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## ADAPTIVE WAVELET ALGORITHMS FOR ELLIPTIC PDE'S ON PRODUCT DOMAINS

CHRISTOPH SCHWAB AND ROB STEVENSON

ABSTRACT. With standard isotropic approximation by (piecewise) polynomials of fixed order in a domain  $D \subset \mathbb{R}^d$ , the convergence rate in terms of the number  $N$  of degrees of freedom is inversely proportional to the space dimension  $d$ . This so-called *curse of dimensionality* can be circumvented by applying sparse tensor product approximation, when certain high order mixed derivatives of the approximated function happen to be bounded in  $L_2$ . It was shown by Nitsche (2006) that this regularity constraint can be dramatically reduced by considering best  $N$ -term approximation from tensor product wavelet bases. When the function is the solution of some well-posed operator equation, dimension independent approximation rates can be practically realized in linear complexity by adaptive wavelet algorithms, *assuming* that the infinite stiffness matrix of the operator with respect to such a basis is highly compressible. Applying piecewise smooth wavelets, we verify this compressibility for general, non-separable elliptic PDEs in tensor domains. Applications of the general theory developed include adaptive Galerkin discretizations of multiple scale homogenization problems and of anisotropic equations which are robust, i.e., independent of the scale parameters, resp. of the size of the anisotropy.

### 1. MOTIVATION AND BACKGROUND

**1.1. Adaptive wavelet algorithms.** For some boundedly invertible linear operator  $A : H \rightarrow H'$ , where  $H$  is some separable Hilbert space with dual  $H'$ , and some  $f \in H'$ , let us consider the problem of finding  $u \in H$  such that

$$Au = f.$$

Specifically, we will be interested in (systems of) elliptic PDEs in variational form. Assuming that we have a Riesz basis  $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$  for  $H$  available, e.g., a wavelet basis, formally viewed as a column vector, by writing  $u = \mathbf{u}^T \Psi$  the above problem is equivalent to finding  $\mathbf{u} \in \ell_2 = \ell_2(\Lambda)$  satisfying the infinite matrix-vector system

$$(1.1) \quad \mathbf{A}\mathbf{u} = \mathbf{f}.$$

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*Key words and phrases.* Elliptic PDE's on (high) dimensional product domains, multiple scale homogenization, tensor product approximation, sparse grids, wavelets, best  $N$ -term approximation, optimal computational complexity, matrix compression.

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Here the stiffness matrix  $\mathbf{A} := \langle \Psi, A\Psi \rangle : \ell_2 \rightarrow \ell_2$  is boundedly invertible and  $\mathbf{f} := \langle \Psi, f \rangle \in \ell_2$ , with  $\langle \cdot, \cdot \rangle$  denoting the duality pairing on  $H \times H'$ . We will use  $\|\cdot\|$  to denote  $\|\cdot\|_{\ell_2}$  or  $\|\cdot\|_{\ell_2 \rightarrow \ell_2}$ .

When one is prepared to spend say  $N$  arithmetical operations and storage locations, the most economical approximation for  $\mathbf{u}$  with respect to  $\|\cdot\|$  is its best  $N$ -term approximation  $\mathbf{u}_N$ , being the vector consisting of the  $N$  largest coefficients of  $\mathbf{u}$  in modulus and zeros elsewhere. Clearly, in the case that not all coefficients of  $\mathbf{u}$  are known, a best  $N$ -term approximation is not practically accessible.

To relate approximation of  $\mathbf{u}$  with that of  $u$ , note that for any  $\mathbf{v} \in \ell_2$ ,  $\|\mathbf{u} - \mathbf{v}\| \approx \|u - \mathbf{v}^T \Psi\|_H$ . Here, and in the remainder of this paper, by  $C \lesssim D$  we mean that  $C$  can be bounded by a multiple of  $D$ , independently of parameters which  $C$  and  $D$  may depend on (sometimes with the exception of the parameter  $n$  that will be introduced in Sect. 1.2). Obviously,  $C \gtrsim D$  is defined as  $D \lesssim C$ , and  $C \approx D$  as  $C \lesssim D$  and  $C \gtrsim D$ .

Using the fact that  $\mathbf{u}$  is given as the solution of (1.1), when for some  $s > 0$  it happens to be the approximation class

$$\mathcal{A}_\infty^s = \{\mathbf{v} \in \ell_2 : \sup_N N^s \|\mathbf{v} - \mathbf{v}_N\| < \infty\},$$

the adaptive wavelet algorithms from [CDD01, CDD02, GHS05] are proven to produce a sequence of approximations that converge with *this rate*  $s$ , requiring a number of operations equivalent to their length, under the assumptions that one knows how to produce approximations for  $\mathbf{f}$  of length  $N$  in  $\mathcal{O}(N)$  operations that converge with rate  $s$ , and that  $\mathbf{A}$  can be sufficiently well approximated by computable sparse matrices. These assumptions are made to be able to control the cost of the successive approximate residual computations inside these iterative algorithms. More specifically, concerning  $\mathbf{A}$ , one assumes that for some  $s^* > s$ , this infinite matrix is  $s^*$ -computable, meaning that

*for each  $q \in \mathbb{N} := \{0, 1, \dots\}$ , an infinite matrix  $\mathbf{A}_q$  can be constructed that has in each row and column  $\mathcal{O}(2^q)$  non-zero entries, computable in  $\mathcal{O}(2^q)$  operations, which satisfies  $\|\mathbf{A} - \mathbf{A}_q\| \lesssim 2^{-q\bar{s}}$  for any constant  $\bar{s} < s^*$ .*

Without the condition concerning the cost of computing the entries,  $\mathbf{A}$  is called  $s^*$ -compressible. For proving  $s^*$ -computability, typically one first shows that it is  $s^*$ -compressible, where the  $\mathbf{A}_q$  are constructed by dropping entries from  $\mathbf{A}$ , and then one shows that the remaining entries can be approximated with suitable quadrature, that keeps the error on level  $\mathcal{O}(2^{-q\bar{s}})$  for any  $\bar{s} < s^*$ , where the *average* number of operations per entry over each row and column is  $\mathcal{O}(1)$ . The reason to expect that  $\mathbf{A}$  is  $s^*$ -compressible are the properties of the wavelets, that is, their vanishing moments and smoothness.

The condition on  $\mathbf{f}$  has to be verified for the right hand side at hand. From  $\mathbf{u} \in \mathcal{A}_\infty^s$  and  $\mathbf{A}$  being  $s^*$ -compressible with  $s^* > s$ , it follows that in any case  $\mathbf{f} \in \mathcal{A}_\infty^s$ , meaning that apart from the cost for creating them, suitable approximations for  $\mathbf{f}$  do exist.

In the above references, it is assumed that  $\mathbf{A}$  is symmetric and positive definite. Other invertible systems can be put into this form by forming the normal equations  $\mathbf{A}^T \mathbf{A} \mathbf{u} = \mathbf{A}^T \mathbf{f}$ . As shown in [Gan06], for mildly non-symmetric or indefinite equations, the algorithms from [CDD01, GHS05] can be applied directly to the original system, avoiding the squaring of the condition number.

Let us now study the value of  $s$  for which  $\mathbf{u} \in \mathcal{A}_\infty^s$  might be expected. To this end, for  $\tau \in (0, \infty)$ , define  $\mathcal{A}_\tau^s = \{\mathbf{v} \in \ell_2 : \sum_{N \in \mathbb{N}} (N^s \|\mathbf{v} - \mathbf{v}_N\|)^\tau N^{-1} < \infty\}$ . This class is thus (slightly) smaller than  $\mathcal{A}_\infty^s$ . We think of  $H$  as a Sobolev space of order  $\ell$  on a  $d$ -dimensional domain, possibly incorporating essential boundary conditions. We have in mind some scalar elliptic PDE of order  $2\ell$  together with appropriate boundary conditions. Then for standard, isotropic, sufficiently smooth wavelets of order  $p > \ell$ , it is known (e.g., [DeV98, Coh03]) that, with  $\tau = (s + \frac{1}{2})^{-1}$ ,

$$(1.2) \quad u \in B_\tau^{sd+\ell}(L_\tau) \iff \mathbf{u} \in \mathcal{A}_\tau^s \quad (s \in (0, \frac{p-\ell}{d})),$$

where  $B_\tau^{sd+\ell}(L_\tau)$  is a certain Besov space measuring “ $sd + \ell$  orders of smoothness in  $L_\tau$ ” with the secondary smoothness parameter also equal to  $\tau$ , possibly incorporating essential boundary conditions. The upshot of this result on non-linear approximation is that  $B_\tau^{sd+\ell}(L_\tau)$  is much larger than the Sobolev space  $H^{sd+\ell}$ , membership of which is needed to get the same rate with standard linear approximation methods of order  $d$ . Since the method of approximation is of order  $p$ , note that by imposing whatever smoothness conditions on  $u$ , for  $s > (p - \ell)/d$ ,  $\mathbf{u} \in \mathcal{A}_\infty^s$  cannot be guaranteed, or actually be expected.

Thinking, as we will do, of wavelets that are *piecewise smooth*, and *globally  $C^r$* , with  $r = -1$  meaning no global smoothness conditions, the smoothness condition on the wavelets reads as  $(p - \ell)/d \geq p - r - 3/2$ , i.e.,  $(p - \ell)/d \geq 1/2$  for spline wavelets where  $r = p - 2$ .

