to problematize the counting of routes and to make clear that counting routes is the same as writing down all strings with three R's and five A's. The fact that the students have worked with the computer simulation, where they could explore at perceptual level, should give them an opportunity to reflect on this with help of the paper and pencil tasks.

	Task	Computer screen
1	Play the game TIC TAC several times and try to answer the following questions with the help of the results: a) How many points do you get on average in one round of five balls? b) What is the probability that a ball comes in the 100 points? c) Can you discover a relation between the probability to come in a certain box and all possible routes to that box?	

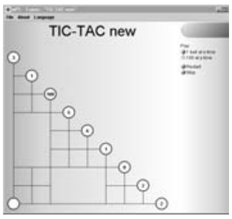
	Task	Computer screen
2	Play the game Plinko several times. a) Which box is the best to click on? Explain clearly why you think so. If you two have different opinions, indicate that. b) Determine the probability to come into the 100 from the second box from the left.	
3	Play the game TIC-TAC NEW several times. What is the influence of the new game board on the probability to come into the 100 points?	
4	With the game GALTON, you can distribute 100 points in total. Of course, the purpose is to do this in such a way that you gain as many points as possible when you play the game. a) Which point distribution gives you most points on the average? b) Determine for each box on the bottom row the probability that a ball comes there. c) You want to calculate which score you get on average with a certain point distribution of the game Galton. How can you use the question above to do so?	
5	Play the game GALTON NEW several times. a) Try five point distributions. Record your average score. b) Which point distribution gives you the most points? Do you have an explanation for that? c) Determine by experimenting for all boxes the probabilities that you come there. Indicate how you determined those probabilities.	
6	In the past lessons you have analyzed several games. Now you will design a new game yourselves. (…) Make a clear explanation (with calculations) how you have to play the game if you want to gain as much points as possible. (…)	

Figure 7. Overview of the collaborative investigation tasks.

7. LEVEL RAISING IN THE DOMAIN OF ROUTES AND PROBABILITIES

We will clarify the operationalization of level raising in the domain of 'Routes and Probabilities' by means of task 1b (see Figure 7) of the learning materials. In task 1a, the students have to calculate the average number of points in one round of the game. In task 1b, they have to discover what the probability is that a ball will get into the 100 points box. Possible answers to this question were:

1) There are nine boxes to reach, so the probability is 1/9.
2) By experimenting we find that out of the 100 balls, 14 of them get into the 100 points, so the probability is 14%.
3) The probability to finish in the middle is bigger than the probability to finish at either end.
4) There are 28 routes to point C, out of the 256 routes from the starting point, so the probability is 28/256 ≈ 10,94%.

Answer (1) is what we call an answer at the perceptual level. In this case it is a misconception, because the students apply prior knowledge in the wrong way. The students are aware of the fact that 'probability = favorable possibilities/total number of possibilities' but they have not yet learned to count routes. A first step towards level raising can be experimenting, like in answer (2). When the students see, by experimenting, that the probability to get to each box is not equal for every box, they have the opportunity to find out that there are a different number of routes leading to each box.

The notion that there is a higher probability to end up in the middle than at the ends, as in answer (3), can be understood as a starting point for the process of level raising too. The next step is to find a way to count the routes. 'By hand' this soon becomes too complex. That is why the method of 'adding in end points' is taught. With this method students can calculate the exact number of routes to a certain box and so they will move on to answer (4), which is the right answer at the conceptual level.

8. RESULTS

8.1 The First Research Question: Are There Differences in Attained Mathematical Level between the Three Conditions?

This question is answered by analyzing the results of the post-test on knowledge of the topic 'Routes and Probabilities'. The test consisted of 13 items, the maximum score was 32 and the minimum score 0. Eight missing values (students who did not participate in the pre-tests or the post-test) were excluded. Table I shows the mean values of the scores of the three conditions.

Table 1. Mean scores (M) and standard deviation (SD) for the post-test

	N	*M*	*SD*
BEFORE	21	12.14	5.96
DURING	20	12.25	6.47
AFTER	19	10.68	5.77
Total	60	11.72	6.02

All the means are not very high compared to the maximum score of 32. The standard deviations are high in all conditions and very comparable. It is clear that the mean scores of this test do not differ substantially. This was confirmed by a covariance analysis with pre-test 1 as a covariate, that showed no significant difference between the three means. ($F = 0.577$, $df = 2$, $p > 0.05$). When we consider the average final learning results in terms of mathematical level raising, it seems to make no difference whether students worked with those tasks either before or after their traditional textbook, or whether they worked with the condition in which textbooklike tasks were integrated with the simulation. In all the conditions, there were some students who did not attain the conceptual level.

8.2 The Second Question: Do Students in the Three Different Conditions Approach the Collaborative Investigation Tasks with the Computer in the Following Ways?

a. do students in BEFORE approach the tasks at the perceptual level?
b. do students in DURING approach the tasks initially at the perceptual level, and at the end at the conceptual level?
c. do students in AFTER approach the tasks at the conceptual level?

A first review and analysis of the written answers of the investigation tasks of all pairs showed that for some tasks there was a clear difference between the answers of the three conditions and for other questions there appeared no such difference. The first analysis also made clear that the students' answers could not always be categorized as 'perceptual' or 'conceptual', but that some answers 'fell in between'. These answers were not correct, but we could see that students had learned something. We categorized these answers as '*start level raising*' or '*semi level raising*', which we will explain below.

Figure 8 shows the four categories of students' answers. With 'perceptual level', we mean that students are playing the game without reflection, or that they apply prior knowledge in an incorrect way (misconceptions). With 'start level raising', we mean that students come up with an answer that makes a start in the process of level raising. Often these were their own constructions. So 'start level raising' denotes a development from the perceptual level. With 'semi level raising', we mean that students wrongly apply the concepts they have been taught. So 'semi level raising' means that students have not attained the conceptual level. There is a difference in quality between 'start level raising' and 'semi level raising', since 'start level raising' means that students try to build up new concepts from their own ideas and experiences, while 'semi level raising' indicates that students are not able to link new learned concepts to their own ideas and experiences. Finally, on the conceptual level students can apply the concept in the correct way; thus conceptual change has taken place. In Table 2 we give as an example the different categories of answers for task 1b.

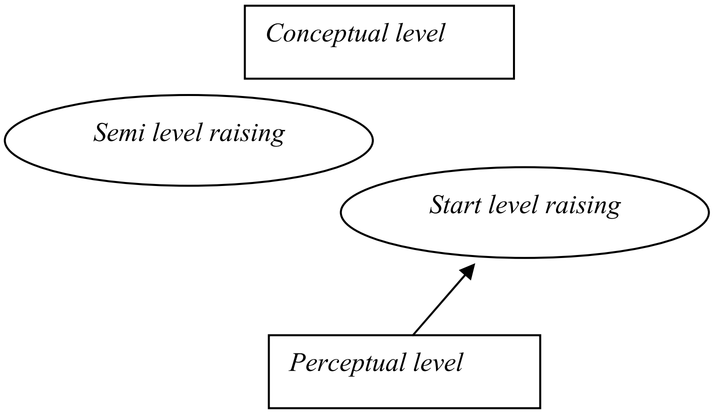


Figure 8. Four categories of students' answers.

Table 2. Categories of answers for investigation task 1b

Task 1b: What is the probability that a ball hits the 100 points?

Perceptual level	1/9; experiments
Start level raising	the probability is not equal for each box, so we have to count routes; own counting strategy
Semi level raising	counting routes wrongly or linking it wrongly to probabilities; probability = 28
Conceptual level	counting routes correctly, 28/128

The analysis of the students' answers was carried out as follows. For each task, we classified the four levels of answers. Then, for each task (see Figure 7), we determined the level(s) of the answer of each pair of students. The dominant level(s) of answers for each task (i.e., the levels that occurred for more than 25% of the pairs) are illustrated in Table 3.

Table 3 makes clear that some tasks show differences in level of answers between the conditions and other tasks show similarities. Before analyzing the pattern of answers in each condition, we first checked whether all tasks could be answered at different levels. It might be that the nature of a certain task explains the differences or similarities in the levels of answers.

Table 3. Dominant level of answer per condition per task:
Percentage of students that answered the question on the particular level..
(Key: P = perceptual level, St = start level raising, S = semi level raising, C = conceptual)

Task	BEFORE N = 21				DURING N = 20				AFTER N = 19			
	P	St	S	C	P	St	S	C	P	St	S	C
1a	100				100				100			
1b	60	40			60	40					36	36
1c	50	40			33	44			27	27	27	
2a	60	40			80				80			
2b	60					40						60
3		80			70	30					55	
4a	90				100				91			
4b	86							50	36		27	27
5a	100				86				100			
5b	30	70			43	-[2]				73		
5c	100				86				91			
Final task	90				38	63			50	50		

Tasks 1a, 2a, 4a and 5a were answered at perceptual level by most of the students in all conditions. These tasks are ‘starters’. They ask students to play a certain game a few times and to draw conclusions about how to win it. Although they could be answered at the conceptual level, these tasks in fact ask for an answer at the perceptual level.

Tasks 1b, 2b, and 4b show the most differentiation in the levels of the answers. All of these tasks ask students to calculate probabilities in a grid, so these are important questions from a conceptual point of view. These tasks give opportunities for level raising.

[2] In this condition, the level of answer was unclear for 57% of the students, probably due to an error in the learning materials.

Task 1c shows homogeneity in answers across all conditions, apart from the fact that 'semi level raising' does not dominate in BEFORE. This was a so-called 'yes/no' question: answers at perceptual level are a simple yes or no without explanation. The aim of this task was to focus students on the fact that the calculation of probabilities has to do with counting routes. The fact that students do not formulate answers at the conceptual level might be because of the fact that the concept 'total number of routes' is not mentioned in the question.

The results of task 3 make clear that for students it is very difficult to count routes in a grid with a different shape. BEFORE students realized that there are fewer routes to the 100-box, however DURING students did not. Most of the AFTER students did not manage to count routes.

The answers to task 4c could not be categorized in levels, so we left it out of our analysis. The aim of the task was to show students how to calculate the expected value when you know the probability to come into a certain box and the amount of points you can win for every box. The students' answers gave little information about the level at which they answered the question. This could be to do with the fact that the question is rather 'closed', i.e., there is no room for the students' own constructions. Besides, it is very difficult from the conceptual point of view. The concept 'expected value' was introduced to give a motivation for the calculation of probabilities for each box, but this might have been too advanced for the students to understand.

In task 5, the conceptual level and 'semi level raising' do not dominate. This can be related to the formulation of the task, especially task 5c, that explicitly asks students to determine the probability by experimenting, in other words, it asks for an answer at perceptual level. The task does not aim at level raising and is in fact not a good investigation task. However, it is interesting that two pairs in AFTER did answer that they did not only experiment, but that they also calculated the probability. In their calculation, they did not take into account the asymmetric probabilities. In the final task, most students answered at perceptual level, or made a start with level raising. The aim of this task was to let students apply what they learned about probabilities by developing their own game and reflecting on it. This did not happen, which might have to do with the formulation of the task.

We will now summarize the results per condition in order to answer the second research question. Tasks 1a, 2a, 4a, 5a and 5c will be left out of the analysis, because they explicitly ask for an answer at the perceptual level, as we just explained. The students in the condition BEFORE indeed answered the investigation tasks at perceptual level and they made a start with level raising. It is interesting that they made a start in level raising in task 1b but not in 2b and 4b. This might have to do with the fact that they attempted task 1b at the beginning of the lessons and that at that time the question was new to them. We expected that students would make a start with level raising by working on the final task, but there they stayed at the perceptual level. It might be that they lacked tasks to evoked level raising, that is, that they had not made any exercises causing them to reflect on the mathematical structures they had been exploring. The students in DURING started at perceptual level and ended up at conceptual level, but not for task 5 and the final task. The students in AFTER answered some tasks at the conceptual level, but it was also clear that many students did not answer at conceptual level, but that semi level raising had occurred, that is that they were not able to apply the concept they learned in the correct way.

8.3 'Start Level Raising' and 'Semi Level Raising'

In this section we will illustrate the levels of answers in students' dialogues. In Table 4, we give an example of start level raising in a dialogue of two students, Susan and Peter, in the BEFORE condition. They are working on task 1b for the game TIC-TAC, and the question was 'What is the probability that a ball ends up in the 100 points box?'

Table 4. Dialogue between Susan and Peter

		Statement	Level
1	S:	There are nine possibilities	*Perceptual*
2	P:	(..) But the chance that it will come here is probably bigger than that it will come here	*Start level raising*
3	S:	O, why?	
4	P:	O, no, that is, yes, of course, because the chance that it will go like this is bigger than that it will go up all the time.	*Start level raising*
5	S:	Ok, but the computer is also doing something, ok	*Perceptual*
6	P:	Comput…, yes, but it is about theories	*Conceptual*
7	S:	Yes, that's true...	*Conceptual*
8	P:	I mean, there are …	*Start level raising*
9	S:	But how can you calculate that?	*Conceptual*
10	P:	For instance, that one can go like that and like that, all kind of routes, but that one only has one route …	*Start level raising*
11	S:	Yes, but look, with this one you have… yes…yes	*Start level raising*
12	P:	O no, that one cannot go like that, well that is very difficult (…)	*Start level raising*
13	S:	Different possibilities, then you have to…all those routes, you know	*Start level raising*

In line 1, Susan counts the total number of possibilities. Maybe she has in mind the idea 'probability = favorable possibilities/total number of possibilities', which was prior knowledge and an answer at perceptual level in this context. In line 2, Peter discusses this answer by expressing his idea that the probability is not equal for all boxes. By asking 'why' in line 3, Susan gives him the opportunity to refine his idea. In line 5, Susan criticizes the idea that one could calculate those probabilities, because the computer has a random influence, but Peter thinks that they have to find an answer at the conceptual level (line 6). Susan is convinced (line 7) and realizes that this is not easy to calculate (line 9). Peter wants to refine his idea (line 10), Susan gets involved in this and she finally realizes that they have to count routes in order to calculate the probability. It is interesting to see that Susan started with an answer at the perceptual level and that at the end of the discussion she has made a start with level raising. Peter, however, started at a higher level and was stimulated to refine his thinking by Susan's questions. In Table 5, we give an example of semi level raising in a dialogue between two students in the AFTER condition, Anouk and Mara. They are also working on task 1b.

Table 5. Dialogue between Anouk and Mara

		Statement	Level
1	A:	1,3,6,10,15,21, 21 plus 7 is 28 chances	Conceptual
2	A:	28, how do you say, it is 1 at the 28 or something, 28, no, there can be more chances, isn't it?	Semi level raising
3	M:	Just 28	Semi level raising
4	A:	Ok	Semi level raising

In line 1, Anouk counts routes in the correct way, which is an answer at the conceptual level with respect to the counting of routes. In line 2, she tries to link the number of routes towards the 100-box to the probability to end up in that box and she mixes up 'number of routes' with 'probability'. We categorize this answer as 'semi level raising' because they seem to approach the problem at the conceptual level, but they are not able to apply the concepts in the right way. In lines 3 and 4, they agree to accept this answer.

9. CONCLUSION AND DISCUSSION

Having compared three conditions of use of the learning materials, what can we conclude about the use of collaborative investigation tasks with the computer for the learning of the mathematical concepts involved? When we consider the average final results in terms of mathematical level raising, it seems to make no difference whether students worked with the tasks either before or after their traditional textbook, or whether they worked with tasks in which textbook-like tasks had been integrated with the computer-based activities. In all conditions, many students did *not* attain the *conceptual level* for all concepts. When looking at the levels of answers to specific investigation tasks, however, we found some significant differences. Students that worked with the textbook-like tasks integrated into the collaborative investigation tasks (the DURING condition) made a start with level raising more frequently (using their own constructions) than students who worked with the investigation tasks before they worked with their traditional textbooks. It may be that the textbook-like (paper and pencil) tasks helped students to elaborate their experiences with the computer simulation. Compared to the students who worked with the investigation tasks with the computer after they worked with the textbook, the DURING students showed less 'semi level raising', that is that they showed less use of a concept without being able to integrate it in their experiences at the perceptual level. It is therefore possible that integration of textbook-like exercises and investigation tasks in the context of the computer simulation encouraged students to try to build up a concept with their own constructions, by reflection on their experiences at the perceptual level. One result of this is that they were less likely to 'learn a trick but not know how to apply it'. We value this as an important conclusion, since this last phenomenon is a well-known problem throughout mathematics education.

We will make some critical remarks on our study and give some outlines for further research. A point of critique could be that the students in the three conditions worked with the same formulation of the investigation tasks. In order to make the learning materials most effective in each of the three conditions it might have been better to adapt the instructional text of the tasks to the state of prior knowledge, which was different for the three conditions.

Another point of critique for all conditions is that students do not get the opportunity to invent the procedure of counting routes ('*adding in end points*') with the help of their own constructions. In the next version of the learning materials, we will try to give opportunity for this with the help of smaller models of the game board in the computer simulation. The work of Kafai et al. (1998) may give ideas on how to improve the final task, where students develop a game by themselves. In further research we will continue with the DURING approach for using the learning materials, and special attention will be given to the calculation of probabilities.

The categories 'start level raising' and 'semi level raising' may help to identify steps in the process of mathematical level raising. 'Start level raising' is in fact the first step in the process of guided reinvention, where students try to build up mathematical concepts from their own experiences. 'Semi level raising' is well known as a problem of 'transfer' in mathematics learning when students know how to apply knowledge in the same context that they learnt it, but are not able to link it properly to prior knowledge, or to apply it in new contexts. Although 'start level raising' and 'semi level raising' both indicate that a student has not attained the conceptual level, we value the 'start' case more highly, since the knowledge is better founded. Such concepts appear to be useful in future research projects to further investigate the process of level raising.

So far, we have paid little attention to the role of the teacher, who gave minimal help when the students were working in pairs. In further research, we will focus on the quality of the teacher help.

Chapter 4

RECONSTRUCTION OF A COLLABORATIVE MATHEMATICAL LEARNING PROCESS[1]

The study focused on the interaction between two secondary school students while they were working on computerized mathematical investigation tasks related to probability theory. The aim was to establish how such interaction helped the students to learn from one another, and how it may have hindered their learning process. The assumption was that interaction is beneficial for students if they can perform certain key activities, namely showing, explaining, justifying, and reconstructing their work. Both students attained mathematical level raising. However, the student who explained frequently and criticized himself attained more mathematical level raising than the student who did not explain her work frequently or criticize herself.

1. INTRODUCTION

One of the difficulties for teachers is to observe the learning process of students who are working collaboratively. The first sight of a classroom with pairs of students talking vividly to one another may be satisfying, but does not tell us if they are learning mathematics. What do they really discuss with one another? How does the interaction develop? Do they support or hinder one another's learning?

The importance of interactions with peers for students' mathematical learning has become evident in recent decades (Teasley, 1995; Webb, 1991; Yackel, Rasmussen & King, 2000; Wood, 1999). Some studies (Roschelle, 1992; Trognon, 1993) analyzed in detail the interaction between two students in relation to the content they learned; Roschelle did so in the field of physics, and Trognon with respect to the 'four cards' problem.

The study reported here aimed at analyzing the collaborative learning process of two students in a situation that was in a way very close to the usual situation in their classroom, i.e. students were working in pairs with a computer and learning materials, and the teacher was playing a minimal role. The learning materials, however, were very experimental and developed especially for these students. Earlier research

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[1] Pijls, M., Dekker, R., & Van Hout-Wolters, B. (2007). Reconstruction of a collaborative mathematical learning process. *Educational Studies in Mathematics, 65*(3), 309-329.

made it clear that, in this type of education, the students are not always motivated to perform investigations with the help of a computer (PRINT,1998; PRENT, 1999). We ascribed this to the fact that many computer simulations on probability theory were rather abstract. That is why we tried to develop simulations in a way that would be very accessible for students, offering them many opportunities to explore the subject on a concrete level. We developed investigation tasks in order to make them reflect on their experiences with the computer games. The tasks were intended for the students to experience the *grid structure* by playing several games. The tasks were presented to the students before they started to work on a chapter in their regular textbook. We expected that the investigative tasks will allow students to experience the properties of a grid and profit from this experience while working on their textbook chapter. The underlying structure of all the different games in the computer simulation was a grid, in several shapes, probabilities and appearance. The idea was allow students to develop a feel for the grid structure, by providing several examples.

We took the learning results of both students as a starting point for the analysis of their utterances during the lessons. Our focus was on events indicating whether or not certain concepts in probability theory were being learned. Our goal was to provide a detailed illustration of the learning process of two students, and to formulate hypotheses about interaction and mathematical learning.

2. THEORETICAL BACKGROUND

2.1 Learning Mathematics

In lessons dealing with the transition from a concrete or practical approach to a problem toward a more abstract approach, it makes sense to provide students with meaningful problems (Freudenthal, 1973) to be collaboratively solved. Inspired by Van Hiele's level theory (Van Hiele, 1986), we defined the transition from the perceptual to the conceptual level as 'mathematical level raising'. In the different stages of their learning process, students approach problems in different ways. Before they learn a concept, they approach it at a perceptual level; that is, they see it 'as it is' and do not recognize properties of a mathematical concept in it. After they have acquired a certain mathematical concept, students will recognize aspects of that concept in a realistic problem and can use these aspects to solve the problem. They thus treat the problem at another level. Level raising has taken place, namely from the perceptual to the conceptual level. An example from our experimental learning materials is given below.

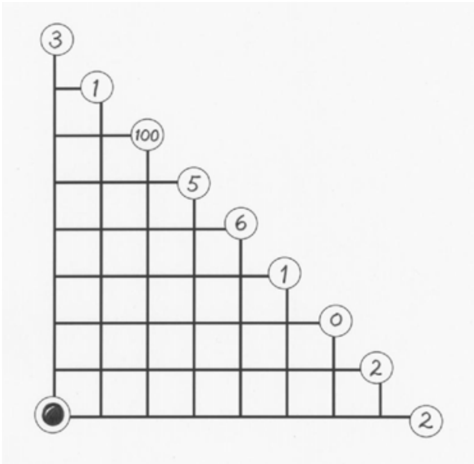


Figure 1. The game TIC-TAC.

Figure 1 shows the game board of the gambling game TIC-TAC. The task related to this game is:

One can play a little ball by clicking with the mouse on its starting point at the bottom left. There is a 50% probability that a ball will go up, and a 50% probability that it will go to the right. The question is: "What is the probability of ending up in the box with 100 points?"

Students' answers were:

1) "There are nine boxes, so the probability is 1/9."
2) "The probability of ending up in the middle is greater than the probability of ending up in an end point."
3) "I enumerated all possible routes toward the box with 100 points; the probability is 28."
4) "I enumerated routes in the grid and the probability is 28 divided by the total number of routes; 28/256."

The first two answers occurred as student answers in the study described in this article. With respect to the concept of calculating probabilities in a grid, we regard the first answer as an answer at the perceptual level, which is actually not correct but shows the prior knowledge of the students. The fourth answer is an answer at the conceptual level. In an earlier study (Pijls, Dekker & Van Hout-Wolters, 2003), we categorized two stages in the learning process between the perceptual and conceptual level, namely 'beginning level raising'[2] and 'semi level raising'. We defined the former as the beginning of concept building with the help of one's own constructions (as shown in the second answer), while the latter occurs when students seem to master certain concepts, but do not really understand them, and apply them incor-

[2] *'Beginning level raising' was indicated as 'start level raising' in (Pijls, Dekker & Van Hout-Wolters, 2003)*

rectly (as in the third answer). We assume that in this case their knowledge is not rooted in own constructions. These two stages, though not necessarily both, may occur in a student's learning process. In both stages, students have not yet mastered a concept; however, we value beginning level raising higher than semi level raising, since in the former case students understand what they know at that moment, while this is not the case with semi level raising.

2.2 Collaborative Learning

When students are trying to learn something difficult, they benefit from sharing their ideas with their peers, especially when their peers have a different point of view. Several studies mentioned differences between students (Kieran & Dreyfus, 1998; Sfard, 2003; Webb, 1991), in particular differences in thinking, as a prerequisite for a discussion in which students can learn. Webb (1991) showed a correlation between learning and giving explanations. Dekker and Elshout-Mohr (1998) developed a process model for the analysis of interaction between two students working on a mathematical task. The point of departure is that the students' mathematical work is different (see Figure 2). The assumption is that the performance of four key activities leads to mathematical level raising. These are: showing one's work, explaining one's work, justifying one's work, and reconstructing one's work. These activities are evoked by peers who perform such regulating activities as asking one to show one's work, to explain one's work and to criticize one's work. The key activities are assumed to be related to reflection, either because they give rise to reflection (showing one's work) or because they are the result of it (justifying and reconstructing). As the middle column of the model shows, the key activities can be observed when students are at work. Confusion may arise between explaining and justifying; one should keep in mind that justification is a reaction to criticism ("My answer is right, because…"). The other two key activities (*i.e.,* to show or to reconstruct one's work) can also cause confusion. If there are indications that a student's answer has evolved from another answer, we call it a reconstruction. Mental activities — such as thinking about and criticizing one's own work — accompany the key activities, but are less observable. Although the process model describes learning, it is also an instrument to analyze those parts of students' dialogues that give rise to mathematical level raising. The idea behind the model is that it describes the activities students can learn from. In principle, one person could perform both regulating and key activities.

A and B are working on the same mathematical problem. Their work is different.

A is working		B is working
A asks B to show his work	*What are you doing?* *What have you got?*	*B asks A to show her work*
A becomes aware of her own work		B becomes aware of his own work
A shows her own work	**I am doing this...** **I have got this...**	**B shows his own work**
A becomes aware of B's work		B becomes aware of A's work
A asks B to explain his work	*Why are you doing that?* *How did you get that?*	*B asks A to explain her work*
A thinks about her own work		B thinks about his own work
A explains her own work	**I'm doing this, because...** **I have got this, because...**	**B explains his own work**
A thinks about B's work		B thinks about A's work
A criticizes B's work	*But that's wrong, because...*	*B criticizes A's work*
A thinks about B's criticism		B thinks about A's criticism
A justifies her own work	**I thought it was right, because...**	**B justifies his own work**
A thinks about her justification		B thinks about his justification
A criticizes her own work	Oh no, it isn't right, because...	B criticizes his own work
A reconstructs her work	**I'll better do it like this...**	**B reconstructs his own work**

bold: key activities,
standard: mental activities,
italic: regulating activities

Figure 2. Process model of Dekker and Elshout-Mohr (1998).

Apart from differences between students, what other prerequisites are there for collaborative learning? What about the learning materials, and the role of the computer? To make it possible for the students to profit from one another, the instructional materials must provoke an interesting discussion. Dekker (1994) formulated four criteria for learning materials for mathematical level raising in collaborative learning. Problems must be: *real or meaningful*, in order to motivate and stimulate the students; *complex*, so that that the students need to work together; *to construct something*, in order to reveal the students' thoughts and to promote discussion; and of course *aimed at level raising.* The use of the computer may support these four criteria.

However, fulfilling these prerequisites does not guarantee that an interaction will be fruitful. Sfard and Kieran (2001) analyzed the discursive interactions of pairs of 13-year-old students on two levels: the level of immediate mathematical content and the level of meta-messages and engagement in the communication. They found the students' communication to be unhelpful because of the ineffectiveness of their communication. Kieran (2001) found different patterns of interaction between pairs of students, and showed that adolescents can experience some difficulty in following their own thinking in novel problem situations.

2.3 Aim of the Study

The aim of the study was to investigate whether we could explain the students' mathematical level raising attained in the post-test by the expected occurrence of key activities according to the process model. For example, if we knew from pretest and post-test that a student had mastered a certain concept, we expected to find in the transcript an event in which the student showed — and, possibly, explained, justified and reconstructed — his or her work related to this concept.

3. METHOD

The students took part in a field experiment on students' collaborative investigations using a computer (see Pijls et al., 2003). The experiment took place in a Montessori school in Amsterdam; the subjects were 16-year-old students who were mainly studying applied mathematics. The students were used to working on their own, while the teacher played the role of a coach. The experiment extended over 10 lessons of 45 minutes each.

3.1 The Theme of Routes and Probabilities

The mathematical concept to be learned is a model to calculate binomially distributed probabilities, namely a *grid structure*. By calculating the number of routes in a grid ending in a node, one creates Pascal's triangle. The aim of the lessons was, firstly, to learn an enumeration of routes procedure (adding the number of routes towards two 'preceding' points, the so-called 'adding in end-points'), and then to calculate probabilities in a grid and to use the grid as a model for solving word problems. The difficulty for students was that the enumeration procedure might become a meaningless 'trick'. This is what we regard as semi level raising, which may result in counting mistakes and not recognizing occasions of applying the grid structure.

3.2 Learning Materials

In order to make the enumeration of routes in a grid meaningful for students, we developed investigation tasks in a computer simulation. The simulation (Pijls, 2001) comprised several computer games with the grid as an underlying model. The idea was that students could explore the probabilities in a grid in an informal way by playing the games. The investigation tasks were intended to encourage students to reflect on their experiences with the computer games and to develop the mathematical concept of enumerating routes in a grid. The investigation tasks were followed by the tasks in their regular textbook (Boer et al., 1998). This textbook is a modern book, in the tradition of realistic mathematics education, teaching new mathematical concepts according to the principles of guided reinvention. In the book, students worked on small, relatively closed tasks in order to reinvent mathematical concepts, which are then summarized in ‘theory blocks’. We assumed that, by providing the students with investigation tasks before they worked with the textbook, they could explore the domain, which, in turn, would help them to attain level raising. During the investigation tasks, the students worked together using one set of activity sheets as well as their own exercise book.

The investigation tasks consisted of five tasks that concern the analysis and playing of a game. The aim of the first task was to raise the question of counting routes. The aim of the second task was to make clear that this game had the same underlying structure; it was made smaller so that students have the opportunity to develop a enumeration strategy. The third task let students explore so-called ‘grids with a hole’. The fourth task let them experiment with probabilities. The idea was that in order to answer the question students should calculate all probabilities. The aim of the fifth task was to let students experience an asymmetric distribution.

The chapter on ‘Routes and Probabilities’ in their mathematics textbook started with exercises that made students enumerate routes in a grid by systematically focusing on the number of paths towards a certain angular point. Secondly, students were shown how to use the grid as a model and the relation between Pascal’s triangle and enumerating routes in a grid. Finally, they were taught to calculate probabilities in a grid and how to use this to solve word problems. Although the didactical rationale behind the textbook exercises was to make students reinvent the concepts they learned, many of the tasks were closed and fragmented, leaving little room for own students’ creativity. Our intention was to create opportunities for students’ own inventions with the investigative tasks and the computer simulation.

3.3 Role of the Teacher

In our experiment, the learning materials were developed for independent learning. The usual role of the teacher at the Montessori school in Amsterdam is to coach the students and to give them help when they ask for it. Since the materials were still in a developmental stage and the researcher was the most familiar with them, we decided to work together as a team, that is, the teacher – who knew the students well – to coach the students, and the researcher to provide help with the mathematics when the students asked for it. The students were used to working self-reliantly in pairs. At the beginning of the lessons, the students were told to execute all tasks together and in what time sequence they had to finish their work. At the end of the lessons, they were to check their work against correction sheets. Since we wanted to stay very close to their normal school routine, the students were not given any special training on interaction.

3.4 Data Collection

We audio-taped the students during all their lessons. We collected written student products and log files to clarify any inaudible remarks on the audiotapes. For the present study, we selected one pair of students by taking samples of the audiotapes of all pairs. The occurrence of an interesting discussion concerning the relevant theme during their first lesson was our rationale for selecting Susan and Peter. They formed a high-average pair: Susan scored high and Peter scored average on prior knowledge. Furthermore, in her pretest on the theme of 'Routes and Probabilities', Susan made extensive elaborations by drawing diagrams, and Peter answered a difficult question correctly. So we chose the students in such a way that we expected them to show their work, criticize one another and attain mathematical level raising.

3.5 Analysis

We read the transcript several times and this led to suppositions about the learning processes of Susan and that of Peter (cf. Dekker, Elshout-Mohr & Wood, 2001). Then we asked ourselves this question: Which passages in the verbatim protocol account for the learning results in their post-test? With the agreement of two coders, we selected four episodes comprising events in which a concept was learned and an event in which this did not happen. We hoped that the latter would help us to understand why some concepts had not been learned. We analyzed the mathematical level of the utterances and the interaction with help of the process model of Dekker and Elshout-Mohr (1998), which we described in section 2.2.

4. RESULTS

4.1 Pre-test and Post-test

The pre- and posttest both consisted of 11 open problems on the theme of 'Routes and Probabilities'. It concerned enumerating routes in a grid, calculating probabilities in a grid, enumeration problems to solve with a grid and probability problems to solve with a grid. The problems could be answered at a perceptual level and at a conceptual level. As an example in a task on enumerating routes in a grid, students were given a part of a map with the question how many shortest paths one could take from point A to point B. When students enumerated routes by drawing possible paths and counted them, we regarded this as an answer at the perceptual level, whereas systematically counting routes by 'adding in end points' or with Pascal's triangle was determined as an answer at the conceptual level.

In her pretest, Susan approached the questions at the perceptual level. Thus, she could solve easy problems, but not those that were more complex. She exhibited a concrete approach to the tasks, by trying to use known procedures and by enumerating concrete routes. In her post-test, Susan showed that she had learned to enumerate routes according to the procedure taught in their textbook, the angular point method. She had also learned to use the grid as a model to solve enumeration problems, although she was not clear about when this method can be used and when not. She knew how to calculate probabilities in a concrete grid by enumerating routes and multiplying by probabilities. Susan had attained the conceptual level for the enumeration of routes and calculating probabilities in a grid, but she had not fully learned to use the grid as a model. She attained semi level raising for this concept, since she showed not to know exactly for what kind of word problems the grid could or could not be used.

In his pretest, Peter's answers demonstrated a preference for formulas. He tackled the enumeration problems and the complex route enumeration problem by applying a formula (either his own or one he had been taught) that was not correct. It is striking that he answered a question about calculating probabilities in a grid correctly. In his post-test, Peter showed that he had learned to count routes and to use the grid as a model to solve enumeration problems. He knew when to use it and when not. He had refined his method in order to solve probability problems in a grid. Peter had attained the conceptual level for all concepts.

To summarize, Susan and Peter both started at the perceptual level. However, Peter showed a tendency to approach problems at a conceptual level, while Susan explored the perceptual level. At the end of the experiment, Susan had attained the conceptual level for two of the concepts, and Peter for all concepts.

4.2 Reconstruction of the Learning Process

In order to reconstruct the learning process, we selected four episodes we considered to be central with respect to mathematical level raising. These were:

1) Susan developed the notion that in order to calculate probabilities in a grid, she and Peter had to enumerate routes. Peter subsequently constructed a formula to enumerate routes.
2) Peter wanted to refine his formula to enumerate routes; Susan did not really respond to this.
3) Susan and Peter reconstructed Peter's formula to enumerate routes.
4) Peter explained to Susan the enumeration formula given in their mathematical textbook.

Analysis showed us that episodes 1, 3, and 4 contain utterances indicating mathematical level raising, and that during episodes 2 mathematical level raising could have taken place but did not. For each episode, we wanted to establish whether there was a correspondence between learning and the interaction as described in the process model. First, we tried to determine the mathematical level of the utterances (analogous to our previous study (Pijls et al., 2003)) and then coded the utterances with the help of the process model. Both codings were performed with the agreement of two coders.

4.2.1 First Episode: the Notion of Enumerating Routes

Susan and Peter were working on the task shown Figure 1. They have been playing the game TIC-TAC on the computer and realized that the little ball hits the boxes in the middle more often. Below we show a part of Susan and Peter's discussion on task 1b; (….) indicates that a part of the dialogue has been deleted for the sake of clarity.

1	S:	There are nine possibilities.
2	P:	(…) But the probability that it will come here is greater than that it will come here.
3	S:	Oh, why?
4	P:	Oh, no, that is, yes, of course, because the probability that it will come here is greater than that it will come here, because the probability is greater that it will go like this, than that it will go up all the time.
5	S:	Okay, but the computer is also doing something… okay.
6	P:	Comput.., yes, but it's about the theories.
7	S:	Yes, that's true…
8	P:	I mean, there are…
9	S:	But how can you calculate that?
10	P:	For example, that one can go like this or like this, or like that, all kind of routes, but that one has only one route.
11	S:	Yes, but look, with that one you then have…
12	P:	Oh no, because that one can't, well that's very difficult (silence) (…)
13	S:	Different possibilities, then you have to… all those routes, you know... (…)
14	S:	You see, it goes as often to the 5 as to the 1.
15	P:	No, I mean: You can see that the ball is going more often to the middle than here and here.
16	S:	Yes...
17	P:	I think that the probability is … I think, er… the greatest probability is that it will go to the 6, because that's in the middle. The probability increases as you go toward the middle. (…) I think the probability is 5% here, and then it doubles, and then it doubles all the time, or something like that…. What's the probability…? (…)
18	S:	So then you have to take percents. 1/9, what's that?
19	P:	1/9?... something like 9 percent, 9.333 percent (…) But we can also… we keep it apart.

Susan showed Peter her answer (line 1), which is in accordance with their prior knowledge of probability theory, namely that in order to calculate a probability, one must first determine all the possibilities. Peter criticized this by showing his answer:

The probability that a ball will end up in the middle is greater than the probability that it will end up at an endpoint. Susan asked Peter to explain this, and he explains (line 4) that the probability that a ball will zigzag is greater than that the probability that it will go upward all the time. Susan criticized this (line 5) by stating that there is no structure: possibly the computer influences the game. In addition, Peter criticized himself (line 6) by saying that there must be some theory behind it. For him, mathematics is about theories. Susan agreed and asked for an explanation. Peter explained his idea by refining it (line 10): A box in the middle can be reached by all kind of routes, while a box at an endpoint can be reached in only one way.

They were both puzzled (lines 11 and 12); this was difficult for them. Then Susan reconstructed her answer at the conceptual level (line 13). She realized that they had to count routes in order to calculate probabilities. After this statement, there was a long silence: They did not know how to enumerate routes. Susan then showed (line 14): The probability is symmetric around the middle. Peter showed that the probability is greater in the middle, and (line 18) that he wanted to calculate the probability. He estimated that it was 5% at an endpoint and then doubled each time it moved closer to the middle. When he started to calculate, so did Susan, and she returned to her original answer of 1/9. Susan proposed writing down this answer and proceeding with the next question. Peter, however, was not satisfied: He wanted to think about it and would return to the question later. They went on with the second task.

Interpretation of the First Episode. Susan formulated an important conclusion: In order to calculate the probability, one has to count routes. She gathered a lot of information about grids and probabilities. However, she did not manage to use this information to answer the question and she relied on her original answer at the perceptual level. Peter was clearly searching for a theory. At the end of the episodes, Susan did not ask for explanations nor did Peter ask Susan why she had decided the answer was 1/9. He criticized himself. In the first part of this episode, mathematical level raising went hand in hand with a discussion in which both Susan and Peter showed their own work and criticized one another's work. In the last part of the episode, they returned to their original answer (1/9), but they no longer criticized one another and they stayed at the same mathematical level.

4.2.2 Second Episode: Peter Reconstructs his Enumeration Procedure

After the first episode Susan and Peter continued with the next investigation tasks, which concern other type of grids. They made calculations to determine optimal chances to win. However, there was no link in their reasoning to the previous task and they did not mention any more the need to enumerate possible routes in order to calculate the probabilities in a grid. At the beginning of the next lesson, Peter returned to the question from the first episode. He has been thinking about this problem outside the lesson and he has reconstructed his enumeration procedure. He now shows his work to Susan. Their dialogue is given in below.

1 P: I assumed that every time that, every time that the ball moves toward the middle, that the probability that it, er, one upward and then this way and that way, so here the probability is 1, 2, 3, 4 and 5, and then I've added those chances, that's 25, and then this is three 25ths, so twelve 100ths

2 S: Yes.

3 P: Is 12 percent.

4 S: Yes.

5 P: I think, but I'm not sure, so it would be, yes, 20% and that 4%. Well, shall we (…)

6 P: You may come up with another theory, but anyway I think it's better than 1/9.

7 S: Okay.

8 P: At least, for me. I don't know what you think.

9 S: Yes, it's okay, add it to our stuff... And do you also have something for this one? Er, have a look.

10 P: Wow. Erm, let me see, er...

11 S: Shall we proceed? Add it… How do you write it?

12 P: Well, I'll write.

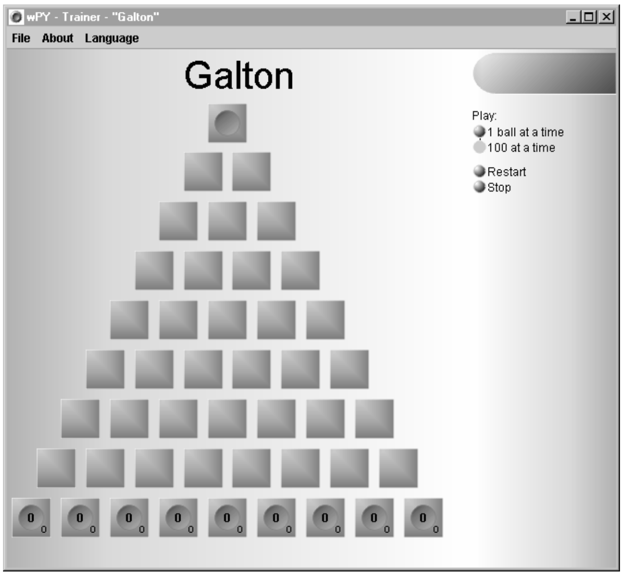
Peter had worked out his idea that the probability increases toward the middle. He showed his new theory to Susan (line 1): If I name the probability at one endpoint 1, and in the next box 2, and so on, then it makes 25 altogether, so the probability that the ball will come land in the 100 box is 3/25, or about 12 %. Susan (lines 2 and 4) did not ask for an explanation, nor did she criticize him; instead, she agreed. Peter then worked it out (line 5): The probability will be 20% in the middle and 4% at an endpoint. Peter seemed to justify his answer (line 6), though not with a content-related argument. Susan agreed with him (line 7), but did not explicitly agree with the content of his work. Peter was not sure of his answer and asked Susan what she thought (line 8). Susan agreed with his idea in a practical way (line 9), and asked whether he could solve another question, too, as she wanted to proceed with the next one. Peter then replaced their original answer (1/9) by 12%, and they went on with the next question.

Interpretation of the Second Episode. Peter came up with an idea he was not sure of. He started to defend it, but since he received no feedback from Susan regarding the content, he could not really explain, justify, or reconstruct it. There was nobody to indicate the weak points of his theory. Since he had offered his idea at the beginning of the lesson, they did not have the actual computer game at hand to test his idea. Susan accepted his theory without criticism. It seems that she was not interested in why his theory would work; instead, she wanted to go on with the next task. In fact, Peter criticized Susan's idea of 1/9.