Returning to the adaptive wavelet schemes, when the differential operator has sufficiently smooth coefficients, and the wavelets have  $\tilde{p} > p - 2\ell$  *vanishing moments*, then, as shown in [GS04], the corresponding stiffness matrix  $\mathbf{A}$  is  $s^*$ -computable for some  $s^* > (p - \ell)/d$ , for  $d > 1$  additionally assuming that  $(r + \frac{3}{2} - \ell)/(d - 1) > (p - \ell)/d$ , which for  $r = p - 2$  reads as  $(p - \ell)/d > 1/2$ . We conclude that under these conditions, for the whole relevant range of  $s$ , the adaptive wavelet algorithms converge with the best possible rate  $s$  in linear complexity.

**1.2. PDE's on (high dimensional) product domains.** In terms of the number of degrees of freedom, the best possible rate of approximation  $(p - \ell)/d$  decreases with increasing space dimension  $d$ . This effect is known as the curse of dimensionality. Moreover, simultaneously the smoothness conditions required to achieve a certain rate increase.

Fortunately, high dimensional PDEs are usually formulated on simple domains  $\Omega$ , being the  $n$ -fold product of component domains  $\Omega_m \subset \mathbb{R}^{d_m}$ ,  $m = 1, \dots, n$ , of dimension  $d = \dim \Omega = \sum_{m=1}^n d_m$ . We mention only mathematical finance (pricing of derivative contracts on baskets of  $n$  assets under stochastic volatility models with  $d_m - 1 \geq 0$  ‘hidden’ volatility drivers for the  $m$ th risky asset with the case  $d_m = 1$  corresponding to deterministic volatility; see, e.g., [HMS05]), multiscale problems (elliptic homogenization problems with  $n$  separated length scales); see, e.g., [HS05], stochastic PDEs (the computation of  $n$ -point correlation functions for random solutions); see, e.g., [vPS06].

In the non-adaptive, linear approximation setting it is known that under circumstances the curse of dimensionality can be circumvented by so called sparse tensor product approximation schemes: with  $H^s(\Omega_m)$  being either the standard Sobolev space, or a closed subspace of it involving essential boundary conditions,

for  $\mathbf{t} \in [0, \infty)^n$ ,  $\ell \in [0, \infty)$ , let

$$H^{\mathbf{t}, \ell}(\Omega) := \bigcap_{k=1}^n \bigotimes_{m=1}^n H^{\mathbf{t}_m + \delta_{mk}\ell}(\Omega_m),$$

with  $\delta_{mk}$  being the Kronecker delta. So, as extreme cases, in particular  $H^{\mathbf{t}, 0}(\Omega) = \bigotimes_{m=1}^n H^{\mathbf{t}_m}(\Omega_m)$ , known as a Sobolev space with dominating mixed derivatives, whereas  $H^{0, \ell}(\Omega)$  is (isomorphic to) the standard Sobolev space of order  $\ell$  on  $\Omega$  (possibly with appropriate boundary conditions). Using suitable approximation schemes of order  $p_m > \mathbf{t}_m + \ell$  in the  $m$ th coordinate space, the (optimized) tensor product approximation with  $N$  unknowns results in an error in the  $H^{\mathbf{t}, \ell}(\Omega)$ -norm of  $\mathcal{O}(N^{-s})$  with rate

$$s = \min_m \frac{p_m - \mathbf{t}_m - \ell}{d_m},$$

which is thus *independent of  $n$* , under the assumption that certain higher order mixed derivatives of the approximated function are bounded in  $L_2(\Omega)$ ; see [GK00] and references cited there.

Recently, in [Nit06], corresponding results have been shown for non-linear approximation using a tensor product wavelet basis for  $H^{\mathbf{t}, \ell}(\Omega)$ , reducing the regularity constraint to (nearly) its minimum: in the  $m$ th coordinate direction, assume we have available a set of sufficiently smooth wavelets  $\{\psi_\lambda^{(m)} : \lambda \in \Lambda_m\}$  of order  $p_m > \mathbf{t}_m + \ell$  such that, with  $|\lambda|$  denoting the level of the corresponding wavelet, for a range of  $\ell' \in \mathbb{R}$  that includes  $\mathbf{t}_m$  and  $\mathbf{t}_m + \ell$ ,

$$(1.3) \quad \{2^{-|\lambda|\ell'} \psi_\lambda^{(m)} : \lambda \in \Lambda_m\} \text{ is a Riesz basis for } H^{\ell'}(\Omega_m).$$

Here, since we need that, properly scaled, the wavelets generate Riesz bases for more than one Sobolev space, we made explicit the scaling that depends on the level and smoothness index  $\ell'$ . This is in contrast to the previous section, where for convenience, we tacitly adapted the scaling of the wavelets to the relevant Sobolev space. As shown in [GO95, GK00], as a consequence of the scaling (1.3) we have, with  $\Lambda := \prod_{m=1}^n \Lambda_m$  and, for any block-multiindex  $\boldsymbol{\lambda} \in \Lambda$  of length  $|\boldsymbol{\lambda}| := (|\boldsymbol{\lambda}_1|, \dots, |\boldsymbol{\lambda}_n|) \in \mathbb{R}^n$ ,

$$(1.4) \quad \{2^{-\mathbf{t} \cdot |\boldsymbol{\lambda}| - \ell \|\boldsymbol{\lambda}\|_\infty} \bigotimes_{m=1}^n \psi_{\boldsymbol{\lambda}_m}^{(m)} : \boldsymbol{\lambda} \in \Lambda\} \text{ is a Riesz basis for } H^{\mathbf{t}, \ell}(\Omega).$$

Now with  $\mathbf{u}$  denoting the representation of any  $u \in H^{\mathbf{t}, \ell}(\Omega)$  with respect to the tensor product Riesz basis, the question is for which  $s$  we might expect  $\mathbf{u} \in \mathcal{A}_\infty^s$ , and under which regularity conditions.

Suppose that the wavelets in each coordinate direction  $m$  are sufficiently smooth such that for  $\ell' \in \{\mathbf{t}_m, \mathbf{t}_m + \ell\}$ , with  $\mathbf{u}_m$  denoting the representation of  $u_m \in H^{\ell'}(\Omega)$  with respect to (1.3), and  $\tau = (s + \frac{1}{2})^{-1}$ ,

$$(1.5) \quad u_m \in B_\tau^{sd_m + \ell'}(L_\tau(\Omega_m)) \iff \mathbf{u}_m \in \mathcal{A}_\tau^s \quad (s \in (0, \frac{p_m - \ell'}{d_m})),$$

which is (1.2) in the coordinate space for two cases. Then, as shown in [Nit06] (the proof given there for  $\Omega_m = (0, 1)$  carries over verbatim to the more general situation under consideration here)

$$(1.6) \quad u \in \bigcap_{k=1}^n \bigotimes_{m=1}^n B_\tau^{sd_m + \mathbf{t}_m + \delta_{mk}\ell}(L_\tau(\Omega_m)) \iff \mathbf{u} \in \mathcal{A}_\tau^s \quad s \in (0, \min_m \frac{p_m - \mathbf{t}_m - \ell}{d_m}),$$

where  $\otimes_\tau$  denotes the so-called “ $\tau$  tensor product” (we refer to [Nit06] for a definition and properties of  $\otimes_\tau$  for  $0 < \tau < 1$ ). For piecewise smooth, globally  $C^{r_m}(\Omega_m)$  wavelets in the coordinate spaces, the aforementioned smoothness conditions are satisfied when  $(p_m - \mathbf{t}_m - \ell)/d_m \geq p_m - r_m - 3/2$ , i.e., when  $(p_m - \mathbf{t}_m - \ell)/d_m \geq \frac{1}{2}$  for spline wavelets.

Compared to (1.2), in (1.6) not only the curse of dimensionality has been removed, but also the regularity condition needed to achieve a certain rate  $s$  has been reduced. Indeed, a comparison for  $\mathbf{t} = 0$ , and, for simplicity,  $\ell = 0$ ,  $d_m \equiv 1$ , shows that only the order of the mixed derivatives involved in the smoothness requirements from (1.6) are equal to those from (1.2); the order of the (primary) directional derivatives involved in (1.6) is independent of  $n$ . E.g., solutions of elliptic PDEs on polyhedra in more than 2 space dimensions generally exhibit anisotropic singularities that, using isotropic basis functions, can be approximated with limited rates only, regardless of the order of approximation. Such solutions, transported as a function on the unit cube, have arbitrary regularity in the scale of spaces at the left hand side of (1.6) (see [Nit06]).

The result (1.6) deals with best  $N$ -term approximations, which are not practically accessible. When  $u$  is given as the solution of  $Au = f$  with  $A : H^{\mathbf{t}, \ell}(\Omega) \rightarrow H^{\mathbf{t}, \ell}(\Omega)'$  boundedly invertible and  $f \in H^{\mathbf{t}, \ell}(\Omega)'$ , the adaptive wavelet methods discussed in the setting of isotropic wavelet bases can be applied verbatim. To achieve the high, dimension independent rates of the best  $N$ -term approximations from tensorized wavelet bases shown in [Nit06], the resulting infinite stiffness matrix  $\mathbf{A}$ , however, has to be  $s^*$ -computable with

$$(1.7) \quad s^* > \min_m \frac{p_m - \mathbf{t}_m - \ell}{d_m}.$$

E.g., thinking of  $p_m = p$  and  $\mathbf{t}_m = 0$ , this is a much stronger condition than is needed with isotropic wavelets of order  $p$ , where  $s^* > (p - \ell)/d = (p - \ell)/\sum_m d_m$  is required.