Although Peter's theory was a good try, it is not correct. He rejected the 1/9 probability and wanted to make a theory based on the conclusion that the probability increases toward the middle and that it is symmetric around the midpoint in a enumeration method. In terms of mathematical level raising, he might have benefited from someone asking him for an explanation. No level raising took place during this interaction.

4.2.3 Third Episode: Reconstructing the Enumeration Procedure

Susan and Peter returned to the question from the first episode during the second lesson, when they were working on the fourth task on a Galton-board-like game (Figure 3). The students had the opportunity to drop a 100 balls at the same time, and so gain insight on the distribution of the balls.



Task 4b
In the Galton game, one can distribute a total of 100 points over the nine boxes. Of course, the purpose is to do this in such a way that you win as many points as possible. Determine for each box on the bottom row the probability that a ball will end up in it.

Figure 3. Task 4b.

The aim of this task was to let the students realize the relation between the distribution of the balls and the probability to reach a certain box. The episode started when Peter mentioned that this question made him think of task 1b, which we have discussed before.

1 S: Here? But do you mean the same as...?
2 P: Yes, erm, I don't know… I think that in fact that theory is a bit vague.
3 S: And here will come… Here comes 2, here comes 0? Yes, that sounds… The

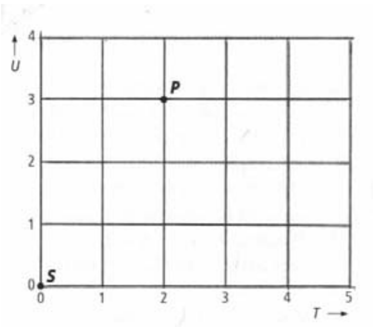
ball comes here the most.
4 P: Yes, I know (…)
5 S: And then here, less… Here it comes less often.
6 P: Yes, but I did 'plus one' all the time, 1, 2, 3, 4, 5, but it's also possible that it doubles all the time.
7 S: Yes.
8 P: Wait a minute (noise of paper)
(silence)
9 S: Like this. 16/46, …
(…)
10 P: This is more likely than this… So I think that the probability doubles all the time.
11 S: Yes, okay.
12 P: What did you have? I mean…
13 S: Yes, erm… I really don't have anything more, but I think your answer is reasonable.
(…)
14 S: Erm, oh. I only calculated it for the box in the middle.
15 P: Yes, as I said, 16/46, 8/46, 4/46. The number is divided by two.

Peter wanted to use the same theory as in task 1b (first episode). Susan asked for an explanation (line 1), and Peter said that he was still not sure of his theory. Susan then repeated the conclusion that the probability of ending up in the middle was greater than the probability of ending up in one of the endpoints (lines 3 and 5). It might be on the basis of experiments with dropping 100 balls at a time that Peter returned to his idea that the probability "doubles all the time". He concluded from the distribution of the balls in the Galton-board, that the probability to reach a certain box increased at a stronger rate than 'adding one'. He looked back at task 1b and now worked it out in such a way that the probability doubled for each box toward the middle. Then the probability of ending up in the middle was 16/46 (≈ 30 %). He reconstructed his answer, and then (line 10) showed his reconstructed work. In line 12 he may have meant that the probability of ending up in the middle was more likely to be 30% than 20%. Peter asked Susan to show him her work (line 12). Susan simply agreed with Peter's work (line 13); she did not really react to the content of what he had said, nor did she show him her work. They then wanted to express the probability in percentages, and here Susan took the initiative in the calculations (line 14). When she had finished the calculations, she realized that she had worked out the probability for only one box. Peter explained that with that probability, one could easily calculate the other probabilities. This was their answer to question 4b. They also reconstructed their answer to question 1b (first episode) in the same way.

Interpretation of the Third Episode. The game made Peter think about task 1b and it gave him the opportunity to reflect on his ideas and to reconstruct his answer. Susan's calculating skills helped him to work out the next version of his theory. Therefore, they reconstructed their answer to question 1b (see Figure 1). Although this was a beautiful construction in terms of probabilities, the theory is not correct. Susan and Peter had made a good start with regard to mathematical level raising, and we believe that these explorations gave them a sense of the distribution of paths and probabilities in a grid. As in the previous episode, Peter asked Susan to criticize his work and to show him her work, but she did not react to the content; thus, Peter did not develop his theory further.

4.2.4 Fourth Episode: Enumerating Routes in their Mathematical Textbook

After the third episode, Susan and Peter finished the investigation tasks, and started to work on the textbook's 'Routes and Probabilities' section. After the first task, in which they had to mention possible paths in a grid, they start with the task shown in Figure 4.



2. The result of 2-3 must be preceded by 2-2 or by 1-3.
   a) Indicate these predecessors in the system of coordinates.
   b) How many routes are there from 0-0 to 2-2?
   c) How many routes are there from 0-0 to 1-3?
   d) How can you obtain the number of routes from 0-0 to 2-3 from the answers to tasks b) and c)?

Figure 4. Task 2 from the students' mathematics textbook.

The aim of this task was for the students to reinvent the procedure for enumerating routes. We expected them to refer to the investigation task in which they realized that they had to enumerate routes in order to calculate probabilities, but this did not happen. The episode below starts while they were working on task 2d.

1 S: Er, 1-3.
2 P: Of course not… Yes, yes. 3-1 I say. 3-1, 2-3, or 2-2, only to come here, you must go along these two sides.
3 S: Huh?
4 P: Don't you understand?
5 S: No, I don't understand this question at all!
6 P: Look, the final score is this.
7 S: Yes.
8 P: To come here, first you have to be here.
9 S: Yes.
10 P: If it'd been here, it would've been 3-1…

(…)(…)(…)
11 P: But I want to find a way to calculate it. Now I count… It's just 2 to the power of 2…. No, er, yes. 2 times 2 times 2, 2 to the power of 3 is it…
(…)(…)(…)
12 P: But how do you calculate it, why does it take so long?
13 S: Now, erm, three upward…
14 P: But you just have to add that and that?
15 S: Huh?
16 P: That one and that one, and then you have that M.
17 S: Okay. And this is 10?
(…)
18 S Very easy indeed…
(…)
19 A: Susan and Peter, how did you calculate that?
20 P: We just made a thing… a grid.
21 S: … grid.
22 A: Is that possible for question 11a?
23 S: That's possible for all questions.
24 P: Yes, that's possible for all questions.

Susan showed her answer (line 1). The answer is correct, but Peter seemed to mix up the 3 and the 1, so he criticized her and started to show Susan his idea. Susan asked him to explain (lines 3 and 5). Peter did so (lines 6, 8, and 10), but still made the error of saying 3-1 instead of 1-3; however, he explained and understood the idea of the angular point method correctly. The task led them to the conclusion that in order to know the routes toward 2-3, one must add the routes to 1-3 and 2-2.

Peter made it clear (line 11) that he was not satisfied with the fact that the angular point method for enumerating routes that was suggested in the book is still recursive: In order to calculate the routes to a certain point, one first has to know the routes to its predecessors. He wanted to find a direct formula for enumerating routes, and he made a guess that it would be 2 to the power of 2. Then they read an explanation of the angular point method in their textbook, and Peter picked up the idea quickly. Susan, however, still enumerated routes.

After some time, Peter realized (line 12) that they were not working on the same task. He asked Susan to show him where she was. Susan showed that she was still enumerating routes (line 13). Peter criticized her method and she asked him for an explanation. Peter explained that you have to add the number of routes toward two endpoints. Susan asked for further explanation. Between lines 17 and 18, there is a long episode (which we have omitted, because it is repetitive) in which Susan asks Peter for an explanation and he explains the angular point method. Susan got the idea and checked her work with Peter. Then they worked it out together. Susan had picked up the angular point method and adopted it very well.

A peer ('A') then asked them to show him their work (line 19) and whether task 11a could be solved with a grid. This task was one of a series of problems that had to be solved with a grid. Susan and Peter were working comfortably with the angular point method of enumerating routes in a grid and in the context of the textbook;

every task could be solved with a grid. This may have led them to the idea that any enumeration problem could be solved with it. Especially the fact that Peter showed Susan the technique but did not explain it, might have perpetuated the misunderstanding and reinforced Susan's misconception.

Interpretation of the Fourth Episode. Peter had understood the idea of the angular point method very quickly. However, he was not careful about the details. Susan did not pick up this concept from the textbook exercises and continued to count routes by hand, which took a lot of time. Peter noticed this and explained how to count routes. She understood the routine and applied it. When they were asked by another student whether the routine could be used for all enumeration problems, they said it could, and this possibly strengthened Susan's idea that enumerating routes could be used to solve enumeration problems, as she showed in her post-test. Especially since no explanation was asked for and no criticism was given, she remained at the stage of semi level raising for the concept of solving enumeration problems with the help of a grid.

5. SUMMARY

The questions here are: Can we account for the concepts Peter and Susan learned through their dialogue? Did they support or hinder one another? With the results of their post-test in mind, we selected four episodes in which we found indications that they had or had not mastered a certain concept. These episodes were analyzed on the occurrence of activities of the process model.

Table 1. Number of key activities in the selected episodes

		Routes		Probabilities		Routes as a model	
Susan	Process	**shows**	3	**shows**	6	**shows**	2
		asks to show	1	**reconstructs**	3		
		asks to explain	3	*asks to show*	1		
				asks to explain	2		
	Attained Level	Conceptual		Conceptual		Semi Level Raising	
Peter	Process	**shows**	2	**shows**	10	**shows**	2
		explains	5	**explains**	4		
		criticizes	2	**justifies**	1		
				reconstructs	3		
				asks to show	3		
				criticizes	2		
				self-criticizes	5		
	Attained Level	Conceptual		Conceptual		Conceptual	

Table 1 presents a summary of the activities that occurred in the episodes, organized according to the concepts to which they are related. With 'attained level' we indicate the level that the students attained in the post-test. As can be seen, both students attained the conceptual level for the concept of routes and probabilities in a grid. Both showed their work and both reconstructed it; both asked one another to show their work and they criticized one another. However, Susan did not explain her work, while Peter did. Actually, Susan asked Peter to explain his work several times, while Peter did not ask her. Another difference is that Peter criticized his own work while Susan did not. So, in general, Peter showed more reflection on his own work than Susan did. This may have been an advantage in attaining the conceptual level for 'routes and probabilities as a model.'

As we know from the results of the post-test both students attained mathematical level raising for the subject of routes and probabilities. The one who both explained and generated self-criticism attained level raising with the subsequent concept, too. The one who showed her work and reconstructed it, without giving explanations, attained semi level raising for the subsequent concept. These observations lead to three hypotheses about key activities and mathematical level raising:

- Giving explanations about a concept may facilitate further level raising;
- Criticizing oneself is crucial for further level raising;
- Reconstruction which is not preceded by explanation may lead to semi level raising in the continuation of the learning process.

As mentioned under 'Learning Materials,' the investigation task on probabilities was an important one. Table 1 shows that a lot of activities took place for this concept. Peter did well in this crucial part of the learning materials, and it helped him later in the learning process.

6. DISCUSSION

This experiment concerned the learning of a part of probability theory. The question is whether we can generalize the results for other domains of mathematics. Since the interactions between Susan and Peter deal with the transition from a concrete toward an abstract approach of a grid, we believe that conclusions about their learning process would hold for any other topic in mathematics where level raising takes place.

One of the students clearly profited more than the other. What happened? What was the role of the learning materials and the composition of the pairs? The learning materials, especially the investigation tasks, indeed generated differences between the students. These differences gave rise to discussion, but Susan did not explain her work. Was it because Peter did not ask her to? Possibly, but when he asked her to show her work, she did not always react. She seemed to think that Peter's answer was more worthwhile than her own. Susan scored higher on prior knowledge than Peter. Nevertheless, she seems to have the feeling that he knows better. She gives the impression of a student who works hard. Her drive to proceed with the tasks now affects her adversely, because for investigation tasks it is important to take as much time as you need to find the answer. What does Susan need to give explanations? A partner who asks her to explain her work and gives her the feeling that her answer makes sense, and in the background a teacher who makes it clear that they can take their time to find the right answer?

Although the students were not explicitly asked to make sense of one another's contributions, the format of the lessons made it clear that they had to solve the problems together. However, nothing was said about how they should do this. We expected them to learn from performing key activities, without telling them that they had to do this, or rewarding this. We did this to evoke key activities by means of the tasks and the composition of the pairs. Since one of the students performed less key activities than the other one, it might be a point of attention in our next experiment to make it explicitly clear to students that they have to make sense of one another's work. This is one of the 'social norms' (Wood, 2001) that may induce meaningful interaction.

In the experimental setting, the role of the teacher was to provide help when the students requested it. This help was mainly focused on the mathematical content of the tasks because of the developmental stage of the learning materials. As shown in Dekker and Elshout-Mohr (2004), it might be useful to have the teacher encourage the performance of key activities. In our next experiment, we will compare the effects of teacher help that is focused only on the mathematical content of the tasks with the effects of a teacher who only ensures that the key activities are performed.

Chapter 5

TEACHER HELP FOR COLLABORATIVE MATHEMATICAL LEVEL RAISING[1]

A field study with 16-year-old students in senior general secondary education was executed with the following research question 'Do students working in pairs on investigation tasks with the computer attain more mathematical level raising when they are supported by a teacher who stimulates their interaction (process help) than when they are supported by a teacher who gives mathematical help (product help)?' Students in both conditions improved, but the two types of help showed no significant difference in level raising. Also, students in both conditions had serious problems with the learning materials, and wanted the teacher to explain and correct more. For students at this level of education, learning with investigation tasks in small groups appears to be very difficult.

1. INTRODUCTION

Guiding students by teachers in a collaborative learning process means creating optimal conditions in which they can learn from one another. Within the context of a mathematics lesson, the issue is whether the teacher should stay away from the mathematical content of what students are saying and focus on their interactions, or whether the teacher should provide mathematical hints when students ask for help. Generally speaking, should a teacher leave the learning content to the students and keep watch that the collaborative learning process stays under way, or, conversely, should the teachers take part in the collaboration by focusing on the content of what has to be learned?

This study is the second part of the research project 'Mathematical investigation tasks with the computer' that deals with the question how students can best learn mathematical concepts when they are working on collaborative investigation tasks with the computer. This question is answered for 16-year-old students in senior general secondary education, for the subject of probability theory. Earlier research has made it clear that the students in senior general secondary education who have to

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[1] Pijls, M., Dekker, R., & Van Hout-Wolters, B. (in press). Teacher help for collaborative conceptual level raising in mathematics. *Learning Environments Research.*

take obligatory courses on applied mathematics were not always motivated to perform investigations with the computer. The findings for these students were opposite to those for students in pre-university education and those for students in senior general secondary education who chose pure mathematics[2] (PRINT, 1998; PRENT, 1999; Niemiec, Sikorski & Walberg, 1996; Schnackenberg, 2000). In an earlier study (Pijls, Dekker & Van Hout-Wolters, 2003) we developed collaborative investigation tasks around a computer simulation, which we made very accessible for students in order to motivate them. In this study we want to investigate which type of teacher help is most conducive to learning mathematics with these materials. Any results may lead to hypotheses on guiding students who are learning collaboratively mathematics, or any other subject.

2. THEORETICAL BACKGROUND

2.1 Teacher Help for Collaborative Learning

Collaborative learning is fruitful for students if it gives them the opportunity to sharpen their own thinking (Davidson, 1990; Dekker, 1994; Webb, 1991; Dekker & Elshout-Mohr, 2004). The quality of students' interactions is influenced by several factors, such as the learning materials, the composition of the groups, and the role of the teacher. In this study we will focus on this last aspect. We will build on a study performed by Dekker and Elshout-Mohr (2004), which we will extensively discuss in section 2.3. They compared the effect on mathematical level raising by a teacher who focused on the students' interactions with the effects by a teacher who paid attention to the mathematical content of the students' work.

Several studies showed the positive effect of instruction on the student's achievements in collaborative learning settings. Kramarski, Mevarech and Arami (2002) concluded that cooperative learning in combination with a metacognitive instruction led to higher learning results than cooperative learning in combination with 'regular' instruction on cooperative learning. Hoek (1998) found positive effects of the instruction of social and cognitive strategies on the learning results for both types of instruction, but no additive effect of the two. In an experiment lasting one year, Hoek and Seegers (2005) found that instructional activities including modeling problem solving, stimulating reflection and giving feedback about the process

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[2] *Secondary education encompasses schools providing pre-university education (VWO), senior general secondary education (HAVO), pre-vocational secondary education (VMBO) and Practical Training (PRO). All four types of secondary education are for children aged twelve and over and all begin with a period of basic secondary education. In 1999 all HAVO and VWO schools introduced set subject combinations and the 'study house' construction, which commences in the fourth course year and requires students to acquire skills and knowledge in a much more independent capacity. We speak of 'obligatory' mathematics, since all subject combinations include applied mathematics, while only the technical and science subject combinations include pure mathematics.*

of collaboration were effective for collaborative problem solving. Hughes and Greenhough (2002) compared children working on a computer task about a turtle following Logo commands. There were four conditions. They worked (a) alone, (b) with a peer, (c) with an adult, and (d) with a peer and an adult. The software used gave feedback to the children if they did not act correctly. The adults let the children make some errors before giving helpful advice. The results of the study showed no significant differences due to the conditions on post-test gains. They concluded that feedback from an adult and collaboration were not additive in their effects on learning. This last study, however, concerned a learning environment with very strong feedback ('error' notifiers in a computer program) about mistakes in the learning materials.

Although the teacher's instruction on group work turns out to be fruitful in several studies, Wood (2001) stresses the fact that teachers should stay away from the students' discussions in order to create situations in which students can construct their own knowledge. This is not an easy job for teachers, since they may experience a strong (human) tendency to explain, as she mentions in the following passage.

> "As advocated in the reform documents, learning mathematics with understanding is thought to occur best in situations in which children are expected to problem solve, reason, and communicate their ideas and thinking to others. Moreover, it is thought that situations of confusion and clash of ideas in which students are allowed to struggle to resolution are precisely the settings that promote learning with understanding. Therefore, (…) teachers must resist their natural inclination to tell students information, make the task simpler, or step in and do part of the task" (p. 116, *ibid*)

Another obstacle for teachers in supporting students' collaborative learning lies in the fact that the teacher cannot hear what the students are saying while working on the task. Brodie (2001) describes this problem. For teachers it is difficult to give appropriate mathematical hints that fit every single student, since they did not follow the discussion taking place between students. A mathematical hint could hinder the students' own thinking process. In order to avoid this, teachers may focus on the process of mathematical reasoning in order to make students explain their work to one another and criticize one another.

Marzi (2003) and Veenhoven (2004) also distinguish between teacher help that focuses on the content of what has to be learned and teacher help that focuses on the students' learning process without guiding the content of what has to be learned. Marzi (2003) investigated the difference between students working collaboratively under the guidance of a *supportive* adult (with the role of giving students feedback about their progress in structuring the task and solving the test, but refraining from intervening in their problem solving) and an *instructive* adult (maintaining students' involvement, but also focusing their attention on the steps required to solve the test and directing their actions to the task goal). It was found that pupils working with the supportive adult showed significantly more clear explanations and theoretical approaches than pupils working with the instructive adult. The adult's abstention from interfering with the content of the group work appeared to be beneficial to the pupils' learning.

In a different educational setting, Veenhoven (2004) examined which components of the teacher's helping behavior contributed to the student's investigation skills. His study was carried out with 16-year-old students in secondary education working on collaborative investigation tasks for the subject of Geography. The students were followed while they were working on these tasks. It was shown that support of the students' interactions led to better learning results than support of the learning content.

The two studies mentioned above show that when an adult does not interfere with the content of the tasks but stimulates the process of interaction between students, the students achieve higher learning results.

2.2 Mathematical Level Raising

We operationalize the learning of mathematics as attaining *mathematical level raising* (cf. Van Hiele, 1986). When a topic is new for students, they will approach it at the perceptual level, according to their prior knowledge or naive knowledge of the domain. By reflecting on both their own and one another's work they will develop a structure in a certain domain and approach the subject at a conceptual level: level raising has taken place. Mathematical level raising is defined for a certain concept, so an approach can occur at the conceptual level with respect to one concept and at the perceptual level with respect to another.

To stimulate the learning process it is important to connect the learning materials to the students' starting level, and to create possibilities for level raising. In our project, we use computer games with an underlying mathematical structure in order to have a connection to the perceptual level. The tasks accompanying computer simulations aim at the conceptual level. Students start with the gambling game 'Plinko' and they finish with calculating probabilities with the help of Pascal's triangle.

In order to give rise to discussions between students, the learning materials must conform to certain criteria (Dekker, 1994). The materials have to be *real* or *meaningful* to motivate and stimulate the students, they have to be *complex* in order to make the students need one another, they have *to construct something*, in order to make the different thoughts visible and object for discussion, and they have to *aim at level raising*. We expect the collaboration with peers to stimulate reflection and to induce level raising. Dekker and Elshout-Mohr (1998) developed a process model in which they describe the interactions between students that lead to mathematical level raising. They distinguish students' *key activities*:

1) to show one's work;
2) to explain one's work;
3) to justify one's work;
4) to reconstruct one's work.

All those activities are evidently related to reflection, either because they give rise to reflection (showing and explaining one's work) or because they are the result of it (justifying and reconstructing). Students can evoke these key activities among one another by regulating activities: to ask one another to show one's work, to ask one another to explain one's work, and to criticize one another's work. The situation described in the process model is that of a small group of students working together on the same mathematical problem. The work of the students is assumed to be different.

2.3 Process Help and Product Help

When students have full opportunities to learn with one another and with the especially developed learning materials, the teacher's help can be minimal. Dekker and Elshout-Mohr (2004) investigated what kind of minimal help from the teacher was most beneficial to students attaining level raising for students in the 5th year of pre-university education. The students were working in heterogeneous triads on investigation tasks on the subject of geometrical transformations. Two types of teacher help were compared:

- *product help*: teacher's help focused on the quality of the mathematical product students are working on (mathematical hints).
- *process help*: teacher's help focused on the quality of the interaction between students.

It turned out that the 'product students' made more products at a conceptual level during the lessons than the 'process students'. However, the process students attained significantly more level raising in the post-test than product students. Dekker and Elshout-Mohr explained this by the fact that the process teacher did not disturb the thinking process of the students. The teacher only kept the interaction going and hence created an opportunity for reflection, but did not interfere with the students' thinking. The product teacher disturbed the students' thinking with mathematical hints, since he/she could not follow the whole discussion of the students. The product teacher focused on the products of the students.

In this study, we will investigate the effect of process and product help for another type of education than Dekker and Elshout-Mohr did. Our research project closely relates to their study. Both projects deal with students working in groups, with the groups composed in such a way that there are differences in mathematical level between students. In both projects learning materials were developed for collaborative learning according to the four criteria mentioned in the previous section (meaningful; complex; to construct something; aiming at level raising). The learning materials contained no 'theory blocks' (*i.e.* sections in which a mathematical concept was shown and explained) and no 'correction sheets' with the correct answers to assess their work afterwards (the students were used to correct their work with the help of correction sheets after finishing the tasks). No classroom discussions took place in either project.

Besides the similarities, there are differences between Dekker and Elshout-Mohr's study and ours. We used a computer simulation in order to give students the opportunity to reflect on concrete experiences, whereas Dekker and Elshout-Mohr used pencil and paper tasks and concrete geometrical shapes. The size of the groups differed, since Dekker and Elshout-Mohr studied triads and we studied dyads. In principle, we expect that in a group of three peers more discussion would occur than in a dyad, but since three students cannot work on one computer at a time, we chose dyads. Another difference lay in the type of education (pre-university versus senior general secondary) and the kind of mathematics (students who chose to study pure mathematics in Dekker and Elshout-Mohr's study versus students who mainly had obligatory applied mathematics in our study). The opinion of many teachers is that students in pre-university education are much more inclined to investigate than students in senior general secondary education. The same holds for the pure math students versus the applied math students. A final difference was the length of the experiment (two lessons of 65 minutes in Dekker and Elshout-Mohr's study versus six lessons of 40 minutes in our study).

2.4 Aim of the Study

We executed this experiment in order to answer the following question:

Do students in grade ten of senior general secondary education who work in pairs on investigation tasks with the computer attain more mathematical level raising when they are supported by a teacher who provides process help than when they are supported by a teacher who provides product help?

This question will be answered by a pretest and post-test analysis. We expect the process help to lead to more mathematical level raising, since process help promotes students to perform key activities (to show, explain, justify, and reconstruct one's work), which in turn leads to mathematical level raising. Based on Dekker and Elshout-Mohr's results, we expect product help to disturb the process of mathematical level raising, since it keeps students from executing key activities and, hence, denies them an opportunity for mathematical level raising.

3. METHOD

3.1 Participants

Two mathematics classes and their teachers from the fourth year of senior general secondary education participated in our experiment. It concerned 52 students (53 minus one student who did not make the post-test). The school, a Montessori school in Amsterdam, had been selected on account of their experience with students working independently and the motivation of the teachers to take part in the experiment. We chose this school in order to have students and teachers who already were used to a situation as close as possible to the experimental roles. Students were used to plan their own work, for example. The experiment took place in the school and consisted of six to eight lessons of 40 minutes (depending on how much time a pair of students needed).

3.2 Pretest, Composition of the Two Experimental Groups and Post-test

One week before the experimental lessons started, the students made a pretest consisting of 12 open-ended paper-and pencil questions, with a total maximum score of 46 points. It contained four items that tested prior knowledge of probability theory (total of 11 points) and eight items about the subject of Routes and Probabilities (total of 35 points). The eight questions about the subject of Routes and Probabilities were spread out over four subtopics, 'Counting routes in a grid', 'Calculating probabilities in a grid', 'Solving counting problems with the help of the grid as a model', and 'Solving probability problems with the grid as a model'. An example of an item on the subtopic 'Calculating probabilities in a Grid' is given in the Appendix. Since these questions tested the knowledge that students had to learn, we did not expect them to answer all these questions correctly. We made this clear in some hints, like "The following question is a bit more complicated. Give it a try!". The reliability of the pretest was sufficient (Cronbach's alpha = .68). One person coded all the pretests and another person coded ten per cent of all the pretests. The interjudge reliability was 96%.

We used the results of the test to assign participants to two condition groups that were comparable with regard to their pretest results and that contained an equal number of students from each class. This was done in order to avoid confounding teacher effects. There were no indications that one of the teachers was more liked by students than the other. We also used the pretest results in order to compose semi-heterogeneous pairs of students: pairs consisting of an average and a high-level student, or an average and a low-level student (in order to have differences between students not being too large). The pairs were composed by the researcher and corrected by the teachers in cases some students were expected not to be able to cooperate with one another (for instance, a very shy girl that had asked her teacher not to be linked up with very extravert students).

At the end of all the lessons the students made a post-test. The open-ended questions in this post-test were very much comparable to the pretest. The number of questions and the division of points were the same and the questions often had very similar contexts. This time, however, we expected the students to be able to make the majority of the tasks. The reliability of the post-test was sufficient (Cronbach's alpha = .77). One person coded all the post-tests. Another person coded ten per cent of all the post-tests and the interjudge reliability was 96%. The results of the pre- and post-tests were used to analyze differences in mathematical level raising between the two condition groups.

3.3 Learning Materials

The learning materials consisted of a computer simulation with accompanying collaborative investigation tasks (to be solved with paper and pencil) on the subject of Routes and Probabilities. Any domain of science or mathematics could have been taken into consideration, but this part of probability theory was chosen because it has both a visual and an abstract component. The computer simulation was composed of simple (gambling) games with the underlying structure of a *grid*, a mathematical concept for counting possibilities and calculating probabilities. An example of such a game is TIC-TAC, as shown in figure 1.

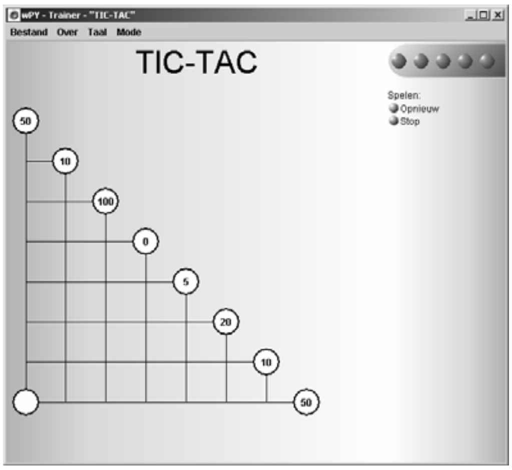

Figure 1. The game TIC-TAC.

Students were asked if they had a winning strategy for the game TIC-TAC, and then they got the investigation task 'What is the probability for a ball to end up in the node with 100 points?'. The idea of this question was that they would realize that the probabilities to end up in a certain box were not equal for all. In the paper-and-pencil task that followed, both students (each drawing on a separate working sheet) had to draw possible paths in the TIC-TAC gameboard from the starting point towards the box with the 100 points. Subsequently the students were asked to describe the paths in terms of RURUUUU (with an R meaning 'to the right' and an U 'upwards') to compare them with their peer and to formulate what struck them. In the following tasks the concept of 'counting routes in a grid' was developed. So the games were meant to create experiences with different grids and the tasks to make students reflect on their experiences with the games and, hence, learn about mathematical grids. The same learning materials were used in both the product-help group and the process-help group.

3.4 Preparing the Roles of the Teachers: Process Help and Product Help

One month before the experiment started, the teachers received the learning materials. They discussed it and gave some useful hints for improvement. During two sessions we prepared the two teacher roles. In the first session the roles of the process and the product teacher were explained. The process teacher provides no content-related help; he stimulates the interaction between students. The initial instruction of process help to all students was:

> I will not help you in content, but I want you to discuss a lot, to show your work to one another, to explain to one another, that's what you learn from, and criticize one another, so that the work improves…

An example of process help to students who asked for help:

> I want you to decide by yourselves, think about it, be critical towards one another's ideas.

The product teacher only provides content-related help when he is asked for it and he gives no hints for the interaction. The initial instruction of product help to all students was:

> You will work on these tasks by yourselves; I am here to assist you.

Two examples of product help to students who asked for help:

> Do you understand the picture?
>
> Yes, 20 % of this, yes, both lines end up in the same box.

The researchers showed the behavior of the two roles by means of role plays. Then we extensively discussed delicate aspects of each role. As a possible difficulty of process help, one of the teachers expected students to find it terrible if the teacher did not provide any explanation. Besides, they mentioned the risk of students not solving the tasks in the correct way. This risk was mentioned for product help too, since the teacher would only help the students if they asked for it. The fact that the learning materials were developed for interaction between students and the fact that the product teacher was neglecting this aspect of the learning process was mentioned as an unnatural aspect of product help. Despite all these possible risks their new roles in the classrooms could bring about, the teachers were very motivated and willing to take part in the experiment. We decided that both teachers should motivate their students to go to work if they were not willing to do this. In the next session we divided the roles and one of the teachers wanted to experiment with process help and the other fully agreed to provide product help. The teachers practiced their roles by means of role plays.

3.5 Data Collection and Analysis

The utterances of both teachers during the experimental lessons were audio-taped and transcribed in order to check whether the help of the teachers was executed correctly. Furthermore it would give us the opportunity to analyze the help of the teacher and the reaction of the students. Did they perform key activities?

One of the researchers attended the experiment, observing alternately in the product group and the process group. The following aspects were observed:

- Did the students interact with each other?
- Did the students ask the teacher for help?

- How did the teacher provide help?
- How did the students react on the teacher's help?
- Did the dyads interact with each other?

Point of attention was the performance of key activities: did the students show, explain, justify or reconstruct their work? The observations were collected and yielded a general impression of the course of the lessons.

At the end of each lesson, the researcher interviewed the teachers during ten minutes. They were asked how they experienced giving process or product help and what reactions they experienced from their students. The answers were written down and used to accomplish the analysis of the teacher's help.

At the end of all mathematical investigation tasks, the students were asked (on paper) what they had learned and whether they wanted to react on the experiment. Their written answers were collected and used to come to know the students' experience of the lessons.'

4. RESULTS

4.1 Course of the Experiment

In general, the students of both conditions experienced the help of the teacher as very different from the normal situation. Although they were used to working independently, they expected the teacher to help them by providing explanations. In the process group, the teacher did not do this at all. With regard to a difficult task in the learning materials, many students arrived at an impasse in their learning process. This task was hard for them and many students were discouraged. The process teacher encouraged them to trust their own thinking. In the product group, the teacher always asked students to show their work before he gave a mathematical hint. For some students, this was new, since they expected the teacher to give explanations when they told him: 'I don't understand.' During the second lesson, many students in the product condition arrived at a difficult task in which they had to reflect on the previous tasks and use the information in a more abstract way. This was hard for them and they asked the teacher for help. He could not help them all at a time and some of them had to wait for a long time. This made the students feel that the teacher did not provide enough help.

4.2 Check of the Teacher Help

By analyzing the transcripts of what the teachers said, we checked whether the two types of help had been executed as we had meant them to. The general conclusion was that this was indeed the case. Both teachers forgot their role only a few times and minimally even then. Both teachers helped their students to keep going. The product teacher did this by giving them explanations, sometimes rather extended ones. The process teacher tried to make his students think for themselves, encouraging them and giving them confidence. He referred to key activities (see section 2.2), especially giving explanations to one another. Sometimes it was very difficult for him to avoid giving mathematical hints, especially when the students said that they both had no idea of what the answer was. Students were concerned about the test and the process teacher reassured them. Both teachers were very dedicated.

4.3 Results of Pretest and Post-test

We analyzed the results of the pretest and the post-test in order to determine whether the students attained mathematical level raising. Each of the tests consisted of 13 questions, with a maximum score of 46 points. As we see in Table 1, the two condition groups were comparable on pretest results (that is how they were constructed) and the mean score of 9.31 points shows that the students had sufficient prior knowledge at their disposal.

Table 1. Mean (M) and standard deviation (SD) of pre- and post-test in both conditions

Condition		Pretest		Post-test	
	N	*M*	*SD*	*M*	*SD*
Process help	26	9.38	4.92	13.46	8.80
Product help	26	9.23	4.35	15.12	7.10
Total	52	9.31	4.60	14.29	8.00

The difference between pre- and post-test in both conditions showed that on average all students' learning results improved (t-test for difference between pre- and post-test: $t = 6{,}367$, $df = 51$, $p < .001$).

We compared the results of the post-test for the process and the product group with a t-test. The test showed that the results of the two groups did not differ significantly ($t = .75$, $df = 50$, $p > .05$). This was confirmed by a covariance analysis with the pretest as a covariate, which showed no significant difference in the two means of the post-test between the two conditions ($F(1,49) = 1.43$, $p > .05$).

The post-test measured level raising for four topics and the total score on the post-test consisted of the sum of the score on prior knowledge, 'Counting routes in a grid', 'Calculating probabilities in a grid', 'Solving Counting problems with a Grid', and 'Solving Probability Problems with a Grid'. Each concept was represented by two questions. In order to have a closer look at the level raising taking place in both condition groups, we analyzed for both condition groups how many students attained the conceptual level for each topic. We counted a student as 'having reached the conceptual level' very strictly, namely when she or he got the maximum score for both questions on the topic. The example in the Appendix shows that the maximum score is given when a student has fully reached the conceptual level. A check of the pretest results showed that no student was at the conceptual level for any topic in the pretest. The results of the 'conceptual level analysis' are shown in Table 2.

Table 2. The number of students who attained the conceptual level for a certain topic in two conditions

Topic	Condition Product	Condition Process
Counting Routes in a Grid	8	8
Calculating Probabilities in a Grid	0	2
Solving Counting Problems with a Grid	1	0
Solving Probability Problems with a Grid	0	1

Nine students in the product condition and 11 in the process condition attained the conceptual level for a concept. In both condition groups eight students attained the conceptual level for the concept of 'Counting Routes in a Grid'. Both groups consisted of 26 students, so only a minority of the students attained this level. The students who attained level raising were average or high-level students. This makes clear that, on average, the learning materials were very difficult for the students in this experiment.

4.4 The Role of the Process Teacher

What did the process teacher do and what did his behavior effectuate with respect to the learning process of the students? In this section we will describe and discuss the role of the process teacher by analyzing episodes from the audiotapes of the lessons; (. . .) indicates that words have been deleted for the sake of clarity.

In the passage below, the process teacher introduced the second lesson. In the introduction of the first lesson he had made it clear that he was not going to help the students with the mathematics. Now he clarifies what he expects them to do.

> I want to make clear to you, my role will be a bit different from usual. So, what it is all about, is that, all the time, you will try to tell one another the meaning of what you did. Show one another what you do, explain it, if you do not understand. That is the object of this collaboration, that you pay attention to that. That means you not only just do what you do, but also explain to one another why you do it. (…) So, if you really disagree, criticize one another. Then you say 'No, that is not correct' or those kind of things and do not hesitate to do so, don't think 'oh, leave it'. If you really think that you disagree, say it. That's what you learn from and, ehm, and then you will see that you will be able to continue. OK.

The teacher *emphasized that he wanted the students to interact*. He encouraged students to show their work, to explain it, and to criticize one another's work. He made it clear that these are activities one can learn from and he put his faith in the students. It turned out that many students got stuck and had nothing to discuss. Although the teacher made it clear that he would not help the students with the mathematical content, they did ask him a lot about the mathematics. How did he react in such a situation? Below we find an example of a typical reaction.

Student:	(…) Here… that must be 0.80, but if we …that 80, no
Process teacher:	No, you have to agree about that by yourselves. He [pointing at peer student] will certainly try to convince you, I guess
Student:	The result is not 100 %, I think 0.8…
Process teacher:	Think about it, see if you can discover the mistake together. I, no, you may, no, you cooperate very well…

The students wanted the teacher to give them a clue and to judge their work. The process teacher *passed back the content-related questions* and he *put his faith in the students' own ideas*. The intention was to stimulate the students to develop their own ideas. Very often this did not happen and students got frustrated, as the following episode illustrates.

Process teacher:	Do you agree with one another?
Student:	No, it is really shit, because if you get no explanation, and you don't know in which direction to go, no, …
Process teacher:	Then you find out for yourselves
Student:	No, after some time, you are really fed up…
Process teacher:	You can do it, you can really do it, I am sure, because if you carefully read the text, then you can do it.

In this passage, the process teacher *stressed the importance* for the students *to come to an agreement about the right answer*. The students were discouraged, since they did not know 'which direction to go'. The process teacher directed them to their own thinking. The students did not want to continue, but the process teacher encouraged them to proceed. This was a very problematic situation for the students, since they really got 'stuck' and they did not have any clue of how to proceed.

In summary, the process teacher stressed the importance of interacting and finding out the right answer by themselves, passed back the content-related questions and put his faith in the students' own ideas. Although the process teacher made very strong efforts to stimulate their own thinking, the majority of the students wanted the teacher to judge their work and give them mathematical hints. In the evaluation of the lessons, the teacher told us: "I found it difficult that I was not allowed to say the things I wanted when they got stuck. I would really have liked to give them subtle hints like 'use the preceding task'."

4.5 The Role of the Product Teacher

We will now describe the role of the product teacher by analyzing his utterances during the lessons. What did the product teacher do and what was the effect on the students' learning process?

Student R:	This is 64, isn't it?
Product teacher:	How did you get that answer?
Student R:	Well, you have five days, 1, 2, 3, 4, 5, and two days off. And here you can choose when.
Product teacher:	But can you… how can you describe that?
Student R:	I don't know.
Product teacher:	Such a week, with two free days, how can you describe that in a practical way?
Student J:	With a tree
Student R:	We have to… Pascal's triangle
Product teacher:	Yes, finally we will come to Pascal's triangle, a tree is a bit long, but imagine that you would make a string, in your day planner or so, if you want to write down that you are free on Tuesday and Friday, how would you describe that?

The students wanted to check their answer. Their answer was not correct. The teacher *asked* student R *to explain* his answer. This explanation was not correct. The teacher's criticism (unlike criticism from peers) did not encourage student R to justify his work. The product teacher tried to guide the students to a higher level by *giving a hint*. The students, however, had some difficulties in understanding this hint.

Student N:	Fred, that string, is it 7?
Product teacher:	Why?
Student N:	Because there are 7 beads and they all have the possibility on 7 places.
Product teacher:	Not all the places can have different colours, the blues are all blue, and the reds are all red. Try it, with your pencil.
Student N:	Uh, I do not understand.
Product teacher:	Whether you transpose two blue ones or not, that does not make a difference.
Student N:	Oh, then it must be 7^6

The students asked the teacher to judge their work. The product teacher *asked* the students *to explain* their work. The explanation of the students was not completely correct. The product teacher *criticized* the part of their work that was not correct and *provided a hint*. The students did not understand the hint and they guessed what the correct answer could be. Subsequently (this is not part of the transcript) the product teacher then *asked* them *to have a look at the previous tasks.* The students realized that they had skipped some tasks and they set to work on them. As we can see, the product teacher *criticized* the students' work without directly telling them the correct answer. He *directed them to the previous tasks* in order to let them make the link and build up the correct answer by themselves.

Student M:	Fred, is this 2 to the power of 8 or not?
Product teacher:	In total? No,…
Student R:	The question is 'How many routes are going to the box with 100 points'

Product teacher:	They say 'Hint: use task 8' What is task 8 about?
Student R:	Yes, then we did this… 2 to the power of
Product teacher:	What do these numbers mean?
Student M:	There are this many routes to this, five…
Product teacher:	How many routes to this point
Student R:	Yes
Product teacher:	What does a route look like?
Student M:	Like this, or like this
Product teacher:	Yes, like this, but always…? [Asks to complete]

As in the previous examples, the students checked their work by asking the teacher. This time, the product teacher *criticized* the answer directly and *referred them to the previous task*. When the students made it clear that they had already considered the previous task, the product teacher *simplified the task by splitting it up in small questions*. He asked the student to complete his sentence, as if it were a little riddle. The product teacher provided very extended, although fragmented, help. From this passage we cannot judge whether student R really profited from this teacher intervention.

In summary, when the students asked the teacher to judge their work, the product teacher made them show and explain their work. When he criticized (implicitly or explicitly) their work, however, they were not stimulated to create their own mathematics and to alter their self-image ('we are no mathematicians'). The interactions between teacher and students were extensive and therefore students sometimes had to wait. In order to provide all students with his help, the product teacher would have liked to give some classical instruction when the students 'en masse' encountered a problem. In general, as the passages above show us, the help of the product teacher took more time than the help of the process teacher.

4.6 Students' Experiences

At the end of all the lessons we asked the students whether they had any hints to improve the lessons. In the process groups seven out of 13 pairs answered that they wanted to have more explanations. One of these pairs mentioned: 'We have learned nothing, because we couldn't ask anything and we only learn if we have made some mistakes and correct them and then make the test.' Although another pair said: 'We have learned better to think for ourselves.' In the product group, eight out of the 13 pairs answered that they wanted to have more explanations. One of them said: 'We we did not learn a lot, yes, we learned to cooperate and to consider the problem in a different way and to look for solutions. We did not get enough explanation and help.'