**1.3. Results from this paper.** A (scalar) PDE that, in variational form, corresponds to a bounded differential operator  $A : H^{\mathbf{t}, \ell}(\Omega) \rightarrow H^{\mathbf{t}, \ell}(\Omega)'$ , has the following general form:

$$\langle Au, v \rangle = \sum_{\{\alpha, \beta : \|\mathbf{max}(|\alpha| - \mathbf{t}, \mathbf{0})\|_1, \|\mathbf{max}(|\beta| - \mathbf{t}, \mathbf{0})\|_1 \leq \ell\}} \int_{\Omega} g_{\alpha, \beta} \partial^{\alpha} v \partial^{\beta} u$$

with  $g_{\alpha, \beta} \in L_{\infty}(\Omega)$ . Here for block-multiindices  $\alpha \in \prod_{m=1}^n \mathbb{N}^{d_m}$ , we set  $\partial^{\alpha} := \otimes_{m=1}^n \partial_m^{\alpha_m}$  with  $\partial_m^{\alpha_m}$  acting in the component domain  $\Omega_m$ , and the vector of their orders  $|\alpha| := (|\alpha_1|, \dots, |\alpha_n|) \in \mathbb{R}^n$ . For  $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^n$ , we set  $\mathbf{max}(\mathbf{s}, \mathbf{s}') := (\max(s_1, s'_1), \dots, \max(s_n, s'_n))$ , and analogously  $\mathbf{min}(\mathbf{s}, \mathbf{s}')$ . Under additional conditions, e.g., coercivity,  $A$  has a bounded inverse. Examples will be given in Sect. 2.

An entry  $\mathbf{A}_{\lambda, \lambda'}$  of the stiffness matrix  $\mathbf{A} = (\mathbf{A}_{\lambda, \lambda'})_{\lambda, \lambda' \in \Lambda \times \Lambda}$  with respect to the tensor product wavelet basis (1.4) reads as

$$(1.8) \quad 2^{-\mathbf{t} \cdot (|\lambda| + |\lambda'|) - \ell (\|\lambda\|_{\infty} + \|\lambda'\|_{\infty})} \\ \times \sum_{\{\alpha, \beta : \|\mathbf{max}(|\alpha| - \mathbf{t}, \mathbf{0})\|_1, \|\mathbf{max}(|\beta| - \mathbf{t}, \mathbf{0})\|_1 \leq \ell\}} \int_{\Omega} g_{\alpha, \beta} \partial^{\alpha} \left( \bigotimes_{m=1}^n \psi_{\lambda_m}^{(m)} \right) \partial^{\beta} \left( \bigotimes_{m=1}^n \psi_{\lambda'_m}^{(m)} \right).$$

We will show that when the parameters  $(p_m, r_m, \tilde{p}_m)$  characterizing the (piecewise smooth) wavelets in the coordinate spaces  $\Omega_m$ , with  $\tilde{p}_m$  being the number of vanishing moments, are chosen to satisfy

$$(1.9) \quad \min_{\{\alpha, \beta: \|\max(\alpha-t, \mathbf{0})\|_1, \|\max(\beta-t, \mathbf{0})\|_1 \leq \ell\}} \left[ \min_m \frac{\tilde{p}_m + \min(|\alpha_m|, |\beta_m|)}{d_m}, \right. \\ \left. \min_{\{m: d_m > 1\}} \frac{r_m + \frac{3}{2} - \max(|\alpha_m|, |\beta_m|)}{d_m - 1} \right] > \min_m \frac{p_m - t_m - \ell}{d_m},$$

the (mixed) derivatives of sufficiently high order of the coefficients  $g_{\alpha, \beta}$  are in  $L_\infty(\Omega)$ , and the PDEs in the coordinate spaces are appended with either periodic or homogeneous Dirichlet boundary conditions, then, with  $s^*$  as in (1.7),  $\mathbf{A}$  is  $s^*$ -computable. With this the adaptive wavelet methods for solving systems  $Au = f$  are shown to converge with the optimal *rate* in linear complexity. Note that the *constant* involved in the error bound for the approximations produced by the adaptive wavelet algorithm might depend on  $n$ .

*Remark 1.1.* The condition (1.9) is in any case satisfied when  $\tilde{p}_m > p_m - t_m - \ell$  and, when  $d_m > 1$ ,  $\frac{r_m + \frac{3}{2} - t_m - \ell}{d_m - 1} > \frac{p_m - t_m - \ell}{d_m}$ , i.e., for spline wavelets,  $\frac{p_m - t_m - \ell}{d_m} > \frac{1}{2}$ .

To arrive at this result on computability of the stiffness matrix, we show that under these conditions, for any  $\alpha, \beta$  in the sum from (1.8), the matrix  $\mathbf{I}^{(\alpha, \beta)} = (\mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)})_{\lambda, \lambda' \in \Lambda \times \Lambda}$  defined by

$$(1.10) \quad \mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)} := 2^{-|\lambda| \cdot |\alpha|} 2^{-|\lambda'| \cdot |\beta|} \int_{\Omega} g_{\alpha, \beta} \partial^{\alpha} \left( \bigotimes_{m=1}^n \psi_{\lambda_m}^{(m)} \right) \cdot \partial^{\beta} \left( \bigotimes_{m=1}^n \psi_{\lambda'_m}^{(m)} \right)$$

is  $s^*$ -computable (Theorem 6.2), so that from

$$(1.11) \quad 2^{-t \cdot (|\lambda| + |\lambda'|) - \ell (\|\lambda\|_\infty + \|\lambda'\|_\infty)} \leq 2^{-(|\lambda| \cdot |\alpha| + |\lambda'| \cdot |\beta|)}$$

for such  $\alpha, \beta$ , the result follows.

For *separable* PDEs, i.e., when each  $g_{\alpha, \beta}$  is a (finite sum of) product(s) of functions on the coordinate spaces,  $\mathbf{A}$  is a finite sum of tensor products of stiffness matrices resulting from PDEs in the coordinate spaces, each of them being  $s^*$ -compressible. As shown in [Nit06], as a consequence,  $\mathbf{A}$  is  $s^*$ -compressible. In this paper, we extend this result to non-separable PDEs, i.e., to coefficients  $g_{\alpha, \beta}$  that are not necessarily of the aforementioned special form.

*Remark 1.2.* Although we consider in this paper only elliptic PDEs, our results on  $s^*$ -compressibility and  $s^*$ -computability of the stiffness matrix also have immediate applications to the efficient solution of parabolic problems by implicit timestepping procedures as discussed, e.g., in [vPS04].

*Remark 1.3.* Generally, (1.11) is not sharp, resulting in compression and quadrature rules that quantitatively can be improved. In order not to (further) complicate the exposition, we did not make an attempt to do so. For the highest order terms, i.e., those with  $\|\alpha\|_1 = \|\beta\|_1 = \|t\|_1 + \ell$ , equality holds if and only if  $|\lambda|_m \equiv \|\lambda\|_\infty$  and  $|\lambda'|_m \equiv \|\lambda'\|_\infty$ . For lower order terms, if present, (1.11) is always crude, resulting in conditions on the number of vanishing moments that are unnecessarily strong. For practical wavelet constructions, however, that usually yield  $\tilde{p}_m \geq p_m$ , this does not do any harm.

The remaining part of this paper is organized as follows. In Sect. 2, we give three examples of applications, the last two involving some extensions of the setting we discussed so far. In Sect. 3, we describe our assumptions on the wavelets in the coordinate spaces, and give some basic estimates. In Sect. 4, we show  $s^*$ -compressibility of the  $\mathbf{I}^{(\alpha,\beta)}$ . After discussing sparse quadrature rules in Sect. 5, in Sect. 6, we show that the  $\mathbf{I}^{(\alpha,\beta)}$  are  $s^*$ -computable. We summarize our results in Sect. 7.

## 2. APPLICATIONS

**2.1. Diffusion problems.** In elliptic and parabolic diffusion problems arising, e.g., in mathematical finance, differential equation of interest is the Dirichlet problem (2.1)

$$-\operatorname{div}G(x)\nabla u = - \sum_{m,m'=1}^n \partial_m g_{m,m'}(x) \partial_{m'} u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where  $\Omega_m = (0, 1)$ , i.e.,  $d_m = 1$ , and so  $\Omega = (0, 1)^n$ , and where the diffusion coefficient  $G(x) \in \mathbb{R}_{sym}^{n \times n}$  is assumed to have smooth component functions  $g_{m,m'}(x)$  and to be uniformly positive definite. The corresponding bilinear form

$$a(u, v) = \sum_{m,m'=1}^n \int_{\Omega} g_{m,m'}(x) \partial_m u \partial_{m'} v$$

defines a boundedly invertible operator  $H_0^{0,1}(\Omega) \rightarrow H_0^{0,1}(\Omega)'$ , where the subscript 0 refers to the incorporation of the homogeneous Dirichlet boundary conditions in the definition of the Sobolev space.

*Remark 2.1.* The diffusion operator (2.1) is (part of) the infinitesimal generator of geometric Brownian Motion which appears in mathematical finance. Other generators  $A$  appear in connection with other Markovian processes; they may be degenerate (see, e.g., [HMS05] and, for the Riesz basis property in this case, [BSS04]), but always fit into our abstract tensor framework.