5. CONCLUSION AND DISCUSSION

5.1 General Conclusions and Explanations for the Results

We investigated what kind of teacher help was most fruitful for students who worked collaboratively with especially developed learning materials and computer simulation. We compared help from the teacher that kept away from the mathematical content and stimulated the interaction process between students with help that addressed the mathematical content of the students' work. It was questioned which kind of help would most lead to mathematical level raising. It turned out that there was no difference in level raising between both conditions. This result differed from the findings of Dekker and Elshout-Mohr (2004). They conducted a similar experiment with a different mathematics task and different type of education, namely pre-university. In their study, teacher help that stimulated the interaction between students and kept away from the mathematical content led to more mathematical level raising for the students. In both studies, students got the same kind of instruction and students were not used to this kind of teaching practices. Students in pre-university education easily grasped the opportunity to solve the mathematical tasks with one another (Dekker & Elshout-Mohr, 2004), while in this study the students in senior general secondary education wanted the teacher to explain the mathematics to them. From the literature it is known that these students were less motivated to perform research by themselves (PRINT, 1998; PRENT, 1999). It is clear that students in this lower type of education are strongly dependent on the teacher's explanations and the teacher's external regulation, while the learning materials presuppose an investigative approach from the students.

In both condition groups, the teachers provided much more help than in Dekker and Elshout-Mohr's experiment. The process teacher encouraged their collaborative investigations and the product teacher took them by the hand with the content matter. Although, the students in this study significantly attained mathematical level raising, the proportion of this level raising was not very impressive: The students improved from 9.31 on the pretest towards 14.29 on the post-test, which had a maximum score of 46 points. So although the teachers made serious efforts, the students did not attain all the learning goals.

Besides the students' difficulties with the learning materials, we question the interaction with their peers. The students were working in couples and this appeared not to be sufficient for a lively collaboration on a difficult subject. They missed new 'input' when they got stuck. The reason for choosing dyads and not groups of three or four students was a practical one: Two students could work with one computer, whereas three could not. A solution for future research may be to create opportunities for peer-groups to share their questions and ideas.

Although the sample size was not extremely large, we do not expect a larger sample size to lead to different outcomes. Furthermore, it might be that different students received or perceived the help in a different way. In future research, it might be interesting to focus on the differences between students' perceptions in relation to the help the teacher provides. Besides, in our study we did not measure the maintenance effects of the lessons.

5.2 Process or Product: How to Proceed?

Analysis of the post-test results did not provide us with a recipe for optimal help. Could the observation and analysis of the lessons give us a clue? We analyzed the observations from the researcher and the teacher interviews with some samples from the audio-taped teacher-students interactions and we came to the following conclusions:

In the process help condition, many students really got stuck when it became difficult and they asked the teacher for help. It turned out that, when students had difficulties with the learning materials, process help did not help to evoke key activities. Both students hardly had any work or ideas to show, so the process of collaboration had already stopped. In the product help condition, the teacher invited students to show and explain their work before providing a mathematical hint. A question or criticism from the teacher, however, does not have the same effect as a question or criticism from a peer. When the teacher criticizes a student's work, a student knows that his or her answer is wrong, while when the work is criticized by a peer a student is challenged to prove that she or he is right. So the product teacher did not evoke the performance of key activities in his direct interaction with the students. What díd happen in the product condition was that students, who were waiting for help from the teacher, meanwhile explained the task to their peers. Thus, product help unexpectedly fostered the performance of some key activities (namely explanations) by students.

When the students were struggling with a difficult task, both the process teacher and the product teacher wanted to provide more explanations than their role allowed them to. The process teacher wanted to give some mathematical hints and the product teacher wanted to insert some whole-class instructions. Nevertheless, the condition-group in which the teacher gave explanations did not perform significantly better than the one in which the teacher did not explain the content matter.

Concerning the possibility to combine process help and product help, we are afraid that if both types of help are offered, students will relapse into their habitual behavior of leaning on the teacher for explanations. Especially since these students do not see themselves as full-fledged mathematicians. So, we think that a combination of both types of help should be considered very carefully.

The question of what type of help works best for students to construct their own mathematical knowledge is not yet answered. We know that although both teachers and students think of the teacher as a source of providing information, this does not increase the students' ability to create their own mathematics. Students may be reluctant to do so, because they are not confident of their own mathematical abilities. Therefore, presenting them with math situations in which they already have acquired a certain level of expertise, e.g. by having them to explain basic math principles to younger students, may help them to gain self-confidence. This, in turn, might help them to rely on their own ideas in their own mathematical lessons and to profit more from the help of the teacher.

APPENDIX

An example of a question on the subject of 'Routes and Probabilities' is shown in Figure 2.

A ball is falling in the pintable. What is the probability that it will fall into the middle box? Show how you arrived at your answer.

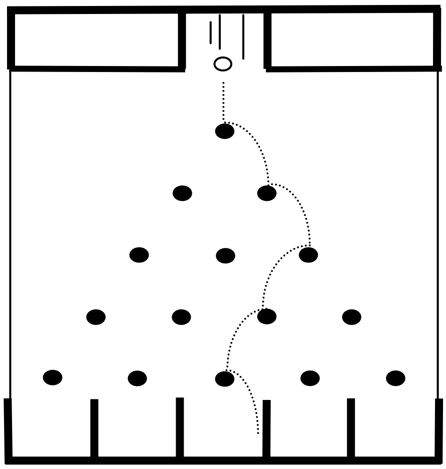

Figure 2. Question on the subject of 'Routes and Probabilities'.

A possible answer to this question is 'There are five boxes, so the probability of falling into the middle box is 1/5'. This answer is in line with the students' prior knowledge, but it neglects the fact that the probability to reach a box is not equal for all five boxes. With respect to the concept of 'Calculating Probabilities in a Grid' it is an answer at the perceptual level. This answer is rewarded with 1 point.

At the conceptual level one can solve this problem by counting routes, as we see in Figure 3. The answer would be 'There are 16 routes to all five boxes and 6 routes to the middle box. The probability is 6/16'. This answer was rewarded with 4 points. Possible answers that were rewarded with 2 or 3 points are 'The probability is more than 1/5' or 'The probability is ½ × ½ × ½ × ½' or 'There are 6 routes to the middle box' (without calculating the probability).

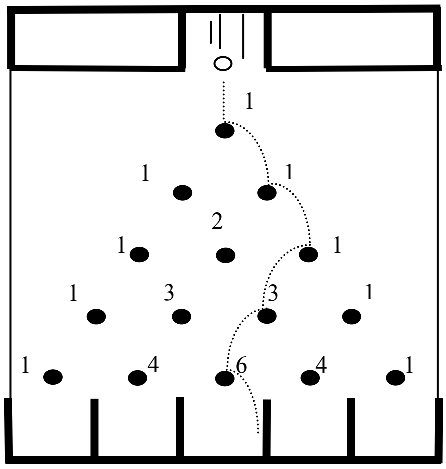

Figure 3. Counting routes.

The previous question was an example for the concept of 'Calculating Probabilities in a Grid'. The pre- and post-test consisted of items for four concepts, two questions for each concept.

Chapter 6

COLLABORATIVE MATHEMATICAL INVESTIGATIONS AND THE ROLE OF THE TEACHER: A CLOSER ANALYSIS OF TWO STUDENTS[1]

We investigated the role of the teacher in the 4th year of senior general secondary education (havo 4) mathematics A course as part of the project on collaborative mathematical investigations using the computer. The students in one of the two conditions were supported only on their interactions (process help) and the teacher provided no technical (mathematical) help. Students had considerable difficulty with this way of working and expected that the teacher would provide more explanation. However, two students also said that they learned 'to think things through more deeply and thoroughly for themselves'. This article illustrates how these two students gained their insights through the process help that was provided. The results were obtained by analysing audio recordings of the students and their written exercises. Both the students and the teacher appeared to find the process difficult. Important aspects for being able to profit from process help seemed to be the students' motivation and metacognitive abilities. The problems and successes of the pair of students and their teacher are followed by various recommendations for supporting collaborative investigation learning in havo 4 mathematics.

1. INTRODUCTION

Two students in an educational experiment in havo 4 mathematics A, in which students collaborated on mathematics tasks using the computer with less support from the teacher than usual, stated that they had learned to think things through more deeply and thoroughly. These students are not representative of the entire student group. The learning results show considerable progress of one and average progress of the other. However, what is striking about these students is the change in their attitude to work. Indeed, the individual construction of mathematical knowledge was a major problem for most of the students. The two students have produced some-

[1] This chapter is a translation of a manuscript which has been submitted to a Dutch journal. Pijls, M., Dekker, R., Van Hout-Wolters, B., & Veenman, M. (submitted). Samenwerkend computerondersteund wiskunde leren en de rol van de docent in havo 4 [Collaborative investigations using a computer and the role of the teacher in grade ten (havo 4)].

thing fundamental that the entire group found difficult. We describe in this article how this change in the attitude to work came about. We wish at the same time to describe the resistance experienced by even these two students and their teacher the first time the teacher refrained from explaining the mathematics to the students. The aim of restoring the students' momentum through their own efforts was ultimately successful. What factors contributed to the success of these two students, but not the others?

1.1 Independent Investigations in Mathematics

The aim of self-regulated learning is to provide students not only with knowledge, but also the resources to gather knowledge for themselves. When students collaborate and learn from each other, the primary role of the teacher is to maintain momentum in the learning process. Self-regulated learning in practice would appear to pose a problem for some students and their teacher.

For instance, that havo 4 students in the 'Economics & Society' and 'Culture & Society' profiles who took part in the first experiments on *practical assignments* in the 'Studiehuis' (PRINT, 1998; PRENT, 1999), found it hard to use the computer for independent investigation. This result is in contrast with those of the 4^{th} year pre-university (vwo) and havo 4 Mathematics B students. In other subjects too, the investigation process of havo 4 students has not met with unconditional success (Rijborz, 2003). With specific reference to mathematics investigation, there was a suspicion that the existing computer simulations on mathematical subjects were too abstract. The mathematical models concerned frequently had too little relevance for these students, and therefore provided insufficient freedom to manoeuvre for investigation. In order nonetheless to facilitate investigation learning in mathematics A for havo 4 students, we developed a computer simulation some years ago in which students were also able actually to play a number of games that provided an opportunity for experimentation (Pijls, Dekker & Van Hout-Wolters, 2000). The aim was to make a subject from probability theory more accessible. Assignments were developed to accompany the games in which students were given an opportunity to reflect on and learn from their experiences in the simulation. We compared various versions of this learning materials in a study in 2001 (Pijls, Dekker & Van Hout-Wolters, 2003) and the results were used to refine the tasks. An investigation was then carried out to establish which type of teacher assistance was the most instructive for students working in pairs on investigation tasks surrounding the computer simulation.

1.2 Research and Question of this Article

Learning performance was measured in before and after tests to establish the *mathematical level raising*, defined as the transition from the perceptual level to the conceptual level (see Section 2.2). The tests consisted of 12 open mathematics questions that could be approached on a variety of different levels. The pairs were chosen with a view to a likely productive learning collaboration, in particular by pairing students who achieved different levels in the before test and who could get along well together. The assistance provided by the teacher was classified into two types, by analogy with a study by Dekker and Elshout-Mohr (2004) on vwo 5 mathematics B. The conditions concerned were *product help* (help provided by the teacher oriented to the mathematical product of students' work) and *process help* (help oriented to the interaction between students). Process help was expected to lead to more mathematical level raising on the part of the students, as with the students in vwo 5 mathematics B. The havo 4 mathematics A results present a different picture (see Pijls, Dekker & Van Hout-Wolters, in press). The students in both conditions attained mathematical level raising to the same extent, and had similar expectations of more explanation from the teacher than was forthcoming when they asked a question. The teachers in both conditions stated afterwards that they had felt inclined to give somewhat more explanation. A striking point is that despite the request from students for more explanation and the teachers' tendency to explain, the level of the students in the condition in which more explanation was given did not increase more. However, the mathematical level raising was only modest in both groups. All the efforts notwithstanding, these students therefore still had difficulty in acquiring mathematical knowledge by themselves. The tendency to view mathematics as something that is explained by the teacher proved obstinate to eradicate, even though more explanation from the teacher did not lead to greater comprehension. The aim of having students construct knowledge independently and thus acquire the concepts relevant for the final examination was not met. This group of students therefore forms an interesting example of the potential conflict between individual constructions and examination requirements, as has been raised in the discussion surrounding New Learning (Wubbels, 2006).

Both the students and the teacher experienced resistance. The aim nonetheless is for the teacher to establish momentum in students' learning processes. At the same time, teachers and students become accustomed to the situation in which the teacher's role is to explain.

We study in this article two students who did state at the end of the project that they had learned to 'think things through more deeply and thoroughly'. Two grains of gold in the desert of troublesome learning. We illustrate how much difficulty these two students and their teacher had in making process help effective (i.e. inviting students to think things through more deeply and thoroughly for themselves). Looking in detail at these two students' and their teacher's learning processes also enables us to suggest why independent investigation learning does not work well for many students.

2. THEORETICAL FRAMEWORK

2.1 Supervising Collaborative Learning.

Collaborative learning can give students the freedom to construct and develop their own ideas (Van der Linden & Renshaw, 2004). They support each other in this process when their ideas differ. Students are able to refine their own ideas in an instructive dialogue. The primary function of the teacher in this process is to ensure that momentum is maintained in the learning process. Various types of assistance in collaborative mathematics learning were investigated.

For instance, Kramarski, Mevarch and Arami (2002) found that metacognitive instruction[2] given to students engaged in collaborative learning led to higher learning results than 'ordinary' instruction on the task itself. Hoek (1998) identified a positive impact on learning results for the instruction of both social and cognitive activities, but no additive effect of the two.

Wood (2001) emphasized that teachers must remain detached from the discussions between students, so that students are truly able to develop their own knowledge. It is important in this regard for students to be allowed to struggle with the problems and to question their own and each other's ideas. It is no easy matter for teachers not to intervene in cases of this kind, because it goes against the grain of the (very human) tendency to explain and assist by taking on some of the burden of the problem. Furthermore, the fact that several different groups are working in a class makes it is impossible for a teacher to hear what all the students have said, and therefore to give every student a hint on his or her own level. A solution to this problem suggested by Brodie (2001) is for the teacher to aim more for general reasoning processes than for the task itself.

The distinction between supporting general processes or assisting in the task itself is also drawn by Marzi (2003) and Veenhoven (2004). Marzi (2003) investigated the difference between primary school pupils who collaborated under the supervision of a teacher who assisted them technically when they asked, and other pupils under the supervision of the teacher who remained detached from the substance of the tasks, but did help them in planning and structuring the task. It would appear that the second form of help from the teacher led to more explanation and a theoretical approach on the part of the pupils than the first. Veenhoven (2004) investigated which components of the teacher's conduct contributed to the pupils' investigation skills. Investigation was carried out in havo 4 and vwo 4 Geography practical assignments. It emerged that pupils who collaborated under the supervision of a

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[2] *The form of metacognitive instruction used was the IMPROVE method, which uses four types of metacognitive question that students were supposed to address to themselves, being comprehension questions ('What is the problem?'), connection questions ('How is this problem different from/similar to what you have already solved?'), strategic questions ('What methods can be used to solve the problem?', 'Why is this method appropriate for solving this problem?, 'How can I organize the information to solve the problem?') and reflection questions ('What am I doing?, 'Does what I am doing make sense?')*

teacher who concentrated on the performance of investigation produced results no better than those of pupils working under the supervision of a teacher who concentrated on the interaction. The two studies demonstrate that if a teacher refrains from intervening in the substance of the task, but concentrates instead on the interaction between the students, the learning results achieved are the same, if not better.

Hoek, Seegers, Gravemeijer & Figueiredo (submitted) conducted developmental research into the role of the mathematics teacher in upper secondary vocational education (mbo). They encountered enormous resistance on the part of students and teachers when the teacher withheld technical assistance. The solution opted for was to hold class discussions in which students were given an opportunity to present their work. Several lessons went by before the teacher succeeded in refraining from technical instruction in the class discussions and allowed students to present their own work. When the teacher finally succeeded, the result was a more investigation-oriented attitude on the part of the students, as evident in closer collaboration between students and more explanation to each other using the graphical calculator.

2.2 Mathematical Level Raising

'Learning mathematics' is operationalized in this research as *mathematical level raising* (cf. Van Hiele, 1986). In other words, if a subject is new for students, they approach it on a perceptual level, in accordance with their prior knowledge or naive knowledge of a given subject. By reflecting on their own and each other's work, they build up structures and knowledge and will start to approach the subject on a conceptual level: the level will then have risen. For instance, upon seeing the game board in Figure 2 (Section 3.2) students may consider that the probability of ending up in the middle square is 1/9, because a ball may end up in nine different squares. This is an approach on a perceptual level with respect to the 'Routes and Probabilities' subject. Students know on a conceptual level that the probability of ending up in the middle is greater than ending up at the edges.

Dekker and Elshout-Mohr (1998) developed a process model in which they describe key activities in the collaboration between two students that contribute to raising the level. These key activities are: *showing* their own work, *explaining* their own work, *justifying* their own work and *reconstructing* it. All these activities are connected with reflection, either because they prompt reflection (showing and explaining), or because they are the result of reflection (justifying and reconstructing). Students can jointly invoke these activities by asking each other to present or explain the work, and offering each other criticism.

The question is how we can prompt students to reflect, or more specifically to perform the above-mentioned key activities. A crucial point in the above is appropriate learning material. Dekker (1991) established four criteria for collaborative learning tasks. The first is that the tasks must be *realistic* or meaningful for students. The material must also be *complex*, so that students need each other. It is also important for students to have to *construct* something, so that their thinking can be made visible and there is something to talk about. Finally, the tasks must be oriented to *mathematical level raising*.

2.3 Process Help and Product Help

When the learning materials and the computer simulation allow students to learn independently and with each other, the teacher's role can be minimal. Dekker and Elshout-Mohr (2004) investigated which type of teacher help led most effectively to raising the level of vwo 5 mathematics B students. The pupils worked in heterogeneous groups of three on tasks related to geometrical transformations. The impact on raising the level of the following two types of teacher help were compared:

1) product help: teacher help oriented to the mathematical product on which students work (mathematical hints);
2) process help: help oriented to the interaction between students, in particular the occurrence of the core activities.

It appeared that the students who received product help while working on the tasks made more products on a conceptual level. Nonetheless, the level of students who were given process help was raised higher. Dekker and Elshout-Mohr say that the teacher that provided process help only sustained the collaboration and thus created an opportunity for reflection, but the process help did not interfere with the students' thinking.

The teachers in the Dekker and Elshout-Mohr investigation held no classroom discussions. The focus was purely on examining how the students used the learning materialsand each other to build up their own knowledge and attain mathematical level raising, and the best way for the teacher to supervise the process.

As mentioned in the introduction, the authors executed a similar investigation into the impact of process help and product help on mathematical level raising for havo 4 mathematics A (see Pijls, Dekker & Van Hout-Wolters, in press). The students in this investigation worked in heterogeneous pairs on investigation tasks with a computer simulation on the subject of probability theory. There appeared to be no difference in the increase in level between the two conditions. The students in both groups requested far more help and also expected more explanation of the teacher than their vwo 5 counterparts in Dekker & Elshout-Mohr (2004). It is surprising from this perspective that the condition in which the teacher explained more fared no better than the one in which the teacher remained detached. As remarked, the independent investigation of a mathematical subject remains difficult for these students. The learning processes would not appear to gain momentum well. Process help appeared not to lead to deeper assimilation and a higher level, so that there was no sign of the differentiation between process help and product help that was apparent in vwo 5.

The subjects of this study were two students who received process help and who responded well to the teaching method, and the aim was to gain insight into the teacher's role in independent mathematics learning for havo 4 mathematics A students. These two students were conspicuous because they stated that they had learned to think things through more deeply and thoroughly. The levels of both also increased, albeit not to the same extent.

3. METHOD

3.1 Selection of the Students

The students mentioned in this article, Roos and Stella, attend a Montessori Lyceum, where students are accustomed to independent learning. They were selected because they had exhibited a change in their attitude to work, which was absent in the other students. Figure 1 shows how they themselves viewed the lessons.

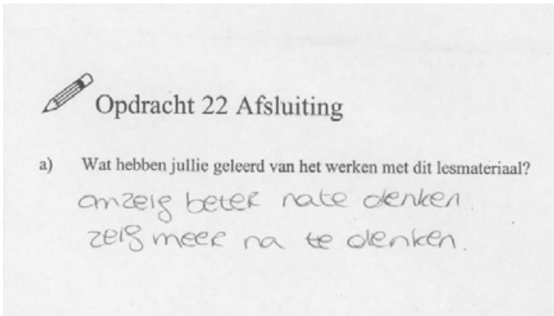

Figure 1. The answers given by Roos and Stella.

[Translation: Question 'What have you learned from working with this learning material?'
Answer: 'Better to think things through for ourselves.
More to think things through for ourselves.']

Roos and Stella are friends. They usually sit together in the mathematics lessons given by Albert, the teacher, who now provides them with process help. They were brought together in the light of somewhat variable results in their before tests (in which a maximum of 46 points could be achieved). Roos had 8 points, which is average, and Stella had 15 points, which is high. We expected that a certain difference in level would lead to different answers and thus to interaction between the students, prompting us to make these students a pair. Another factor in the composition of the pairs was whether the students could get on together well, which these two could. The overall impression of their effort in the task book is excellent. They attended all the lessons.

In the post-test (in which a maximum of 46 points could be achieved) Roos had 11 points and Stella 28 points. Stella thus achieved the second highest result of the class and also a substantial increase in level. Roos advanced only little in the after test, which, however is average for the group. Furthermore, the exercise book kept by Roos and Stella and the audio recordings showed that in her learning process

Roos' level in this subject did initially increase. She built on her own prior knowledge and compared her answer on a perceptual level with that of Stella on a conceptual level. This article does not address the results of the post-test and the differences in them in detail. The focus is rather on the change in attitude to work brought about in these students, and how it came about.

3.2 The Learning Materials

The learning materials consist of several games of chance on the computer with associated investigation tasks. The tasks were oriented to teaching students the concept of *counting routes in a grid*. This is an approach to solving certain counting problems, including the following:

> Joost has 5 blue and 2 red beads.
> How many different ways can he thread them together in a chain?

A grid is a way of visualising the blue beads in steps upwards and the red beads in steps to the right. Joost's counting problem is thus equivalent to asking how many routes exist from the starting point S to the cell with 100 points. Each route consists of two steps to the right and five steps upwards. The object therefore is to count all the different routes. Students find using a grid as a model for solving counting problems difficult. They generally find it fairly easy to solve a counting problem with a grid, but the difficulty is identifying which problems are and are not amenable to solution with the grid.

The learning materials first derive the counting of routes in a grid. Students in the first tasks play trivial games of chance on the computer, which have the grid as underlying model. An important game is TIC-TAC, the screen of which is shown in Figure 2.

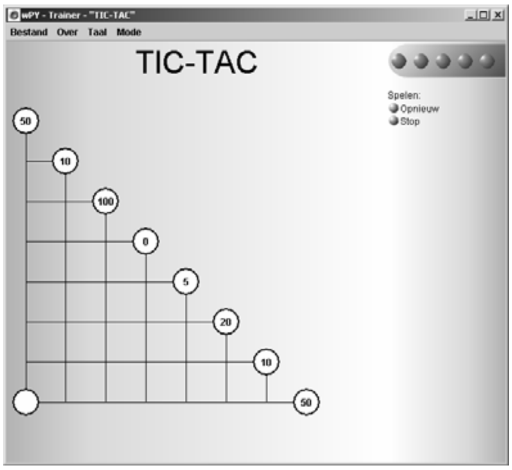

Figure 2. The TIC-TAC game board.

If you click on the bottom left, a ball appears that seeks a path to one of the round cells marked with scores. You win this number of points. The line is shown in red for each 'step' to the right and blue for each step upwards. This creates a route consisting of blue (B) and red (R) steps. The objective of the game is for students to comprehend that there are more different routes to the cells in the middle than to the cells at the edges.

After familiarizing with the game, a pen and paper task is given in which the students, each on their own worksheet, are required to draw multiple routes to the cell with 100 points. They identify the routes in a form like BRBBBBR (where B= upwards and R = to the right) and they compare with each other the various possibilities they have found. They then experiment with different game boards, each with the same underlying grid. In the process of performing the task, they develop the concept of counting routes in a grid. The process is extremely gradual and is easy for the students to follow. The moment then arrives when students are invited to connect the model with the solution of counting problems, like the example above. They are first requested to solve Joost's counting problem. This is a problem that the students are able to solve systematically using their prior knowledge of tree diagrams. However, it is a very complex problem to solve in this way. The students are then asked to play the TIC-TAC game again and to see how to use route counting in the game to solve Joost's problem. We had expected that students would spot the analogy between R= red and R= right and thus understand the connection between routes and probabilities in a counting problem. This did not happen, however. Indeed, students invariably come to a standstill in this task.

Later tasks covered counting problems that can be solved with a grid and the calculation of probabilities with a grid.

The structure of the methodology and the sequence of the tasks is such that the students working in pairs can increase their levels independently.

3.3 The Role of the Teacher

The teacher who participated in the investigation has taught at the Montessori lyceum for many years. He was keen to experiment with this new way of supervising students. It was not natural for him to provide process help exclusively. Product help is more common in everyday school practice. This form of mentoring was practised by means of role play in a number of meetings. There was also some discussion of the teacher's thoughts on the possible pitfalls of process help. The first to be raised was that students might wrongly develop, or fail to learn, a given concept. The teacher was willing to accept the risks of this educational experiment.

The process help was structured as follows. The teacher gave instructions at the start of the lesson: "I will not be helping you with the actual maths. I want you to talk together a lot, to show each other your work, and to explain things to each other, which is how you will learn. Offer each other criticism, so that the work improves." When students were working together and requested help the teacher would say: "I want you to decide for yourselves, to think about and to be critical of each other's ideas."

3.4 Data Gathering and Analysis

An initial insight into what happened in the lessons was obtained by studying the students' exercise book. The criteria used were:

- Did the students work carefully?
- Did the students formulate their own ideas?
- Did the students improve their answer?
- Did the students reach the conceptual level?

The audio recordings of the lessons of Stella and Roos were then played back, summarized and partially transcribed by the researcher. The criteria used were:

- the student's attitude to work;
- the interaction between the students;
- the interaction with other groups;
- the interaction with the teacher.

A picture was built up in this way of how the lessons proceeded. We then investigated which elements were important in the lessons for the change in attitude that occurred with Roos and Stella. We describe this underlying theme below.

4. DESCRIPTION OF THE LEARNING PROCESS OF ROOS AND STELLA

4.1 First Lesson

The students are able to cope with process help, and when they encounter a mathematical problem they can solve it for themselves.

Regarding the *learning material*, they work in this first lesson with several familiarizing tasks on the computer: playing games, followed by pen and paper exercises with which they can explore the underlying structure of the game (see 3.2). The students become familiar with the computer program.

Roos and Stella both have an active *attitude to work*. Their written tasks show that they worked carefully, they experimented and made notes. The audio recordings reveal that they comprehended the questions well, took considerable time to experiment with the computer simulation, and that they thought through what they were doing. When Stella says that she doesn't like an task much, they move on immediately. They are therefore well motivated. A striking aspect of their working methods is that they strike a balance between doing and thinking. They maintain the momentum, while thinking about what they have done.

We can also hear evidence of active *collaboration*. They both formulate their ideas, listen to each other attentively, are critical and augment each other.

A *different group* asks which task they have reached. Therefore, the various groups also keep track of each other.

We see an exploration of the teacher's role in the interaction with the *teacher*.

Stella is unsure about calculating the probabilities and wants to ask the teacher.

1 S: Are we allowed to ask you some questions? …
2 t: I, no, you can, no you are working well together, carry on as you are, you can find out everything yourselves.
3 S: Yes, but are we also allowed to ask you some questions
4 t: What do you mean, what do you want to ask me, then?
5 S: Well, in b) here, we don't really understand how we have to calculate that
6 t: Oh yes, no, but that's right, try to think of something together where you can say something about it.
7 S: Yes, we have thought of something, but…
8 t: Well, if you think that's the right answer, that's fine as far as I'm concerned. If you agree with each other that that's your answer, you have to write it down.

The students are clearly not accustomed to a mathematics teacher giving them no hint whatever on the calculations to be performed. They appear to be unsure how to deal with the fact that they are receiving no help with their calculation. At the same time the absence of explanation from the teacher does not yet completely stall the students, because the tasks are not prohibitively difficult. We can observe the teacher wrestling with giving an answer without addressing the mathematical content of the question. He emphasized that the students themselves had to find a solution, and tries to encourage them.

4.2 Second Lesson, First Part

The students are still able to cope with process help when working on the tasks.

They work in the first part of the lesson on the tasks in the *learning materials* in which the foundation is laid for the concept of route counting in a grid. This involves many 'do tasks' with a direct relationship with the computer simulation. They are also requested to perform calculations for determining probabilities.
The *collaboration* between Roos and Stella is still intense, they work together extremely actively, and when they do, they regularly show each other their work, although there is little explanation or discussion. They talk a great deal in terms of 'you have to do this', or 'you have to do that'.
When they got stuck on a particular question, Stella asked the *teacher* for help.

1 S: Count the number of routes from the cells with the same colour. Should you do it in the normal way? Is it 1, 2, 3…
2 t: Do these have the same colour?
3 S: Ah, I don't mean those, I mean these… 1, 2, 3, 4, 5, 6…
4 t: What do the numbers you have filled in mean?
5 R: Oh, do you have to add the numbers together?
6 t: Yes, at least, …
7 S: Oh, yes, there happen to be five ways of getting to here.

8	t:	Do you understand? Between the two of you, you can also get there, because you had already realized that, as far as I can see.
9	R:	Yes, so 1 + …. 4 + 4 … = 8
10	S:	And this is 1 + 4 … 16
11	R:	7 + 8

There is still a tendency when things get really tough to ask the teacher for help. And the teacher accidentally gives a few small hints that are actually outside the scope of process help, because they have something to do with the content. Roos then formulates the answer. The students exchange ideas very actively to move towards the right answer, except they do not really ask themselves why. The teacher Albert says to them: “Between the two of you, you can also get there” thus emphasizing the students’ own contribution.

4.3 Second Lesson, Second Part

Roos and Stella are able to employ much of their own prior knowledge in the task, and put considerable effort into working on an task to prepare for the concept of counting routes as a model.

Stella reads the task in the learning materials on the counting problem of Joost’s chains (see § 3.2). This is a complex counting problem, which they can solve using with their prior knowledge (probability trees and systematic counting). This task is intended for future reflection, when it will help the investigation that the problem can also be solved with a grid.

The two students each suggest their own solution method. Each of them has a worksheet of their own, and are thus explicitly invited to start by formulating their own answer. They subsequently compare their work with each other, as requested in the tasks. Roos approaches the question on the level of their prior knowledge (‘then we will know for certain’) and draws a large tree diagram (see Figure 3). Stella has an inkling that the task would be enormous and starts looking for a different approach. She counts systematically, out loud, the places where the blue beads can go. She tries to remember a formula for counting the possibilities. Stella’s thinking is mainly evident on the audio recording, but there is little to be seen on paper. On the other hand, Roos runs out of space on her own sheet and draws part of her tree diagram on Stella’s sheet.

They put a great deal of energy into this task and both look for the right answer (‘what is the number of possibilities’) and a satisfactory solution method. In the process, they constantly show each other their work and are critical of both themselves and each other.

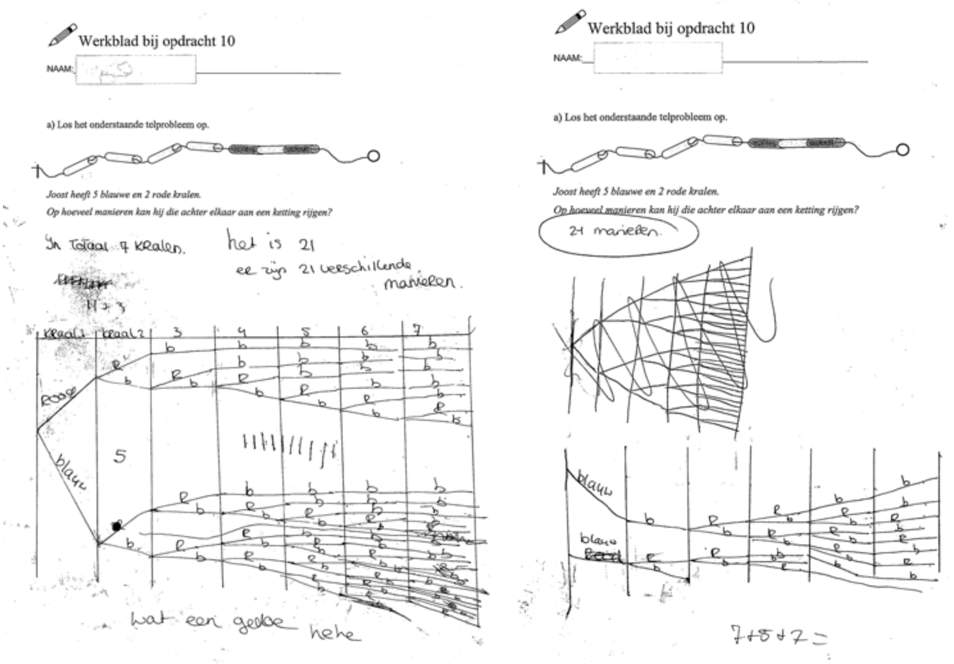

Figure 3. The answer to question 10a) from both Roos (left) and Stella (right).

4.4 Third Lesson, First Part

The students are stuck and don't like the fact that the teacher will not answer their technical question.

At the start of the lesson Stella works alone on the tasks, because Roos has not yet arrived. She is stuck on question 10d), which is:

> How can you use question 9a) to solve the counting problem of Joost's chain?

And question 9a) is:

> How many routes go to the cell with 100 points?

This question is therefore about the connection between the TIC-TAC game and Joost's chains. Stella does not understand the question. She asks the teacher for help, who says she has to solve the problem herself, pointing out that it is difficult for her to find the answer alone, and suggesting she should discuss it later with Roos. Stella sounds out another group to see how they answered this question, but this does not help her to proceed. Roos enters. They tackle task 10d) together. They are unsuccessful, and cannot comprehend what the counting problem is, start talking about something different and consider abandoning the task. They then ask the teacher for help.

1 R: Can I ask a question: is this for the final mark or something, what is this?
2 t: No, you should try to learn from this.
3 R: But why can't you help, then?
4 t: Now, it is important that you find the answers yourselves, because then you will understand it properly, and the test, which is right at the end, is what you will get a mark for.
5 R: Yes, but
6 t: Yes, but then you have to give each other a lot of help and then you have to read the text very carefully
7 S: Yes but, we are doing that, if we don't know, then we just don't know, right?
8 t: Yes, but the idea is that you keep thinking about it, again and again.
9 S: Yes, but, look, but yesterday, at the end, all we had done is this.
10 R: Yesterday we were on question 8
11 t: Fine
12 R: and we have now ended with question 10.
13 t: That's not bad at all.
14 R: Yes it is. Last time we got from 1 to 8.
15 t: No, no.
16 R: Now, I think it is far too slow. But what is, we don't know how yet… there just isn't any counting problem,
17 t: Oh, …
18 R: What is the problem then?
19 t: Here, this is the problem, I now see…
20 S: That five and two, that there are five blue and five red
21 t: No, no, there's a question mark there, so this must be the problem. Do you understand?
22 R: Yes but how many ways are there of arranging them after each other.
23 S: Yes but, that is impossible with this.
24 t: Now, try to work it out for yourselves… what that has to do with that.
25 S: Yes, that's what I have been doing.
26 R: Hey, wait a minute, I get it, these are also 21 routes, and this has 21 ways, so, that is the same

Because they know that the teacher is not supposed to help them, Roos asks him for something other than direct help on the task. She asks the teacher whether the tasks are assessed, and whether they will be given marks for them. The teacher says no. He says that these tasks are intended for them to learn from. Roos asks why the teacher is not allowed to help if they learn that way. Apparently 'learning for yourself without the help of the teacher' is associated with 'being assessed', and 'learning' is likewise associated with 'being helped', or rather 'the teacher explaining things'. The teacher's answer was that it is important for them to find out for themselves, because then they will know it properly. We can also detect signs of unease in his attitude. The students say that they are having no success in finding out for themselves, that they are stuck and that their progress is too slow by their own standards.

The teacher continues to motivate and encourage. We see here that that they are well capable of monitoring their own pace of work. These students are really accustomed to planning their own work.

And then Roos tries to add something about the content and to ask 'What is the counting problem, then?' and the teacher makes a slip of the tongue 'Here, this is the problem, I see now.'

We can see here that Stella does not see the connection between Joost's chains and the routes in a grid. Roos does express the connection: 'By making routes to, I don't know, by making routes to red, to five times red and two times blue, no, five times blue and two times red.'

4.5 Third Lesson, Second Part

The students have great difficulty with having no explanation from the teacher and the teacher's persistence in his role.

Roos and Stella are working on a question on how Pascal's triangle can be used to solve a counting problem on free days in a week. They cannot answer the question and both of them are convinced neither of their own answer, nor that of the other person. They decide to step back from the question, and take a break. After the break they summon the *teacher*.

1 R: Albert, could you tell us one thing? (…) How can you apply Pascal's triangle if you have two days free, which is the same principle as five blue and two red, but we get a different answer.
2 t: Well, you have to decide together which of the two is right.
3 R: Yes, but they are both right.
4 t: So put both of them down.
5 R: Yes, but that can't be right.
6 t: Yes, but you can't, can't you decide between yourselves?
7 R: Yes, but, is it all right to do this? To take the sequence of 7 and then for example to have 128 ways.
8 t: I don't quite understand what you are saying now, but you have obviously thought about it together, why all that?
9 R: This all together.
10 t: Oh that, yes, that's what it's about.
11 R: Ohrrrr (groan), it is so irritating when someone won't tell you whether it is right or not.
12 t: Yes but, I won't. I'm not going to read it, you two have to read it. You have to decide together.
13 R: Yes, we have read it, but it can.. it is so strange that you get a completely different answer here from there, although we made a whole tree diagram here and it's right, and that's right, too.
14 t: Have you also thought carefully about what those numbers mean?
15 S: Oh, it's not right at all
16 R: Oh

Then the teacher goes away. He keeps in contact with the students without telling them the right answer. They can express their doubts and at the same time confidence is maintained in their ability to answer the question themselves. It is also clear that these students continue to think and not be fixated on hints from the teacher. The teacher has got into his role well. He supports the students at the same time as remaining detached from the content. As soon as the students are on track again (rule 15) he withdraws immediately.

4.6 Third Lesson, Third Part

The students arrive at an answer together and are aware of their ability to arrive at a satisfactory answer.

Stella and Roos are working on the use of Pascal’s triangle in a counting problem on distributing two holidays in a week.

1 S: This one is not right at all, the other one is right, because look…you can’t go to school, here, here, here, here, here and here… you can’t go to school for seven days.
2 R: No,
3 S: So it’s not right at all
4 R: So it is simply 21, so you don’t have that triangle at all, you can’t do it with that triangle… or do you have to add these together, let’s see, what would you add together then? The ones you should add together would be…
5 S: Now simply, go on five days, don’t go on two days, you can do it all like that, then you always get 21 days. Go, don’t go, go, go, go, don’t go, go. Like that, you always get… ok,
6 R: And how are we going to formulate that?
7 S: We just don’t formulate it. Ahem…
8 R: Every time we ask him something, a little later we say ‘oh yes’, that’s stupid, we won’t ask him any more questions, we find out for ourselves.

Stella explains Roos’ answer and uses it to reconstruct her own answer. They continue to puzzle and Roos becomes aware that they made progress by thinking things through for themselves. Soon afterwards the teacher asks: ‘Now, has the euro dropped?’ ‘Yes, two actually’, Stella says. This is a turning point in their learning process. They became aware that thinking things through for themselves, and reconsidering a problem from a different angle, enabled them to arrive at a solution.

4.7 Third Lesson, Fourth Part

The students discuss their various answers intensively and that the lecturer maintains contact with them.

Stella and Roos continue with task 13, which applies the solution of counting problems using a grid or Pascal’s triangle. Their work differs and a discussion ensues. Stella has achieved the conceptual level and Roos is making a start on increasing the level in the subject ‘Grid as a model’.

The teacher comes along and asks ‘Are you back on track again?’, ‘Yes’ they reply. The teacher maintains contact with the students, even now that they are not asking for his help.

4.8 Fourth Lesson

The students are working independently, but can still use some process help.

In this last lesson the students worked independently. They work on a number of tasks with the computer on a new topic. They are actively engaged, establishing links with what they learned earlier. They are critical of their own and each other’s work. They no longer ask the teacher for help. When they have difficulty or something is unclear, they have an air of indifference. Whereas in the first lessons they asked for clarification from the teacher, they no longer do so. They thus occasionally lack a crucial step towards the answer and therefore a part of the theory. Students miss the criticism of each other’s contribution. In that respect, they can still use a degree of process help.

4.9 Summary of the Findings

If we survey the lessons we see that the first lessons were concerned with preparing for learning. Many activities and experiments were initiated that were reflected upon later. The teacher’s role was also explored. Both the students and the teacher had to become accustomed to the teacher providing no mathematical help. The activities in the second lesson develop in depth, and the teacher has a modest role in the background. The third lesson introduces the ‘difficult learning’. The students do their utmost to explain the situation to the teacher. The teacher constantly plays the question back to the students and he supports them in puzzling the answer out for themselves. The students also actually move one step further. They realize this at some point, and decide to rely on their own and each other’s ideas. They have learned something: to think things through more deeply and thoroughly. In doing so they are critical, of themselves and each other. In the fourth lesson they continue to work on tasks on a new topic and they are less critical.

5. DISCUSSION AND RECOMMENDATIONS

Two students in an educational experiment in havo 4 mathematics A, in which students collaborated on mathematics tasks using the computer, while receiving less support from the teacher than usual, stated that they had learned to think things through more deeply and thoroughly. These students were not representative of the entire student group, but they exhibited a change in their attitude to work that we consider essential for learning mathematics, and a good indicator of where the other students ran into difficulties. We have shown with this article how that happened. The results are not specific to mathematics. It is about ‘learning by doing’, learning by wrestling with problems, difficult and laborious learning.

After a smooth start in which the students explore the learning materialsand the new role of the teacher, they ran up against a number of technical difficulties in the tasks. This made them feel more dependent on the teacher's explanation and it was very frustrating for them when it was not forthcoming. It was also difficult for the teacher to leave the students to wrestle with the subject matter at a time when they had appeared to have run out of ideas. Teacher and students persisted and eventually the students became aware that they were able to proceed without help from the teacher.