**2.2. Elliptic  $n$ -scale homogenization.** In [All92, AB96], the following class of scalar elliptic periodic homogenization problems with  $n \geq 2$  scales was considered: let  $D \subset \mathbb{R}^{d_1}$  be a bounded domain and let  $Y_2, \dots, Y_n \subset \mathbb{R}^{d_1}$  be fundamental periods or “cells”. Further, let

$$B(x, y_2, \dots, y_n) \in L^\infty(D \times Y_2 \times \dots \times Y_n; \mathbb{R}_{sym}^{d_1 \times d_1})$$

be a matrix function depending on  $n$  variables taking values in the space  $\mathbb{R}_{sym}^{d_1 \times d_1}$  of symmetric matrices of size  $d_1 \times d_1$ ;  $B$  is assumed periodic with respect to  $y_m$  with period  $Y_m = [0, 1]^{d_1}$  for each  $m = 2, \dots, n$  (different cells  $Y_m$  of possibly different dimensions on each scale would not cause any difficulties). We assume that  $B$  is bounded and uniformly positive definite, i.e., there is a constant  $\gamma > 0$  such that for all  $\xi \in \mathbb{R}^{d_1}$

$$(2.2) \quad \gamma |\xi|^2 \leq \xi^\top B(x, y_2, \dots, y_n) \xi \leq \gamma^{-1} |\xi|^2,$$

uniformly in the “slow variable”  $x \in D$  and in the fast variables  $y_m \in Y_m$ ,  $m = 2, \dots, n$ .

For a scale parameter  $\varepsilon > 0$ , we consider the Dirichlet problem

$$(2.3) \quad -\operatorname{div}B^\varepsilon \nabla u^\varepsilon = f \quad \text{in } D, \quad u^\varepsilon = 0 \quad \text{on } \partial D.$$



The  $d_1 \times d_1$  matrix  $B^\varepsilon$  is assumed to depend on  $\varepsilon$  with multiple scales in the following sense: There are  $n - 1$  positive scale functions  $\varepsilon_1(\varepsilon) > \varepsilon_2(\varepsilon) > \dots > \varepsilon_{n-1}(\varepsilon)$  of  $\varepsilon$  that converge to 0 monotonically when  $\varepsilon \rightarrow 0$  and that are *scale separated*:

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon_{m+1}/\varepsilon_m = 0, \quad m = 1, \dots, n-2,$$

so that for all  $x \in D$  it holds that

$$B^\varepsilon(x) = B\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right).$$

When  $n = 2$ , the scale separation assumption (2.4) is void and we have the classical two-scale homogenization problem

$$(2.5) \quad -\operatorname{div} B\left(x, \frac{x}{\varepsilon}\right) \nabla u^\varepsilon = f,$$

which is dealt with thoroughly in the book by Bensoussan, Lions and Papanicolaou [BLP78]. The purpose of homogenization is the study of the limit of  $u^\varepsilon$  when  $\varepsilon$  converges to 0 and to get an asymptotic expansion of  $u^\varepsilon$  to infer its behaviour when  $\varepsilon > 0$  is small.

It was shown in [All92, AB96] that for small, positive  $\varepsilon > 0$  the homogenization limit and all oscillations of the solution  $u^\varepsilon$  can be described to leading order by a high-dimensional,  $\varepsilon$ -independent limit problem, the so-called  $n$ -scale limit problem. It is posed on the tensor domain  $\mathbf{\Omega} = D \times Y_2 \times \dots \times Y_n$ , which is a special case of our general setting with  $d_m = d_1$  for all  $m$  and thus of total dimension  $d = nd_1$ . For notational convenience, in the following we write  $Y_1$  instead of  $D$ , and  $y_1$  instead of  $x$ .

For the variational formulation of the  $n$ -scale limit, with  $\mathbf{t}^{(m)} := (0, \dots, 0, 1) \in \mathbb{N}^m$ ,  $m = 1, \dots, n$ , we define the space

$$(2.6) \quad \mathbf{V} = \prod_{m=1}^n H^{\mathbf{t}^{(m)}, 0}(Y_1 \times \dots \times Y_m),$$

and endow it with the natural product norm

$$(2.7) \quad \|\{\phi_m\}\|^2 = \sum_{m=1}^n \|\phi_m\|_{H^{\mathbf{t}^{(m)}, 0}(Y_1 \times \dots \times Y_m)}^2.$$

Here, following the definitions given in Sect. 1.2,

$$H^{\mathbf{t}^{(m)}, 0}(Y_1 \times \dots \times Y_m) = \left( \bigotimes_{k=1}^{m-1} L_2(Y_k) \right) \otimes \tilde{H}^1(Y_m) \sim L^2\left(Y_1 \times \dots \times Y_{m-1}; \tilde{H}^1(Y_m)\right)$$

with  $\tilde{H}^1(Y_1) = H_0^1(Y_1)$ , and for  $m > 1$ ,  $\tilde{H}^1(Y_m) = H_{\#}^1(Y_m)$  being the subspace of  $H^1(Y_m)$  of functions whose 1-periodic extension to  $\mathbb{R}^{d_1}$  belongs to  $H_{\text{loc}}^1(\mathbb{R}^{d_1})$  and whose mean over  $Y_m$  vanishes.

Note that  $\mathbf{V}$  is the product of function spaces over tensor product domains of dimension  $d_1, 2d_2, \dots, nd_1$ .

**Proposition 2.2** ([All92, AB96]). *The bilinear form*

$$a(\{u_m\}; \{\phi_{m'}\}) = \int_{Y_1 \times \dots \times Y_n} B(y_1, \dots, y_n) \left( \sum_{m=1}^n \nabla_{y_m} u_m \right) \cdot \left( \sum_{m'=1}^n \nabla_{y_{m'}} \phi_{m'} \right) dy_1 \dots dy_n$$

is bounded and coercive on  $\mathbf{V} \times \mathbf{V}$ . For given  $f \in L_2(Y_1)$ , with  $U = (u_1, \dots, u_n)$  denoting the unique solution of the variational problem: find  $\{u_m\} \in \mathbf{V}$  such that

$$(2.8) \quad a(\{u_m\}; \{\phi_{m'}\}) = \int_{Y_1} f \phi dy_1, \quad \forall \{\phi_{m'}\} \in \mathbf{V},$$

the solution  $u^\varepsilon$  of the problem (2.3) converges weakly to  $u_1$  in  $H_0^1(\Omega)$  and the gradient  $\nabla u^\varepsilon$   $n$ -scale converges to  $\sum_{m=1}^n \nabla_{y_m} u_m$ .

For a proof and the definition of  $n$ -scale convergence, we refer to [AB96, Th. 2.11, eq. (2.9), and Def. 2.3]. Note that the problem (2.8) is independent of  $\varepsilon$ , and can therefore be solved numerically and approximately at a robust, i.e.,  $\varepsilon$ -independent convergence rate.

We are interested in solving (2.8) by adaptive wavelet methods. Therefore, below we transfer results announced in Sect. 1.3 for scalar equations, to the system (2.8) of equations. To this end, for  $m = 1, \dots, n$  we let  $\{\psi_\lambda^{(m)} : \lambda \in \Lambda_m\}$  be a collection of wavelets of order  $p$ , such that for  $\ell' = 0$  or  $\ell' = 1$ ,  $\{2^{-|\lambda|\ell'} \psi_\lambda^{(m)} : \lambda \in \Lambda_m\}$  is a Riesz basis for  $L_2(Y_m)$  or  $\tilde{H}^1(Y_m)$ , respectively, and, with  $\mathbf{u}_m$  denoting the representation of  $u_m \in L_2(Y_m)$  or  $u_m \in \tilde{H}^1(Y_m)$ , respectively, such that for  $s \in (0, \frac{p-\ell'}{d_1})$  with  $\tau = (s + \frac{1}{2})^{-1}$ ,  $u_m \in B_\tau^{s d_1 + \ell'}(L_\tau(Y_m))$  if and only if  $\mathbf{u}_m \in \mathcal{A}_\tau^s$ , i.e., (1.5). Then

$$\{2^{-|\lambda_m|} \bigotimes_{k=1}^m \psi_{\lambda_k}^{(k)} : \lambda \in \prod_{k=1}^m \Lambda_k\} \text{ is a Riesz basis for } H^{(t^{(m)}, 0)} \left( \prod_{k=1}^m Y_k \right),$$

and so the Cartesian product for  $m = 1, \dots, n$  of these bases is a Riesz basis for  $\mathbf{V}$ . With  $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$  denoting the representation of  $U = (u_1, \dots, u_n)$  with respect to this basis, one easily verifies that  $\mathbf{U} \in \mathcal{A}^s$  if and only if  $\mathbf{u}_m \in \mathcal{A}^s$  ( $1 \leq m \leq n$ ). This, in turn, as we have seen in (1.6), holds for  $s \in (0, \frac{p-1}{d_1})$  if and only if

$$u_m \in \left( \bigotimes_{k=1}^{m-1} B_\tau^{s d_1}(L_\tau(Y_k)) \right) \bigotimes_{\tau} B_\tau^{s d_1 + 1}(L_\tau(Y_m)).$$

Moreover,  $\mathbf{U} \in \mathcal{A}^s$  for  $s > \frac{p-1}{d_1}$  cannot be expected by imposing whatever smoothness conditions.