What was it that enabled these two students to proceed so far? The first thing to notice is that their *motivation* is considerable. Without doubt, the fact that they are friends plays a role, but so does their task orientation. The majority of what they say to each other during the lessons is about mathematical tasks (approximately 20 minutes off-task in all lessons). Also, if both of them reach a point where they have had enough of the tasks, they soldier cheerfully on. Finally, it would appear from the way in which (in the first part of the third lesson) they tried to coerce the teacher via a detour to give them a hint anyway, that they were doing their utmost to arrive at the right answer. A role also appears to be played by the assumption that they can arrive at the right answer by questioning persistently and having the teacher answer them.

In their collaboration they sustain a dialogue, and frequently think out loud. They feel sufficiently at liberty to be critical of each other's work and at the same time to be critical of their own work. They show each other their work, they explain to and criticize each other, and they justify or reconstruct their work. We also see that the students use their prior knowledge and thus create an opportunity to build up genuinely new knowledge. The students keep an eye on their rate of working, stand back from an task in order to be able to make progress, and sometimes, conversely, continue to work on an task in order to be sure to arrive at the right answer. In other words, these students are well able to regulate their own learning process, which is to say they have *metacognitive skills*. Veenman, Prins and Elshout (2002) have demonstrated that metacognitive skills are more important than intelligence in a complex learning environment in which students themselves build up knowledge by experimenting (inductive learning). What the two students in this survey do is what Bereiter (1985) referred to as 'bootstrapping'. He was referring to breaking the 'learning paradox', which is the dilemma that you cannot learn something yourself if you don't know anything about it.

Dekker, Elshout-Mohr and Wood (2006) show in an analysis of a collaborating pair in arithmetic at primary school, how these pupils succeed in regulating their collaborative learning. The process is entirely comparable with how this pair works. The difference in the primary school situation is that the teacher pays constant attention to the standards for the collaboration and the discussion of solutions. This is perhaps something that is easier to achieve in the primary school where the teacher spends all day with the schoolchildren than in the maths lesson in a secondary school (see also Gravemeijer, 1995).

What was the *role of the teacher* in all of this? What the students run up against is that they are accustomed to asking the teacher for help when they get into difficulties. The discussion seems to pick up mainly when the students' work differs. This usually indicates a difference in level, and that one is approaching the problem on a perceptual level while the other is approaching it on a conceptual level. It is difficult if they both approach the question on a perceptual level and are not capable of reflection. This situation did not often arise with this pair, and when it did, they called in the help of the teacher. It is a struggle for the students to get used to the teacher not helping them technically, and letting them puzzle on. They observe that not all the tasks proceed at the same pace. Another new aspect for them is that no one tells them whether what they are doing is right or wrong. The fact that the teacher does maintain contact with the students, despite not helping them technically, was very important. Support from an expert is crucial for pupils. Students may be motivated to continue with the mathematics tasks if a mathematics teacher tells them that they are working well and can do it themselves. This is because the students are aware that the teacher possesses the knowledge they wish to acquire, and he also knows how to acquire this knowledge. When the teacher encourages the students to continue as they are, he is actually also telling them which is the right attitude to work: in this case continuing to puzzle for themselves and building on their own ideas. He thus stimulates them to continue independently. At the same time it might just be that the couple of tiny mathematical hints that the teacher accidentally gave in this case, were just enough to prompt the students to carry on. When working in pairs, ideas may dry up sooner than in small groups of three or four students. The fact that this mathematical help was given so casually while emphasizing the students own thinking may well have been a master stroke, which made the students aware of their own ability.

The teacher could have handled the fact that the students stopped asking for help (and adopted an air of indifference) in the last lesson by addressing the class again and clarifying his expectations, while encouraging the students to be critical of each other.

The question that remains is why the other students in this survey showed no change in their attitude to work. We saw firstly that many students lacked the focus to attain a deeper understanding. Furthermore, most students in the group gave up as soon as they sensed that they were getting stuck, exhibiting less task persistence than the two described. There is no doubt that this is related with fact that many students in havo 4 in the profiles E&M and C&M do not regard themselves as mathematical problem-solvers, which is evident from their frequent comments. They also lacked motivation. However, Stella and Roos did exhibit the behavior of mathematical problem-solvers. The other students were quick to accept that they didn't understand something. They then expected an explanation from the teacher, and because one was not forthcoming, they gave up.

What lesson can be learned from this? It is a widespread phenomenon: if learning is difficult, people want an explanation from an expert, and in precisely that sort of situation it is important for students themselves to have the motivation and to possess the necessary metacognitive skills. It may help students to acquire these metacognitive skills by temporarily withholding technical support. Teachers can count on resistance from students, and on their own inclination to provide explanation and technical help. It is also important to aim for a situation in which the students can struggle and at the same time to provide enough sustenance and incentive in the form of process help to allow further progress.

Chapter 7

CONCLUSIONS AND DISCUSSION

1. COLLABORATIVE MATHEMATICAL INVESTIGATIONS USING A COMPUTER: MAIN RESULTS

In this section we give a brief outline of the research project and the main results. We investigated if and how collaborative investigation tasks using a computer can foster mathematical level raising (Van Hiele, 1986; Freudenthal, 1973; Elshout-Mohr & Dekker, 2000) for 16-year-old students in higher general secondary education (havo) 4 mathematics A. Earlier research made it clear that, in this type of education, the students are not always motivated to perform investigations with the help of a computer (PRINT, 1998; PRENT, 1999). This we ascribed to the fact that many computer simulations on probability theory were rather abstract. This is why we attempted to develop simulations in a way that would be very accessible for students, offering them many opportunities to explore the subject on a concrete level. We opted for the subject of 'Routes and Probabilities', since this topic of probability theory has both a visual and an abstract component. It is considered to be a problematic subject. The central concept in this domain, a procedure to *count routes in a grid* is easily adopted by students. However, when they have to make use of this procedure in order to solve word problems, students find it difficult to identify which problems are and are not suitable for solving with a grid.

We conducted a pilot study to test whether two investigation tasks based on the gambling game 'Plinko' (with the underlying structure of a grid) could be used to stimulate mathematical level raising into the subject of 'Routes and Probabilities'. The tasks were done by two pairs of students at a secondary school. It appeared that one of the tasks was approached at different levels: one pair approached it at the perceptual level, and the other pair at the conceptual level (for an explanation of these concepts see Chapter 1, section 1.2). However, this task did not stimulate mathematical level raising. This is why we decided to design a computer simulation in which students could experiment with the different aspects of a grid. In this way students at the perceptual level could make a start with mathematical level raising.

In the first field study (n= 60) we compared the difference in level raising between three versions of the learning materials: investigations with the computer *before, during* or *after* the learning of a mathematical concept. It was shown that there was no significant difference in the final mathematical level that students attained in the three conditions. Yet there were differences in the mathematical level on which students approached the investigation tasks. Qualitative analysis showed that students in the *during* condition appeared more often to make a constructive start with level raising based on their own ideas than the students working on corresponding tasks in the *before* condition did. Moreover, in the *during* condition, the students showed less often that they used concepts that they did not really understand, than students working on corresponding tasks in the *after* condition.

Subsequently, we studied the learning process of two students from the first study in more detail, while they were working on investigation tasks with the computer. This was done by analyzing the written transcripts of all their lessons. The study focused on the interaction between these students. The aim was to establish how such interaction helped the students to learn from one another, and how it may have hindered their learning process. The assumption was that interaction with peers is beneficial for students if they can perform certain key activities (Dekker & Elshout-Mohr, 1998), namely showing, explaining, justifying, and reconstructing their work. Both students attained mathematical level raising. However, the student who explained more frequently and criticized himself attained more mathematical level raising than the student who did not explain her work frequently or criticize herself.

Given the positive experiences we described above, we continued with the 'during version' of the learning materials in which investigation tasks and closed tasks were interwoven. A point of criticism on this version from experts in mathematics education was that students did not get the opportunity to reinvent the procedure of counting routes in a grid, which is why we developed a new version of the learning materials in which we addressed this point.

The next field study, in the same type of education (n= 52), was conducted with the following research question: 'Do students working in pairs on investigation tasks using a computer attain more mathematical level raising when they are supported by a teacher who stimulates their interaction (process help), than when they are supported by a teacher who gives mathematical help (product help)?' Students in both conditions improved, but the two types of help showed no significant difference in level raising on the post-test. Students in both conditions also had serious problems with the learning materials, and wanted the teacher to explain and correct more. For students at this level of education, mathematical level raising with investigation tasks in small groups seems to be very difficult. At crucial moments the learning processes often did not get going and neither process help nor product help worked. These findings were contrary to the results of Dekker and Elshout-Mohr (2004) with students in pre-university education mathematics B. The learning processes for these students were going well and process help stimulated the students to interact, while in some cases product help disrupted their thinking processes.

Although the students in our study had huge difficulties and though in both conditions they expected the teacher to give more explanation, there were two students in the process condition who mentioned in their evaluation that they had learned ‘to think better for themselves and to think more by themselves’. By analyzing the recorded discussions from all their lessons and their written materials, we could show how they arrived at this insight into interaction with each other and their teacher. It appeared to be a difficult process both for the students and for the teacher in which the motivation and the metacognitive skills of the students were very important. These observations may explain why learning mathematics with the help of investigations was not self-evident for all students in this group.

2. DISCUSSION OF THE PROJECT

Collaborative investigation tasks using a computer were used to foster mathematical level raising for students in havo 4 mathematics A. The students in our two experimental studies achieved mathematical level raising, although not much. No significant differences in mathematical level on the post-test (product results) were found between the three versions of the learning materials (before, during or after). However, qualitative analysis of the learning materials and audio recordings showed differences between the learning processes in the three conditions. In the second study, there were no significant differences in mathematical level on the post-test between the two types of teacher help (process help or product help) that we compared. A qualitative analysis of the learning process of two students gave us an insight into the learning processes of these students.

An important question is: How can we explain that there were no significant differences in level raising on the post-test between the condition groups in both experiments? And why were the learning results lower than we expected? Were there, retrospectively, any weak points in the experimental setting? To answer these questions we now discuss the organisation of the (collaborative) learning process, the learning materials, the role of the teacher and the students. Finally, we discuss the key question as to whether students at this school level can truly benefit from making investigations in their mathematical learning process.

Before starting the following sections, we should first point out that the two experiments had different goals. The first one could be seen as a study to improve the learning materials which were then used in the second experiment. We therefore focus on the latter. When we speak about ‘the learning materials’ or ‘the role of the teacher’, we refer to the second experiment, unless stated otherwise.

2.1 The Collaborative Learning Process: Group Size and Types of Help

The aim of collaborative learning is to give students the opportunity to share their work, to discuss it and to learn from doing so. This requires the learning materials that we discuss in the next section. We focus here on the interaction with peers and the guidance of the teacher. In retrospect, what could we have done differently in the design of the collaborative learning environment?

First, the students worked in dyads. What we can ask here is whether the stagnation of the discussions and the fact that the two types of help did not function optimally can be ascribed to this particular number of students. Students working in triads might possibly have been able to share more ideas. All students in triads may be involved in the discussion, but they also have the opportunity to take a step back and reflect on what the other two are saying. However, we worked with dyads, since the most practical way to work collaboratively with a computer is with two students. Moreover, we linked up to the daily practice in many schools, where investigation tasks with the computer are conducted in dyads.

Secondly, as far as the help of the teacher is concerned: the aim of the two types of help was to make students learn independently and from each other as much as possible. We expected process help to maximize the students' interaction and not to disrupt the students' thinking. It appeared that many students got stuck when a specific task was too abstract for them, and as a result the interaction stopped. Looking back, we wonder what the process teacher could have done at this stage without harming the principles of process help. Could a classical discussion, in order to make students share their work, have possibly helped the learning processes continue? And what about product help? The aim of product help was to provide mathematical hints that would fit in well with the learning process of the students, paying no attention to their interaction. A problem with this type of help was the fact that all the students asked for hints at the same time in the learning process. We could possibly have solved this by getting the teacher to discuss the questions of one student in a whole class setting. Having said that, in this research project we did not integrate classical discussion in the instructional process in order to connect to the research of Dekker and Elshout-Mohr (2004) and in order to give students every opportunity to build up their own concepts.

A final point is the fact that both types of help were in a way artificial in an environment for collaborative mathematical learning. The product teacher ignored the process of interaction and the process teacher kept away from the mathematics. Separating process help and product help would not occur easily in daily educational practice. Nevertheless, we did not want to apply a mix of both types of help in this research project, since we wanted to prevent students from waiting passively for the explanations to come later in the lessons when they were working with process help. We really wanted to make them wrestle with the problems, in order to force a breakthrough.

2.2 The Learning Materials

Looking back at the learning materials we wonder whether the whole series of tasks or some specific tasks were not particularly well chosen. The aim of the learning materials was to make students construct their knowledge about the subject of 'Routes and Probabilities' by themselves and with each other, and to gain an insight into the concept of *counting routes in a grid*. It appeared that the task that required them to transfer what they had learned about this concept was particularly tricky for most of the students (task 10d)). Could we perhaps have foreseen this problem? And why did we not arrive at this conclusion when trying out this version of the learning materials? In an earlier stage of the design process we conducted try-outs with single investigation tasks. In order to screen the whole new version of the learning materials, we needed at least six lessons and students with very specific prior knowledge of probability theory. This was not a straightforward matter.

In other tasks, the concept of calculating probabilities in a grid was developed by posing a complex problem divided up into smaller sub-problems. This kind of task is very common in mathematical textbooks: a ‘level jump’ divided into smaller steps. Seen from the conceptual level, these steps are logical, but students at the perceptual level do not often see the rationale. When they miss a link, level raising will not take place. Correcting their own work with solution sheets could have made students aware of the link they had missed. The absence of these solution sheets rendered these textbook-like tasks open to misinterpretation. However, we did not provide any kind of elaborated answers in order to make students build up the concepts on their own.

One could question what it was that made the students learn, because they apparently improved. They worked well with the first part of the learning materials in which investigation tasks with the computer were alternated with closed pencil-and-paper tasks in which they could work out their experiences with the computer simulation. Hence, the notion of enumerating routes in a grid was built up and students could work well with these tasks.

Finally, we have to point out that the problems encountered might have something to do with the mathematical content rather than with the design of the tasks. We have to realize that it involved a difficult part of mathematics. The transition from enumerating concrete routes towards the enumeration of routes as a means to count possibilities and to calculate probabilities demands rather a lot from the students’ powers of abstraction.

2.3 The Teachers

And what was the influence of the teachers themselves in the research? We only compared two teachers, each with their own specific qualities. Can we ascribe the results that we found to the type of help that was provided or do other differences between the two individual teachers play a role? We address these issues in this section.

Both were experienced teachers, with a sound mathematical basis and good contact with the students. Undoubtedly, they both had their own teaching style, notions of teaching mathematics and habits which could have played a role. But we have no indication that these aspects were decisive in the effect of process help and product help.

How the teachers were trained may have been an influencing factor. The teachers had been trained during role play to provide either process help or product help. Practicing this with real students must have been very different. We could sometimes sense the awkwardness in their voices, and students might have observed it too. Although both types of help were given correctly and the teachers had good contact with the students, the teachers were not always convinced that what they were doing was the best way for the students to learn mathematics. This could have been frustrating for both teachers and students.

Another point is that both teachers were not teaching their own class, which could have made things less comfortable for teachers and students. Both teachers did not know all the students in their group equally well. However, we did this in order to create two experimental groups whose results for the pre-test were comparable.

The awkwardness for both students and teachers might have had to do with the fact that expressing your own mathematical ideas and criticizing peers was not part of their *socio-mathematical norms* (Gravemeijer, 1995; Wood, 2001). Changing these habits might be a drastic process which takes time (Hoek et al., submitted).

2.4 The Students: Can Students in Havo 4 Mathematics A really Benefit from Making Investigations in their Learning Process?

What can we say about the students that has influenced the results of this project? At the beginning of the project we chose to focus on havo 4 mathematics A students because they had not performed investigations with the computer in previous projects.

The project involved a group of students who do not have a very strong basis of prior knowledge, compared with their peers in higher general secondary education mathematics B and in pre-university education. These are students who have opted for a set of subjects with very little mathematics. We know from observations and audio recordings that these students are not particularly motivated to find out how things in maths work. They tend to see mathematics as simply carrying out procedures ('What do we have to do?' is a question that is often heard) instead of something logical. It seems that their metacognitive skills are well developed when, for instance, it involves planning their work, but not for gaining insight. This might be linked to our impression that they do not see themselves as mathematicians. One student could often be heard saying: 'I am not going to find out by myself, I am not a mathematician.'

We wonder whether it is possible for *these* students to gain mathematical insight and attain mathematical level raising by such an experimental way of learning. Working with a peer on open tasks around computer games, interlinked with more closed tasks, developing concepts of probability theory, with the teacher having a minimal role, is that a realistic ask of these students?

Some of the students did attain mathematical level raising for the concept of enumerating routes in a grid, but hardly any of them attained it for the more abstract concepts of the domain (see Chapter 5, section 4.3). This was the case for both condition groups, in which the learning of abstract concepts was very difficult.

The above makes us doubt whether students in havo 4 mathematics A can attain mathematical level raising with the help of mathematical investigation tasks. However, we still have to be careful with the statement that they cannot. The students in the most 'investigative' condition (in which the teacher did not provide any explanation) did not perform any worse than the students in the condition in which the teacher did provide some explanations. The students improved, although not much. We still have to realize that our tests were aiming at mathematical level raising. A standard test at school includes some items at the conceptual level and some items to test procedures and routines and therefore gives more opportunities for students to show what they have learned.

3. IMPLICATIONS AND SUGGESTIONS FOR FURTHER RESEARCH

What do the results mean for further research on learning mathematics, with the help of investigations, using a computer, collaborative learning and for the role of the teacher?

3.1 Learning Mathematics by Investigations

We assumed in this study that students can achieve better mathematical level raising when they have the opportunity to structure the information they have to deal with by themselves, than when the theory is presented. For students in havo 4 mathematics A, however, we have to add critical observations to this supposition. We have seen a difference with the students in pre-university education: students in havo 4 mathematics A are less inclined to investigate an abstract concept.

We think that with this study we have arrived at the limit of what is possible, starting from the idea that students construct their own knowledge. We are convinced that students can only learn meaningfully when they are actively involved in the matter and when they are continuously able to link what they are learning with their prior knowledge. When a new concept is too abstract for students, *semi level raising* may possibly occur (see also Chapter 3, section 8.3). This notion was developed to indicate a stage between the perceptual and conceptual level in which students use a certain concept, but do not link it to their prior knowledge, which often leads to misconceptions. It is opposite to *start level raising*, the situation in which students make a start with level raising in connection with their own prior knowledge. They develop ideas that relate to the concept, but do not attain the conceptual level. In chapter 3 we evaluated start level raising better than semi level raising, since we still see the connection of new concepts with one's own ideas as a crucial part of learning.

By using investigation tasks and not expounding theory, we expect that less semi level raising will occur, since students are not offered concepts at all, so the concepts cannot be misinterpreted. However, in normal lessons semi level raising cannot always be avoided because of time constraints and the demands of exams. An interesting issue for further research would be how to guide students from semi level raising to the conceptual level? Do they have to go back to the perceptual level? How can they be stimulated to reflect on their learning process? What is the role of the teacher in this process?

3.2 Learning Materials and the Role of the Computer

We think that the openness of the task is not the most important factor in creating opportunities for students to explore the visual level. The use of concrete materials seems to be crucial in making a topic accessible. For the beginning of level raising, concrete games on a computer with an (open) investigation task, followed by some more closed tasks to elaborate the experiences with the computer games might be recommended. The use of a computer is very helpful for explorations at the visual level. It seems that for the process of level raising, well chosen tasks and collaborative learning activities are important but not sufficient for these students. They will appeal to their teacher for help during this process. Therefore, in further research with these kind of learning materials, the role of the teacher should get special attention (see section 3.4 of this chapter).

3.3 Collaborative Mathematical Learning

In order to learn from lively discussion with each other, students need enough ideas to share and to criticize. It became clear that the discussion easily stopped when students in havo 4 mathematics A worked on difficult tasks in pairs (as described in Chapter 5, section 4.1). Groups of three students may provide more input and opportunities to take a step back. It is important that this is taken into account in further research.

The process model appeared to be a valuable instrument for analyzing the students' learning processes. We expected four key activities, to show, explain, justify, and reconstruct one's work (Dekker & Elshout-Mohr, 1998), to be important for the mathematical learning process. We found indications that providing explanations and criticizing one's own work is particularly important for mathematical level raising in the continuation of the learning process, in other words, for the learning of concepts building on the concept that has been explained. This hypothesis could be tested in a new research project. Special attention could be paid to giving explanations by students with little prior knowledge. Explaining their work is not a familiar situation for these students, since they are seldom the mathematical experts.

Another aspect that could be given attention in further research is what makes students explain their work to each other, how can a teacher encourage this, and what are the effects on the motivation and learning results of the students?

3.4 The Role of the Teacher

We observed that the students in our study, who are not strong when it comes to prior knowledge, motivation and metacognitive skills, leaned very much on their teacher. This is contrary to the experiences with process help in pre-university education (Dekker & Elshout-Mohr, 2004). These students benefited from a teacher who only stimulated their interaction process. It seems that enhancing mathematical level raising through collaborative learning in havo 4 mathematics A requires a different approach from the one in pre-university mathematics B. Our havo students did perform equally well in the product and process condition, while the pre-university students of Dekker and Elshout-Mohr performed better in the process condition.