To show that the adaptive wavelet methods for solving (2.8) realize the same convergence rates as that of the best  $N$ -term approximations, we have to show that the stiffness matrix corresponding to the bilinear form  $a$  with respect to the basis of  $\mathbf{V}$  is  $s^*$ -computable for some  $s^* > \frac{p-1}{d_1}$ . This stiffness matrix has a natural  $n \times n$  block partitioning with the  $(m, m')$ th block being the stiffness matrix corresponding to the bilinear form

$$\sum_{\{\alpha \in \mathbb{N}^{d_1} : |\alpha|=1\}} \int_{Y_1 \times \dots \times Y_n} B_{mm'} \partial_m^\alpha u_m \partial_{m'}^\alpha \phi_{m'} dy_1 \dots dy_n$$

with respect to the bases for  $H^{t^{(m)}, 0}(\prod_{k=1}^m Y_k)$  and  $H^{t^{(m')}, 0}(\prod_{k=1}^{m'} Y_k)$ , respectively. In the case of  $m \leq m'$ , this matrix is the sum over  $|\alpha| = 1$  of matrices

$[\dots]_{\boldsymbol{\lambda} \in \prod_{k=1}^m \Lambda_k, \boldsymbol{\lambda}' \in \prod_{k=1}^{m'} \Lambda_k}$ , with the  $(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ th entry given by  $2^{-(|\boldsymbol{\lambda}_m| + |\boldsymbol{\lambda}'_{m'}|)}$  times

$$\int_{\prod_{k=1}^{m'} Y_k} B_{mm'} \left( \left( \bigotimes_{k=1}^{m-1} \psi_{\boldsymbol{\lambda}_k}^{(k)} \right) \otimes \partial_m^\alpha \psi_{\boldsymbol{\lambda}_m}^{(m)} \right) \cdot \left( \left( \bigotimes_{k=1}^{m'-1} \psi_{\boldsymbol{\lambda}'_k}^{(k)} \right) \otimes \partial_{m'}^\alpha \psi_{\boldsymbol{\lambda}'_{m'}}^{(m')} \right) dy_1 \dots dy_{m'}.$$

Setting  $\boldsymbol{\alpha} := (0, \dots, \alpha, \dots, 0) \in \mathbb{N}^{d_1 m'}$ , i.e.,  $\alpha$  on the position of the  $m$ th coordinate direction, and  $\boldsymbol{\beta} := (0, \dots, \alpha) \in \mathbb{N}^{d_1 m'}$ , without the mean value is zero condition of functions in  $H_{\#}^1(Y_m)$ , the last matrix would be a submatrix of  $\mathbf{I}^{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = (\mathbf{I}^{(\boldsymbol{\alpha}, \boldsymbol{\beta})})_{\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in (\prod_{k'=1}^{m'} \Lambda_{k'})^2}$  from (1.10), with  $g_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  reading as  $B_{mm'}$ , by leaving all entries with  $(\boldsymbol{\lambda}_{m+1}, \dots, \boldsymbol{\lambda}_{m'}) \neq 0$ . With wavelets in the coordinate spaces that are piecewise smooth, globally  $C^r$ , and that have  $\tilde{p}$  vanishing moments, and for a  $B_{mm'}$  that is sufficiently smooth, in Theorem 6.2 it will be shown that this  $\mathbf{I}^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$  is  $s^*$ -computable with  $s^* = \frac{\tilde{p}+1}{d_1}$  when  $d_1 = 1$ , and  $s^* = \min(\frac{\tilde{p}+1}{d_1}, \frac{r+\frac{1}{2}}{d_1-1})$  otherwise. Since a submatrix of an  $s^*$ -computable matrix is  $s^*$ -computable, we conclude that for  $\tilde{p} > p - 1$  and, when  $d_1 > 1$ ,  $\frac{r+\frac{1}{2}}{d_1-1} > \frac{p-1}{d_1}$ , i.e.,  $\frac{p-1}{d_1} > \frac{1}{2}$  for spline wavelets, the adaptive wavelet methods for solving (2.8) converge with the best possible rate  $s$  in linear complexity. More importantly, however, at small, positive  $\varepsilon > 0$  the physical solution  $u^\varepsilon$  can be approximated in  $H^1(D)$  in terms of  $u_1, u_2, \dots, u_n$ : as was shown in [AB96], Theorem 2.14, it holds (under the assumption of  $H^2$ -regularity of all components  $u_i$ ) that, as  $\varepsilon \rightarrow 0$ ,

$$u^\varepsilon(x) - \left[ u_1(x) + \sum_{m=2}^n \varepsilon_{m-1} u_m \left( x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{m-1}} \right) \right] \rightarrow 0 \quad \text{strongly in } H^1(D).$$

**2.3. Anisotropic problems.** We consider the model anisotropic problem

$$- \sum_{m=1}^n \partial_m c_m g_m(x) \partial_m u + c_0 u = f \quad \text{in } \boldsymbol{\Omega}, \quad u = 0 \quad \text{on } \partial \boldsymbol{\Omega}$$

where  $\Omega_m = (0, 1)$ , i.e.,  $d_m = 1$ , and so  $\boldsymbol{\Omega} = (0, 1)^n$ ,  $g_m, 1/g_m \in L_\infty(\boldsymbol{\Omega})$  and sufficiently smooth, and the constants  $c_0 \geq 0$ ,  $c_m > 0$  ( $m = 1, \dots, n$ ). Here the interest is to derive results that are valid uniformly in  $\mathbf{c} = (c_0, \dots, c_m)$ .

The corresponding bilinear form

$$a_{\mathbf{c}}(u, v) = \sum_{m=1}^n \int_{\boldsymbol{\Omega}} c_m g_m(x) \partial_m u \partial_m v + \int_{\boldsymbol{\Omega}} c_0 g_0(x) uv$$

defines a uniformly bounded invertible operator from

$$V_{\mathbf{c}} := \bigcap_{k=1}^n \bigotimes_{m=1}^n c_m H_0^{\delta_{mk}}(0, 1) \cap c_0 L_2(\boldsymbol{\Omega})$$

to its dual, where  $H_0^0(0, 1) = L_2(0, 1)$ , and where  $cH$  denotes the space  $H$  equipped with the scaled norm  $c^{\frac{1}{2}} \|\cdot\|_H$ . Assuming that for  $m = 1, \dots, n$ , and for  $\ell' = 0$  or

$\ell' = 1$ ,  $\{2^{-|\lambda|\ell'} \psi_\lambda^{(m)} : \lambda \in \Lambda_m\}$  is a Riesz basis for  $L_2(0,1)$  or  $H_0^1(0,1)$ , respectively,

$$\left\{ \left[ c_0 + \sum_{m=1}^n c_m 4^{|\lambda_m|} \right]^{-\frac{1}{2}} \bigotimes_{m=1}^n \psi_{\lambda_m}^{(m)} : \boldsymbol{\lambda} \in \boldsymbol{\Lambda} \right\} \text{ is a uniform Riesz basis for } V_{\mathbf{c}}.$$

For a proof, we refer to [GO95].

As a consequence, the resulting stiffness matrix  $\mathbf{A}_{\mathbf{c}}$  defines a uniformly boundedly invertible mapping  $\ell_2 \rightarrow \ell_2$ . Its  $(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ th entry is given by

$$\left[ c_0 + \sum_{m=1}^n c_m 4^{|\lambda_m|} \right]^{-\frac{1}{2}} \left[ c_0 + \sum_{m=1}^n c_m 4^{|\lambda'_m|} \right]^{-\frac{1}{2}} a_{\mathbf{c}} \left( \bigotimes_{m=1}^n \psi_{\lambda_m}^{(m)}, \bigotimes_{m=1}^n \psi_{\lambda'_m}^{(m)} \right).$$

Since  $[c_0 + \sum_{m=1}^n c_m 4^{|\lambda_m|}]^{-\frac{1}{2}} [c_0 + \sum_{m=1}^n c_m 4^{|\lambda'_m|}]^{-\frac{1}{2}} c_0 \leq 1$ , and, for  $m = 1, \dots, n$ ,  $[c_0 + \sum_{m=1}^n c_m 4^{|\lambda_m|}]^{-\frac{1}{2}} [c_0 + \sum_{m=1}^n c_m 4^{|\lambda'_m|}]^{-\frac{1}{2}} c_m \leq 2^{-|\lambda_m|} 2^{-|\lambda'_m|}$ , the forthcoming Theorem 6.2, dealing with computability of  $\mathbf{I}^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ , shows that with wavelets in the coordinate spaces of order  $p_m \equiv p$ , with  $\tilde{p}_m \equiv \tilde{p} > p - 1$  vanishing moments,  $\mathbf{A}_{\mathbf{c}}$  is uniformly  $s^*$ -computable with  $s^* = \tilde{p} > p - 1$ , the latter being the maximum rate of convergence that can be expected for best  $N$ -term approximations. So whenever for some  $0 < s < s^*$ , the representation  $\mathbf{u}$  of the solution  $u$  is in  $\mathcal{A}^s$ , then given a tolerance  $\varepsilon > 0$ , the adaptive wavelet algorithms produce approximations to  $\mathbf{u}$  within this tolerance taking  $\mathcal{O}(\varepsilon^{-1/s} [\sup_N N^s \|\mathbf{u} - \mathbf{u}_N\|]^{1/s})$  operations uniformly in  $\mathbf{c}$ .

Finally, membership of  $\mathbf{u} \in \mathcal{A}^s$  is implied by that of  $\mathbf{u} \in \mathcal{A}_\tau^s$ , which for  $s \in (0, p - 1)$  and  $\tau = (\frac{1}{2} + s)^{-1}$  is equivalent, uniformly in  $\mathbf{c}$ , to membership of  $u \in \bigcap_{k=1}^n \bigotimes_{m=1}^n c_m B_\tau^{s+\delta_{mk}}(L_\tau(0,1)) \cap c_0 \bigotimes_{m=1}^n B_\tau^s(L_\tau(0,1))$ , as follows from a generalization of (1.6).

### 3. WAVELETS AND BASIC ESTIMATES IN THE COORDINATE SPACES

For  $1 \leq m \leq n$ , let  $\Omega_m$  be a domain in  $\mathbb{R}^{d_m}$ . Let  $\{\psi_\lambda^{(m)} : \lambda \in \Lambda_m\} \subset L_2(\Omega_m)$  be a collection of functions (wavelets), such that for some  $\tilde{p}_m \in \mathbb{N}$ ,

$$(3.1) \quad \left| \int_{\Omega_m} u \psi_\lambda^{(m)} \right| \lesssim 2^{-|\lambda|(\frac{d_m}{2} + s)} \|u\|_{W_\infty^s(\text{supp } \psi_\lambda^{(m)})}, \quad s \in [0, \tilde{p}_m],$$

where  $|\lambda| \in \mathbb{N}$  denotes the level of  $\psi_\lambda^{(m)}$ . The estimate (3.1) can be shown when  $\|\psi_\lambda^{(m)}\|_{L_2(\Omega_m)} \lesssim 1$ , the wavelets are *locally supported*, in the sense that  $\text{vol}(\text{supp } \psi_\lambda^{(m)}) \lesssim 2^{-|\lambda|d_m}$ , and when if  $\tilde{p}_m > 0$ , possibly after some smooth transformation of coordinates,  $\psi_\lambda^{(m)}$  for  $|\lambda| > 0$  has  $\tilde{p}_m$  *vanishing moments*, meaning that  $\psi_\lambda^{(m)} \perp_{L_2(\Omega_m)} P_{\tilde{p}_m-1}$ . Actually, when  $d_m > 1$ , with some constructions  $\text{supp } \psi_\lambda^{(m)}$  in (3.1) should read as a neighbourhood of  $\text{supp } \psi_\lambda^{(m)}$  with a comparable diameter. For convenience we ignore this fact, but our results extend trivially to this situation.

In addition to being locally supported, we assume that the wavelets are *piecewise smooth* in the following sense: For all  $l \in \mathbb{N}$ , there exists a collection  $\{\Omega_m^{(l,v)} : v \in \mathcal{O}_l^{(m)}\}$  of disjoint, uniformly shaped regular, open subdomains, such that  $\bar{\Omega}_m = \bigcup_{v \in \mathcal{O}_l^{(m)}} \bar{\Omega}_m^{(l,v)}$ ,  $\bar{\Omega}_m^{(l,v)}$  is the union of some  $\bar{\Omega}_m^{(l+1,\tilde{v})}$ ,  $\text{diam}(\Omega_m^{(l,v)}) \approx 2^{-l}$ ,  $\text{supp } \psi_\lambda^{(m)}$  is connected and is the union of a uniformly bounded number of  $\bar{\Omega}_m^{(l,\tilde{v})}$ , and each  $\bar{\Omega}_m^{(l,v)}$  has a non-empty intersection with the supports of a uniformly

bounded number of  $\psi_\lambda^{(m)}$  with  $|\lambda| = l$ . So, in particular, the singular support of  $\psi_\lambda^{(m)}$  is part of the boundaries of the  $\Omega_m^{(|\lambda|, v)}$ .

Finally, for some  $r_m \in \mathbb{N} \cup \{-1\}$ , we will assume that

$$(3.2) \quad \|\psi_\lambda^{(m)}\|_{W_\infty^s(\Omega_m)} \lesssim 2^{|\lambda|(\frac{d_m}{2}+s)}, \quad s \in [0, r_m + 1],$$

and that for *any*  $s \geq 0$ ,

$$(3.3) \quad \|\psi_\lambda^{(m)}\|_{W_\infty^s(\Omega_m^{(|\lambda|, v)})} \lesssim 2^{|\lambda|(\frac{d_m}{2}+s)}.$$

These estimates follow from a homogeneity argument assuming that *the wavelets are globally in  $C^{r_m}(\Omega_m)$*  which, for  $r_m = -1$ , is to be understood as possibly discontinuous wavelet functions.

In the periodic setting that was discussed in Subsect. 2.2 for the coordinate spaces  $Y_i$  for  $i > 0$ , the above conditions on the wavelets should be read as to hold for their periodic extensions. In a non-periodic setting, in the following Lemma 3.1 we will additionally assume homogeneous Dirichlet boundary conditions that are satisfied in the applications discussed in Sect. 2.

For convenience, in the following we restrict ourselves to the common situation that

$$\tilde{p}_m \geq r_m + 2.$$

With this we have presented all conditions on the wavelet we will need in our investigation of computability of stiffness matrices of PDEs with respect to tensor product wavelets. To use wavelets as a suitable discretization tool, however, more properties are required. Basically, for having (1.3) and (1.5), one needs that the wavelets have some order  $p_m$ , meaning that locally any polynomial of degree  $p_m - 1$  can be reproduced by the wavelet basis, that the wavelets generate a Riesz basis for  $L_2(\Omega_m)$ , and that the dual wavelets, which have order  $\tilde{p}_m$ , have some smoothness, i.e., satisfy a Bernstein inequality like (3.2). For a detailed discussion, we refer to [Coh03]. In the literature, one can find various constructions of suitable wavelets on the interval (e.g., [DKU99, Pri06]), or cubes, and, usually via some domain decomposition approach, on general domains (e.g., [DS99a, CTU99, DS99b, DS99c, HS04, Ste04]). From the introduction, recall that the larger the orders  $p_m$  are, the better compressible the stiffness matrix should be, which will induce stronger conditions on the values of the  $\tilde{p}_m$  and  $r_m$ .

For  $(\lambda, \lambda') \in \Lambda_m \times \Lambda_m$ , we define the indicator  $i_m(\lambda, \lambda') \in \{0, 1\}$  by

$$i_m(\lambda, \lambda') := 0 \text{ when } \begin{cases} \psi_{\lambda'}^{(m)} \text{ vanishes on } \text{singsupp } \psi_\lambda^{(m)} \text{ when } |\lambda'| \geq |\lambda|, \text{ or} \\ \psi_\lambda^{(m)} \text{ vanishes on } \text{singsupp } \psi_{\lambda'}^{(m)} \text{ when } |\lambda'| < |\lambda|, \end{cases}$$

and  $i_m(\lambda, \lambda') := 1$  otherwise; see Figure 1. For  $\alpha \in \mathbb{N}^{d_m}$ , let  $\partial_m^\alpha$  denote the canonical partial differentiation operator on  $\Omega_m$  of order  $|\alpha| = \sum_{k=1}^{d_m} \alpha_k$ .

**Lemma 3.1.** *For some, sufficiently smooth, real valued function  $g_m$  on  $\Omega_m$ ,  $\lambda, \lambda' \in \Lambda_m$  with  $|\lambda| \leq |\lambda'|$ , and  $|\alpha|, |\beta| \leq r_m + 1$ , for*

$$(3.4) \quad I_{\lambda, \lambda', m}^{(\alpha, \beta)} := 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \int_{\Omega_m} g_m \partial_m^\alpha \psi_\lambda^{(m)} \partial_m^\beta \psi_{\lambda'}^{(m)},$$

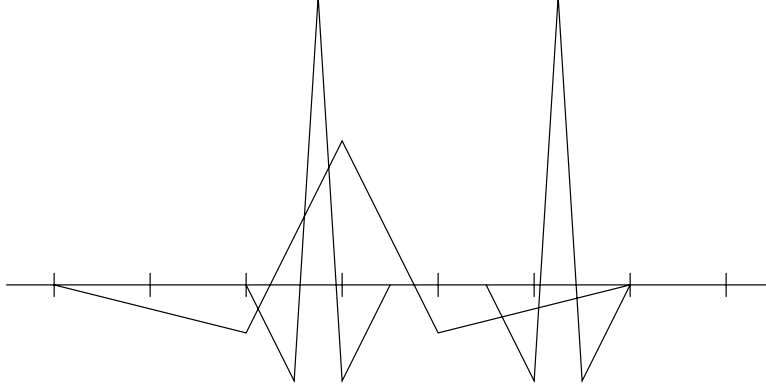


FIGURE 1. Wavelets  $\psi_{\lambda'}^{(m)}$  that vanish (right case), i.e.,  $i_m(\lambda, \lambda') = 0$ , or do not vanish (left case), i.e.,  $i_m(\lambda, \lambda') = 1$ , on the singular support of a wavelet  $\psi_{\lambda}^{(m)}$  with  $|\lambda| \leq |\lambda'|$ . The subdivision of  $\Omega_m$  into  $\Omega_m^{|\lambda|,v}$  is also indicated

we have

$$|I_{\lambda, \lambda', m}^{(\alpha, \beta)}| \lesssim \begin{cases} 2^{(|\lambda| - |\lambda'|)(\frac{d_m}{2} + r_m + 1 - |\alpha|)} \|g_m\|_{W_{\infty}^{r_m + 1 - |\alpha|}(\text{supp } \psi_{\lambda}^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} & \text{for any } \lambda, \lambda' \in \Lambda_m, \\ 2^{(|\lambda| - |\lambda'|)(\frac{d_m}{2} + \tilde{r}_m + |\beta|)} \|g_m\|_{W_{\infty}^{\tilde{r}_m + |\beta|}(\text{supp } \psi_{\lambda}^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} & \text{when } i_m(\lambda, \lambda') = 0. \end{cases}$$

For this to hold in a non-periodic setting, we will assume that all wavelets satisfy homogeneous Dirichlet boundary conditions  $\partial^{\eta} \psi_{\lambda}^{(m)} = 0$  on  $\partial\Omega_m$  for all  $\eta \leq \beta$  with  $\eta \neq \beta$ .