It must be stressed that there are indications that process help could help havo students to think more, and more deeply. This type of help could be elaborated in further research. As said before, process help is somewhat artificial and barely occurs at all in the daily practice of mathematics education. It would be interesting to develop a way of stimulating students to interact with each other that could be integrated into current teaching practices for collaborative mathematical learning. For this purpose we would like to study and describe how teachers in these current practices provide help when students work collaboratively on mathematical tasks. Subsequently, we would like, in a cyclical process, together with the teacher, to reflect on the lessons and possibly improve the process help in order to help students learn more from their interactions.

4. IMPLICATIONS FOR EDUCATIONAL PRACTICE

And finally… what does this study mean for students and teachers? Teachers may recognize the description of havo mathematics A students as not having an investigative approach. Are there any guidelines that follow from this research project?

It is not easy to give guidelines for the learning materials (chapter 3). It seems to make no difference with respect to mathematical level raising when 'theory blocks' in mathematical textbooks are replaced by investigation tasks using a computer. However, we found that carrying out mathematical investigations during their learning process instead of studying theory in their textbook may prevent students from 'meaninglessly learning a trick'.

Another point for giving guidelines is giving process or product help. When students have to learn something they find difficult and which they experience as being too abstract, it is very likely that they will ask the teacher for help. The most common way of providing help is to give mathematical hints or explanations. It is interesting for teachers to realize that this may help just as well as stimulating the students to discuss their work with each other. We cannot provide a recipe for mathematical level raising with students who have little prior knowledge and self-confidence. In order to learn something difficult, students will have to show an active attitude. But since it may be equally effective to provide product and process help, we encourage teachers to experiment with process help. They have to make it clear to students that they will not give mathematical hints and will encourage students to collaborate on a difficult task. We advise trying to work in groups of three with enough differences between the students; focussing on the process of interaction, and when one of the students is no longer involved in the work, asking him or her: ‘What do you think about it?’. Even when students struggle with the subject matter, they will be encouraged to formulate their own mathematical ideas.

A final glance at the project shows us that students in havo 4 maths A are actually able to execute mathematical investigation tasks with a computer, as long as the computer simulation is accessible and not too abstract. Abstract thinking and attaining mathematical level raising seems, in any case, to be difficult for these students. Nevertheless, the two case studies show us that certain dyads discussed the mathematical tasks intensively. In addition to the fact that pre-university students learned more from process help than from product help, we are in favour of teachers going to the limit when encouraging students to express their own ideas and to learn from one another. However, this is far from obvious, since the natural inclination to provide help in terms of an explanation when students ask for one, can be very, very strong. Perseverance is needed to break through this vicious circle and to make students discuss their mathematical ideas. When this process succeeds, students will achieve a more active mathematical working attitude. And it might be very interesting to listen to the ideas they come up with!

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SUMMARY

The main purpose of this project was to investigate if and how collaborative investigation tasks using a computer can foster mathematical level raising for 16-year-old students in higher general secondary education (havo) 4 mathematics A classes. Earlier research revealed that students, in this type of education, are not always motivated to perform investigations using a computer (PRINT, 1998; PRENT, 1999). This we ascribed to the fact that many computer simulations on probability theory tend to be rather abstract. This is why we attempted to develop simulations in a way that students would find accessible, and which would offer them many opportunities to explore the subject on a concrete level. We opted for the subject of 'Routes and Probabilities', since this topic of probability theory incorporates both visual and abstract elements. It is considered to be a problematic subject for students. The students easily adopt the procedure to 'count routes in a grid', a central concept of this domain. However, when they are required to use this procedure to solve word problems, students find identifying those problems that are suitable for solving with the grid difficult. One could say that they have not fully reached the conceptual level for the concept of counting routes in a grid. We refer to the transition from the perceptual level to the conceptual level as *mathematical level raising*. For students, reflection on their own (mathematical) ideas may play a central role in attaining mathematical level raising. This process of reflection may take place in interaction with their peers. According to the process model as developed by Dekker and Elshout-Mohr (1998), the following *key activities* are important for attaining mathematical level raising: to show one's work, to explain one's work, to justify one's work and to reconstruct one's work. The learning materials should stimulate students to discuss their work and the teacher also plays an important role in encouraging the students to talk about their mathematical ideas. We therefore expect the learning materials including the computer simulation, the interaction with peers and the role of the teacher to be important for the students to be able to attain mathematical level raising. In the studies reported in chapters 2 through 7 we focus on how we can attune learning materials and the role of the teacher for collaborating students in havo 4 mathematics A to reach this goal.

CHAPTER 2

We conducted a pilot study to test whether two investigation tasks based on the gambling game ‘Plinko’ (with the underlying structure of a grid) could be used to stimulate mathematical level raising in the subject of ‘Routes and Probabilities’. The tasks were conducted by two pairs of students at a secondary school. It appeared that one of the tasks was approached on different levels: one pair approached the task at the perceptual level and the other pair at the conceptual level. However, the task did not stimulate mathematical level raising. This is why we decided, in the continuation of the research, to design a computer simulation in which students could experiment with the different aspects of a grid. In this way students at the perceptual level could make a start with mathematical level raising.

CHAPTER 3

In the first field study (n= 60) we compared the differences in level raising between three versions of the learning materials: investigations using a computer *before, during* or *after* the learning of a mathematical concept. In the condition *during,* students worked with investigation tasks integrated with textbook-like tasks. In contrast to what the students were used to, these learning materials did not include any ‘theory blocks’. In the other two conditions (*before* and *after*), students carried out the investigation tasks before or after they had worked on tasks in their regular textbook. Students made a pre- and a post-test on mathematical level raising. During all lessons, we made audio recordings of the dialogues between selected pairs of students. The written learning materials of all students and the log-files from their operations on the computer were also collected. We expected that students in the *during* condition would benefit most from the investigations for attaining mathematical level raising.

It was shown that there was no significant difference in the final mathematical level that students attained in the three conditions. Yet, qualitative analysis of the students’ learning materials showed that students in the *during* condition appeared more often to make a constructive start with level raising based on their own ideas than students working on corresponding tasks in the *before* condition. Moreover, in the *during* condition, it was less often the case that the students showed that they had used concepts that they did not really understand, than students working on corresponding tasks in the *after* condition.

CHAPTER 4

We subsequently studied the learning process of two students from the first study in more detail, while they were working on investigation tasks with the computer. This was done by analyzing the written transcripts of their interactions in all their lessons. The aim was to establish how such interaction helped the students to learn from one another, and how it may have hampered their learning process. The assumption was that interaction with peers is beneficial for students if they can perform the key activities showing, explaining, justifying, and reconstructing their work (Dekker & Elshout-Mohr, 1998). We selected four episodes comprising events in which a concept was learned and an event in which this did not happen. In these passages we counted the occurrence of key activities for both students. It appeared that the student who explained more frequently and criticized himself attained more mathematical level raising than the student who did not explain her work frequently or criticize herself.

CHAPTER 5

We continued with the '*during* version' of the learning materials in which investigation tasks and closed tasks were interwoven. A point of criticism on this version from mathematics education experts was that students did not get the opportunity to reinvent the procedure of counting routes in a grid. This is why we developed a new version of the learning materials in which we addressed this point.

The field study described in this chapter, in the same type of education (n = 52), was conducted with the following research question: 'Do students working in pairs on investigation tasks using a computer attain more mathematical level raising when they are supported by a teacher who only stimulates their interaction (*process help*) than when they are supported by a teacher who gives mathematical help (*product help*)?' Students in both conditions improved, but the two different types of help showed no significant difference in level raising on the post-test. Students in both conditions also had serious problems with the learning materials, and wanted the teacher to explain and correct more. For students at this level of education, mathematical level raising with investigation tasks in small groups appears to be very difficult. The learning processes often did not get going at crucial moments and neither process help nor product help worked. These findings were contrary to the results of Dekker and Elshout-Mohr (2004) with students in pre-university education mathematics B. The learning processes for the latter type of students were going well and process help stimulated them to interact, while in some cases product help disturbed their thinking processes.

CHAPTER 6

Although the students in our study had considerable difficulty with this way of working, and though in both conditions they expected the teacher to give more explanation, there were two students in the process condition who mentioned in their evaluation that they had learned 'to think better for themselves and to think more by themselves'. By analyzing the recorded discussions from all their lessons and their written materials, we could show how they gain their insights in interaction with each other and with their teacher. It appeared to be a difficult process both for the students and for the teacher. The motivation and the metacognitive abilities of the students seemed to be very important for their being able to benefit from process help. These observations may explain why learning mathematics with the help of investigations was not self-evident for all students in this group.

CHAPTER 7

In this chapter we summarize the results of this research project. Why were the learning results lower than we expected? Were there, retrospectively, any weak points in the experimental setting? To answer these questions, we discuss the organization of the (collaborative) learning process, the learning materials, the role of the teacher and the students.

We question whether students at this level of education could benefit from conducting investigations in their learning process. It seems that attaining mathematical level raising (which we measured rather precisely, as opposed to school tests where students usually get the opportunity to show more than level raising) is difficult for these students anyway. However, the groups with the most investigative versions of learning materials and teacher help did not perform any worse than the groups who received more explanations in their learning materials or from their teacher.

Finally, some implications for educational practice are set out. As far as the learning materials are concerned, we found indications that carrying out mathematical investigations during their learning process instead of studying theory in their textbook may prevent students from ‘meaninglessly learning a trick’.

Because there are no indications that more explanation will help these students we would like to invite teachers to experiment with refraining from providing explanations. Students may learn to develop their own mathematical ideas and to discuss them with peers when they are prevented from relying on their habitual behavior of asking their teacher to explain the mathematics.

SAMENVATTING

Het voornaamste doel van dit project was om te onderzoeken of en, zo ja, hoe onderzoeksopdrachten met de computer voor samenwerkend leren kunnen leiden tot niveauverhoging bij leerlingen in 4 havo bij wiskunde A. Uit eerder onderzoek bleek dat studenten in dit type onderwijs niet altijd gemotiveerd waren om onderzoeksopdrachten met de computer uit te voeren (PRINT, 1998; PRENT, 1999). Dit schreven we toe aan het feit dat veel computersimulaties over kansrekening nogal abstract waren. Daarom hebben we een simulatie ontworpen, die toegankelijk was voor leerlingen en die leerlingen de gelegenheid zou geven om het onderwerp op een concreet niveau te exploreren. We kozen het onderwerp 'Routes en Kansen' omdat dit onderwerp uit de kansrekening zowel een visuele als een abstracte component heeft. Het wordt als een moeilijk onderwerp voor leerlingen gezien. Het routes tellen in een rooster, het centrale concept in dit domein, leren de leerlingen gemakkelijk aan. Ze vinden het echter moeilijk te bepalen welke tel- en kansproblemen wel en welke niet met een rooster kunnen worden opgelost. Je zou kunnen zeggen dat ze vaak niet helemaal het conceptuele niveau van het concept 'routes tellen in een rooster' bereiken. We verwijzen naar de overgang van het perceptuele niveau naar het conceptuele niveau met de term wiskundige *niveauverhoging*. Voor leerlingen speelt bij het bereiken van niveauverhoging reflectie op hun eigen (wiskundige) ideeën een centrale rol. Het proces van reflectie kan plaatsvinden in interactie met medeleerlingen. Volgens het procesmodel dat Dekker en Elshout-Mohr (1998) ontwikkelden, zijn voor het bereiken van wiskundige niveauverhoging de volgende kernactiviteiten van belang: het eigen werk laten zien, uitleggen, verdedigen en reconstrueren. Het lesmateriaal moet leerlingen stimuleren over hun werk te praten en te discussiëren. Ook speelt de docent een belangrijke rol om de leerlingen aan te moedigen over hun wiskundige ideeën te praten. Daarom verwachten we dat zowel het lesmateriaal, waaronder ook de computersimulatie en de samenwerking met medeleerlingen, als de rol van de docent belangrijk zijn voor het bereiken van niveauverhoging. In de studies welke in de hoofdstukken 2 tot en met 7 worden beschreven, besteden we aandacht aan de vraag, hoe het lesmateriaal en de rol van de docent zo kunnen worden afgestemd, dat dit doel bij samenwerkende leerlingen in 4 havo bij wiskunde A kan worden bereikt.

HOOFDSTUK 2

We hebben een pilotstudie uitgevoerd om te testen of twee onderzoeksopdrachten, gebaseerd op het kansspel 'Plinko' (met de onderliggende structuur van een rooster), konden worden gebruikt om bij het onderwerp 'Routes en Kansen' wiskundige niveauverhoging te stimuleren. De opdrachten werden uitgevoerd door twee tweetallen leerlingen op een middelbare school. Het bleek dat een van de opdrachten op verschillende niveaus werd benaderd: het ene tweetal benaderde de opdracht op het perceptuele niveau en het andere tweetal deed dit op het conceptuele niveau. De opdracht lokte echter geen niveauverhoging uit. Daarom besloten we, in het vervolg van het onderzoek, een computersimulatie te ontwerpen, waarin leerlingen konden experimenteren met de verschillende aspecten van een rooster. Op deze manier zouden studenten op het perceptuele niveau een begin kunnen maken met niveauverhoging.

HOOFDSTUK 3

In het eerste veldexperiment (n=60) vergeleken we de verschillen in niveauverhoging tussen drie versies van het lesmateriaal: onderzoeksopdrachten met de computer *voor*, *tijdens*, of *na* het leren van een wiskundig concept. In de tijdens-conditie, werkten studenten met onderzoeksopdrachten welke geïntegreerd waren met opdrachten, die leken op reguliere opdrachten uit de methode. In tegenstelling tot wat studenten gewend waren, bevatte dit lesmateriaal geen 'theorieblokken'. In de andere twee condities ('voor' en 'na'), voerden leerlingen onderzoeksopdrachten uit voor- of nadat zij werkten aan taken in hun reguliere wiskundeboek. De leerlingen maakten een voor- en natoets op wiskundige niveauverhoging. Tijdens de lessen maakten we geluidsopnamen van de dialogen tussen geselecteerde tweetallen. De geschreven lesmaterialen van alle leerlingen en de log-files van hun manipulaties op de computer werden ook verzameld. We verwachtten dat leerlingen in de tijdens-conditie het meest zouden profiteren van de onderzoeksopdrachten om wiskundige niveauverhoging te bereiken.

Er bleek geen significant verschil in het uiteindelijke wiskundige niveau dat de leerlingen in de drie condities bereikten. Wel bleek uit kwalitatieve analyse van het lesmateriaal dat leerlingen in de tijdens-conditie vaker een constructief begin maakten met niveauverhoging, gebaseerd op hun eigen ideeën, dan leerlingen die werkten aan corresponderende taken in de voor-conditie. Bovendien kwam het in de tijdens-conditie minder vaak voor dat leerlingen lieten zien dat ze een bepaald concept gebruikten zonder het echt te begrijpen, dan in de na-conditie bij leerlingen die werkten aan corresponderende opdrachten.

HOOFDSTUK 4

Vervolgens bestudeerden we het leerproces van twee leerlingen uit de eerste studie meer in detail. Dit werd gedaan door de uitgeschreven protocollen van hun uitspraken in alle lessen te analyseren. Het doel was om vast te stellen hoe de interactie de studenten hielp om van elkaar te leren en in hoeverre het hun leerproces gehinderd had. De aanname was dat samenwerking met medeleerlingen leerzaam is voor leerlingen als ze de volgende kernactiviteiten kunnen uitvoeren: tonen, uitleggen, verantwoorden en reconstrueren van het eigen werk (Dekker & Elshout-Mohr, 1998). We hebben vier episodes geselecteerd waarin een bepaald concept werd geleerd en één waarin dit juist niet gebeurde. In deze passages hebben we het voorkomen van kernactiviteiten voor beide leerlingen geteld. Het bleek dat de leerling die meer uitleg gaf en zichzelf vaker bekritiseerde in het vervolg van het leerproces meer wiskundige niveauverhoging bereikte dan de leerling die haar werk minder vaak uitlegde en zichzelf minder vaak bekritiseerde.

HOOFDSTUK 5

In het vervolg van het onderzoek hebben we de tijdens-versie van het lesmateriaal gebruikt, waarin onderzoeksopdrachten en gesloten opdrachten met elkaar verweven waren. Een punt van kritiek van wiskundedidactici op deze versie van het lesmateriaal was, dat leerlingen niet de gelegenheid kregen om de procedure van routes tellen in een rooster te 'heruitvinden'. Dat is de reden waarom we een nieuwe versie van het lesmateriaal ontwikkelden waarin we aandacht besteedden aan dit punt.

Het in dit hoofdstuk beschreven onderzoek, ook in havo 4 bij wiskunde A (n=52), werd uitgevoerd met de volgende vraag: 'Bereiken leerlingen, die samen werken aan onderzoeksopdrachten met de computer, meer niveauverhoging wanneer zij worden ondersteund door een docent die uitsluitend hun interactie stimuleert (*proceshulp*) dan wanneer zij worden begeleid door een docent die hun wiskundige hints geeft wanneer zij daarom vragen (*producthulp*)?' Studenten in beide condities gingen vooruit, maar de verschillende types hulp lieten op de natoets geen significant verschil in niveauverhoging zien. Leerlingen in beide condities hadden moeite met het lesmateriaal en wilden dat de docent hun meer uitleg gaf en hen meer corrigeerde. Blijkbaar is het voor deze leerlingen heel moeilijk om niveauverhoging te bereiken door samen onderzoeksopdrachten met de computer uit te voeren. Op cruciale momenten kwamen de leerprocessen vaak niet op gang. Dan hielp proces- noch producthulp. Deze bevindingen waren tegengesteld aan de resultaten van Dekker en Elshout-Mohr (2004) met leerlingen vwo wiskunde B. De leerprocessen voor deze leerlingen bleven goed op gang en proceshulp stimuleerde hen om samen te werken, terwijl producthulp soms hun eigen denken verstoorde.

HOOFDSTUK 6

Hoewel de leerlingen in ons onderzoek aanzienlijke moeite hadden met deze manier van werken en hoewel in beide condities de leerlingen van de docent meer uitleg verwachtten, waren er twee studenten in de proces-conditie die in hun evaluatie noemden dat ze geleerd hadden om 'zelf meer en beter na te denken'. Door de tijdens alle lessen opgenomen gesprekken van deze leerlingen met elkaar en met hun docent, alsmede het lesmateriaal, te analyseren, hebben we laten zien hoe ze, in interactie met elkaar en hun docent, tot dit inzicht kwamen. Het bleek een moeilijk proces, zowel voor de leerlingen als voor de docent. De motivatie en de metacognitieve vaardigheden van de leerlingen bleken heel belangrijk te zijn voor het kunnen profiteren van proceshulp. Deze observaties kunnen misschien verklaren waarom wiskunde leren met behulp van onderzoeksopdrachten voor de andere leerlingen in deze groep niet vanzelfsprekend was.

HOOFDSTUK 7

In dit hoofdstuk vatten we de resultaten van dit project samen. Waarom waren de leerresultaten lager dan verwacht? Waren er, achteraf gezien, zwakke punten in de experimentele opzet? Om deze vragen te beantwoorden, bespreken we de organisatie van het (samenwerkend) leerproces, het lesmateriaal, de rol van de docent en de leerlingen.

We vragen ons af of leerlingen in dit type onderwijs in hun leerproces kunnen profiteren van het uitvoeren van onderzoeksopdrachten. Het lijkt erop dat het bereiken van wiskundige niveauverhoging (wat we redelijk precies hebben gemeten, anders dan in schooltoetsen waar leerlingen meestal de gelegenheid krijgen om iets anders te laten zien dan alleen niveauverhoging) sowieso moeilijk is voor deze leerlingen. Hoe dan ook, de groepen met de meest onderzoeksmatige versie van het lesmateriaal en docenthulp deden het niet slechter dan de groepen, die meer uitleg ontvingen in hun lesmateriaal of van de docent.

Tenslotte wordt een aantal implicaties voor de schoolpraktijk gegeven. Voor zover het het lesmateriaal betreft, hebben we aanwijzingen gevonden dat het uitvoeren van onderzoeksopdrachten, in plaats van het bestuderen van theorie in hun wiskundeboek, studenten ervan kan weerhouden om ‘betekenisloos een truc te leren’.

Omdat er geen aanwijzingen zijn dat meer uitleg deze leerlingen zal helpen, zouden we docenten willen uitnodigen om eens te experimenteren met het achterwege laten van uitleg. Wanneer leerlingen niet meer in hun gewoonte kunnen vervallen om de docent om uitleg te vragen, krijgen ze wellicht de gelegenheid om eigen wiskundige ideeën te ontwikkelen en die met medeleerlingen te bediscussiëren.

CURRICULUM VITAE

Monique Pijls was born in Amsterdam on 22 May 1970. In 1988 she got her pre-university degree (gymnasium B) at the RSG Purmerend. After propaedeutics in physical geography, she studied mathematics at the Universiteit van Amsterdam. Her Master's thesis focused on learning materials for geometric transformations for secondary school mathematics classes (1996). Monique has been a member of 'Vierkant voor Wiskunde', where she developed learning materials, and prepared and assisted at mathematical camps for young people. From 1996 to 1997 she took a teacher training course at the Universiteit van Amsterdam. She was a maths teacher at the Lyceum Sancta Maria in Haarlem from 1996 to 1998. Monique took her PhD at the Graduate School of Teaching and Learning at the Universiteit van Amsterdam between 1998 and 2007. Monique currently works as a researcher at the Graduate School of Teaching and Learning in the Dutch Centre of Expertise for Teacher Training Institutes (ELWIeR) and as a researcher on the use of digital video in higher education (DiViDossier). In addition to her work as a PhD student, she has also taken several courses on energy, intuitive development, and sound therapy. In 2006 she obtained a degree at CICO, the Centre for Intuitive Development in Utrecht.

Eerder verschenen proefschriften in deze reeks zijn:

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