*Proof.* To prove the first estimate, we distinguish between two cases. When  $|\beta + \alpha| \geq r_m + 1$ , select  $\gamma \leq \beta$  with  $|\gamma + \alpha| = r_m + 1$ . Integration by parts, where we use the homogeneous Dirichlet boundary conditions in the non-periodic setting, and (3.2) for both  $\psi_{\lambda}^{(m)}$  and  $\psi_{\lambda'}^{(m)}$  show that

$$\begin{aligned} |I_{\lambda, \lambda', m}^{(\alpha, \beta)}| &= 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \left| \int_{\text{supp } \psi_{\lambda}^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)}} (-1)^{|\gamma|} \partial_m^{\gamma} (g_m \partial_m^{\alpha} \psi_{\lambda}^{(m)}) \partial_m^{\beta - \gamma} \psi_{\lambda'}^{(m)} \right| \\ &\lesssim 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \|g_m\|_{W_{\infty}^{r_m + 1 - |\alpha|}(\text{supp } \psi_{\lambda}^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} \\ &\quad \times 2^{-|\lambda'|d_m} 2^{|\lambda|(\frac{d_m}{2} + r_m + 1)} 2^{|\lambda'|(\frac{d_m}{2} + |\beta| - r_m + 1 + |\alpha|)} \\ &= \|g_m\|_{W_{\infty}^{r_m + 1 - |\alpha|}(\text{supp } \psi_{\lambda}^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} 2^{(|\lambda| - |\lambda'|)(\frac{d_m}{2} + r_m + 1 - |\alpha|)}. \end{aligned}$$

When  $|\beta + \alpha| \leq r_m + 1$ , by integration by parts, (3.2), and (3.1) with  $s + |\alpha| + |\beta| = r_m + 1$ , where we used that  $r_m + 1 - |\alpha| - |\beta| \leq r_m + 1 \leq \tilde{p}_m - 1 \leq \tilde{p}_m$ , we have

$$\begin{aligned} |I_{\lambda, \lambda', m}^{(\alpha, \beta)}| &= 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \left| \int_{\Omega_m} (-1)^{|\beta|} \partial_m^\beta (g_m \partial_m^\alpha \psi_\lambda^{(m)}) \psi_{\lambda'}^{(m)} \right| \\ &\lesssim 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \|g_m\|_{W_\infty^{r_m+1-|\alpha|}(\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} \\ &\quad \times 2^{|\lambda|(\frac{d_m}{2} + r_m + 1)} 2^{-|\lambda'|(\frac{d_m}{2} + r_m + 1 - |\alpha| - |\beta|)} \\ &= \|g_m\|_{W_\infty^{r_m+1-|\alpha|}(\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} 2^{(|\lambda| - |\lambda'|)(\frac{d_m}{2} + r_m + 1 - |\alpha|)}. \end{aligned}$$

When  $\psi_{\lambda'}^{(m)}$  vanishes on  $\text{singsupp } \psi_\lambda^{(m)}$ , from (3.1) and (3.3) we have

$$\begin{aligned} |I_{\lambda, \lambda', m}^{(\alpha, \beta)}| &= 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \left| \int_{\Omega_m} (-1)^{|\beta|} \partial_m^\beta (g_m \partial_m^\alpha \psi_\lambda^{(m)}) \psi_{\lambda'}^{(m)} \right| \\ &\lesssim 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \|g_m\|_{W_\infty^{|\beta| + \tilde{p}_m}(\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} \\ &\quad \times 2^{|\lambda|(\frac{d_m}{2} + |\alpha| + |\beta| + \tilde{p}_m)} 2^{-|\lambda'|(\frac{d_m}{2} + \tilde{p}_m)} \\ &= 2^{(|\lambda| - |\lambda'|)(\frac{d_m}{2} + \tilde{p}_m + |\beta|)} \|g_m\|_{W_\infty^{\tilde{p}_m + |\beta|}(\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})}, \end{aligned}$$

which is the second estimate.  $\square$

All our results concerning compressibility and computability of differential operators with respect to tensorized wavelet bases will be based on this Lemma 3.1. Unfortunately, in order to get optimal results, we cannot rely solely on the first estimate, but instead, we have to use that for the vast majority of pairs  $(\lambda, \lambda') \in \Lambda_m \times \Lambda_m$  the stronger second estimate is valid.

#### 4. COMPRESSIBILITY

We will use the following notations, some of them already introduced in Sect. 1.2:  $\Omega := \prod_{m=1}^n \Omega_m$ ,  $\tilde{\mathbf{p}} := (\tilde{p}_1, \dots, \tilde{p}_n)$ ,  $\mathbf{r} := (r_1, \dots, r_n)$ ,  $\mathbf{1} := (1, \dots, 1)$ ,  $\mathbf{d} := (d_1, \dots, d_n)$ , and  $\Lambda := \prod_{m=1}^n \Lambda_m$ . For  $\alpha \in \prod_{m=1}^n \mathbb{N}^{d_m}$ , we set  $\partial^\alpha := \bigotimes_{m=1}^n \partial_m^{\alpha_m}$ , and  $|\alpha| := (|\alpha_1|, \dots, |\alpha_n|) \in \mathbb{R}^n$ . For  $\lambda \in \Lambda$ , we set  $|\lambda| := (|\lambda_1|, \dots, |\lambda_n|) \in \mathbb{R}^n$ . For  $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^n$ , we set  $\min(\mathbf{s}, \mathbf{s}') := (\min(s_1, s'_1), \dots, \min(s_n, s'_n))$ , analogously  $\max(\mathbf{s}, \mathbf{s}')$ , and  $|\mathbf{s} - \mathbf{s}'| = (|s_1 - s'_1|, \dots, |s_n - s'_n|)$ . For  $(\lambda, \lambda') \in \Lambda \times \Lambda$ , we set

$$\mathbf{i}(\lambda, \lambda') = (i_1(\lambda_1, \lambda'_1), \dots, i_n(\lambda_n, \lambda'_n)) \in \{0, 1\}^n$$

inducing a decomposition of  $\Lambda \times \Lambda$  into  $2^n$  disjoint sets.

**Theorem 4.1.** *Let  $\alpha, \beta \in \prod_{m=1}^n \mathbb{N}^{d_m}$  with  $\max(|\alpha|, |\beta|) \leq \mathbf{r} + \mathbf{1}$ . For  $g$  being a real-valued function on  $\Omega$  with, for  $\gamma \in \prod_{m=1}^n \mathbb{N}^{d_m}$ ,*

$$(4.1) \quad \partial^\gamma g \in L_\infty(\Omega), \quad |\gamma| \leq \tilde{\mathbf{p}} + \min(|\alpha|, |\beta|),$$

let the infinite matrix  $\mathbf{I}^{(\alpha, \beta)} = (I_{\lambda, \lambda'}^{(\alpha, \beta)})_{\lambda, \lambda' \in \Lambda}$  be defined by

$$I_{\lambda, \lambda'}^{(\alpha, \beta)} := 2^{-|\lambda||\alpha|} 2^{-|\lambda'||\beta|} \int_{\Omega} g \cdot \partial^\alpha \bigotimes_{m=1}^n \psi_{\lambda_m}^{(m)} \cdot \partial^\beta \bigotimes_{m=1}^n \psi_{\lambda'_m}^{(m)}.$$

Defining, for  $\mathbf{i} \in \{0, 1\}^n$ ,  $\mathbf{z}^{(\mathbf{i})} = (z_1^{(\mathbf{i})}, \dots, z_n^{(\mathbf{i})}) \in \mathbb{R}^n$  by

$$(4.2) \quad z_m^{(\mathbf{i})} = \begin{cases} \tilde{p}_m + \min(|\alpha_m|, |\beta_m|) & \text{when } i = 0, \\ r_m + \frac{3}{2} - \max(|\alpha_m|, |\beta_m|) & \text{when } i = 1, \end{cases}$$

for any  $q \in \mathbb{N}$  we define the compressed matrix  $\mathbf{I}_q^{(\alpha, \beta)}$  as follows: we drop the entry  $\mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)}$  when

$$(4.3) \quad \left| |\lambda'| - |\lambda| \right| \cdot z^{(i(\lambda, \lambda'))} > q.$$

Then the resulting error in the matrix can be bounded by

$$\|\mathbf{I}^{(\alpha, \beta)} - \mathbf{I}_q^{(\alpha, \beta)}\| \lesssim q^{n-1} 2^{-q},$$

and, with

$$(4.4) \quad s^* := \left[ \max_{1 \leq m \leq n} \max \left( \frac{d_m}{\tilde{p}_m + \min(|\alpha_m|, |\beta_m|)}, \frac{d_m - 1}{r_m + \frac{3}{2} - \max(|\alpha_m|, |\beta_m|)} \right) \right]^{-1},$$

the number of non-zero entries per row and column of  $\mathbf{I}_q^{(\alpha, \beta)}$  can be bounded on some absolute multiple of

$$q^{n-1} 2^{q/s^*},$$

in particular showing that  $\mathbf{I}^{(\alpha, \beta)}$  is  $s^*$ -compressible.

Note that by  $\tilde{p}_m \geq r_m + 2$ , the more coordinate directions  $m$  there are in which the interior of the support of one of  $\psi_{\lambda_m}^{(m)}$  or  $\psi_{\lambda'_m}^{(m)}$  has non-empty intersection with the singular support of the other, the more stringent is the dropping criterion. Furthermore, note that with the definition of (4.2), Lemma 3.1 can be reformulated as

$$(4.5) \quad |I_{\lambda, \lambda', m}^{(\alpha, \beta)}| \lesssim 2^{-|\lambda| - |\lambda'|} \left| \left( \frac{d_m}{2} + z_m^{(i_m(\lambda, \lambda'))} - \frac{i_m(\lambda, \lambda')}{2} \right) \right. \\ \left. \times \|g_m\|_{W_\infty^{z_m^{(i_m(\lambda, \lambda'))} - \frac{i_m(\lambda, \lambda')}{2}}(\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})} \right|.$$

To prove Theorem 4.1 we start with a lemma.

**Lemma 4.2.** For any  $q \in \mathbb{N}$ , let  $\mathbf{B}_q = ((\mathbf{B}_q)_{\lambda, \lambda'})_{\lambda, \lambda' \in \Lambda}$  be a matrix, partitioned into blocks  $(\mathbf{B}_q)_{l, l'} = ((\mathbf{B}_q)_{\lambda, \lambda'})_{|\lambda|=l, |\lambda'|=l'}$ , such that for some fixed  $\mathbf{w} \in (0, \infty)^n$  and  $\bar{s} > 0$ , the maximal number of non-zero entries in each row of  $(\mathbf{B}_q)_{l, l'}$  or column of  $(\mathbf{B}_q)_{l', l}$  is bounded by some absolute multiple of

$$2^{\max(l' - l, 0) \cdot \mathbf{w} / \bar{s}},$$

and

$$(\mathbf{B}_q)_{l, l'} = 0 \quad \text{when } |l' - l| \cdot \mathbf{w} > q.$$

Then the number of non-zero entries in each row and column of  $\mathbf{B}_q$  is for  $q \rightarrow \infty$  bounded by  $\mathcal{O}(q^{n-1} 2^{q/\bar{s}})$ .

*Proof.* It is sufficient to count the number of non-zero entries  $(\mathbf{B}_q)_{\lambda, \lambda'}$  for any fixed  $\lambda$  and all  $\lambda'$  with  $|\lambda'| \geq |\lambda|$ , which can be bounded by some absolute multiple of

$$\sum_{\{l': 0 \leq (l' - |\lambda|) \cdot \mathbf{w} \leq q\}} 2^{(l' - |\lambda|) \cdot \mathbf{w} / \bar{s}} \lesssim q^{n-1} 2^{q/\bar{s}}. \quad \square$$

*Proof of Theorem 4.1.* Suppressing the dependence on  $(\alpha, \beta)$  in the notation, we write  $\mathbf{I} = \sum_{i \in \{0, 1\}^n} \mathbf{I}^{(i)}$  ( $\mathbf{I}_q = \sum_{i \in \{0, 1\}^n} \mathbf{I}_q^{(i)}$ ), where  $\mathbf{I}^{(i)}$  ( $\mathbf{I}_q^{(i)}$ ) contains all non-zero entries with indices  $\lambda, \lambda'$  with  $i(\lambda, \lambda') = i$ . By the dropping rule (4.3), the non-zero part of  $\mathbf{I}_q^{(i)}$  consists of those blocks  $\mathbf{I}_{l, l'}^{(i)} = (\mathbf{I}_{\lambda, \lambda'}^{(i)})_{|\lambda|=l, |\lambda'|=l'}$  with  $|l' - l| \cdot z^{(i)} \leq q$ .



By the local supports and the piecewise smoothness of the wavelets, the number of non-zero entries in each row of  $\mathbf{I}_{\mathbf{l},\mathbf{l}'}$  or column of  $\mathbf{I}_{\mathbf{l}',\mathbf{l}}$  can be bounded on some absolute multiple of

$$(4.6) \quad 2^{\max(\mathbf{l}'-\mathbf{l},0)\cdot(\mathbf{d}-\mathbf{i})}.$$

By definition,  $s^*$  is the largest constant such that  $\mathbf{d}-\mathbf{i} \leq \mathbf{z}^{(i)}/s^*$ . By an application of Lemma 4.2, we conclude that the number of non-zero entries in each row or column of  $\mathbf{I}_q^{(i)}$  is  $\mathcal{O}(q^{n-1}2^{q/s^*})$ .

By (4.5) and the assumption (4.1), a tensor product argument (e.g., [LC85]) shows that

$$\begin{aligned} |\mathbf{I}_{\lambda,\lambda'}^{(\alpha,\beta)}| &\lesssim 2^{-|\lambda'|-|\lambda|}\cdot\left(\frac{q}{2}+\mathbf{z}^{(i(\lambda,\lambda'))}-\frac{\mathbf{i}(\lambda,\lambda')}{2}\right) \\ &\quad \times \max_{|\gamma|\leq\mathbf{z}^{(i(\lambda,\lambda'))}-\frac{\mathbf{i}(\lambda,\lambda')}{2}} \|\partial^\gamma g\|_{L^\infty(\prod_{m=1}^n \text{supp } \psi_{\lambda_m}^{(m)} \cap \text{supp } \psi_{\lambda'_m}^{(m)})}. \end{aligned}$$

By bounding  $\|\mathbf{I}_{\mathbf{l},\mathbf{l}'}^{(i)}\|^2$  on the product of its maximal absolute row-sum and its maximal absolute column-sum (Schur lemma), we find that

$$\|\mathbf{I}_{\mathbf{l},\mathbf{l}'}^{(i)}\| \lesssim 2^{-|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{z}^{(i)}}.$$

From  $\|\mathbf{I}^{(i)} - \mathbf{I}_q^{(i)}\|^2 \leq \max_{\mathbf{l}} \sum_{\{\mathbf{l}':|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{z}^{(i)}>q\}} \|\mathbf{I}_{\mathbf{l},\mathbf{l}'}^{(i)}\| \times \max_{\mathbf{l}'} \sum_{\{\mathbf{l}:|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{z}^{(i)}>q\}} \|\mathbf{I}_{\mathbf{l},\mathbf{l}'}^{(i)}\|$ , we conclude that

$$\|\mathbf{I}^{(i)} - \mathbf{I}_q^{(i)}\| \lesssim q^{n-1}2^{-q},$$

which completes the proof.  $\square$

When, say for  $|\lambda'_m| \geq |\lambda_m|$ ,  $i_m(\lambda_m, \lambda'_m) = 1$ , i.e.,  $\text{supp } \psi_{\lambda'_m}^{(m)} \cap \text{sing } \text{supp } \psi_{\lambda'_m}^{(m)} \neq \emptyset$ , in the above proof we had to apply the less favourable first bound from Lemma 3.1. On the other hand, since  $\text{sing } \text{supp } \psi_{\lambda'_m}^{(m)}$  is only  $(d_m - 1)$ -dimensional, the larger  $\sum_{m=1}^n i_m$  is, the more sparse the block  $\mathbf{I}_{\mathbf{l},\mathbf{l}'}^{(i)}$  is, meaning that we could keep more of these blocks and still respect the complexity bound (this explains the factor  $d_m - 1$  instead of  $d_m$  in the expression  $\frac{d_m-1}{r_m+\frac{3}{2}-\max(|\alpha_m|,|\beta_m|)}$  from (4.4)). This better sparsity also resulted in a sharper estimate for  $\|\mathbf{I}_{\mathbf{l},\mathbf{l}'}^{(i)}\|$  in terms of an upper bound for its entries using the Schur lemma (this explains the factor  $r_m + \frac{3}{2}$  instead of  $r_m + 1$  in the expression  $\frac{d_m-1}{r_m+\frac{3}{2}-\max(|\alpha_m|,|\beta_m|)}$ , as well as in the definition of  $z_m^{(i)}$  in (4.2)).

## 5. QUADRATURE

We study the problem of approximating the (remaining) entries  $\mathbf{I}_{\lambda,\lambda'}^{(\alpha,\beta)}$  with numerical quadrature. We assume that for  $1 \leq m \leq n$  and  $a_m, b_m \geq 0$ , to approximate  $\mathbf{I}_{\lambda,\lambda',m}^{(\alpha,\beta)}$  ( $\lambda, \lambda' \in \Lambda_m$ ) from (3.4), where  $g_m(y) := g(x_1, \dots, x_{m-1}, y, x_{m+1}, \dots, x_n)$  with fixed  $x \in \Omega$ , we have available a family of quadrature rules  $(Q_{\lambda,\lambda',m,N}^{(\alpha,\beta)})_{N \in \mathbb{N}}$ , where  $Q_{\lambda,\lambda',m,N}^{(\alpha,\beta)}$  has  $\mathcal{O}(N)$  abscissae, such that for some  $k_m \in \mathbb{N}$ ,

$$(5.1) \quad \begin{aligned} &|\mathbf{I}_{\lambda,\lambda',m}^{(\alpha,\beta)} - Q_{\lambda,\lambda',m,N}^{(\alpha,\beta)}| \\ &\lesssim N^{-a_m} 2^{-|\lambda|-|\lambda'|}\left(\frac{d_m}{2}+b_m\right) \|g_m\|_{W_\infty^{k_m}(\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)})}. \end{aligned}$$

Before continuing, we verify this assumption in a common situation. Let us assume that  $|\lambda'| \geq |\lambda|$ . Suppose that for any  $l' \in \mathbb{N}$  and  $v' \in \mathcal{O}_{l'}^{(m)}$ , there exists a sufficiently smooth transformation of coordinates  $\kappa_m$ , with derivatives bounded uniformly in  $l'$  and  $v'$ , such that for some  $e_m \in \mathbb{N}$ , and all  $|\lambda'| = l'$ ,

$$\psi_{\lambda'}^{(m)} \circ \kappa_m|_{\kappa_m^{-1}(\Omega_m^{(l',v')})} \in P_{e_m-1}.$$

In the following, without loss of generality, for notational convenience, we take  $\kappa_m = \text{id}$ .

To approximate an integral  $\int_{\Omega_m^{(l',v')}} f$ , for any  $k \in \mathbb{N}$  we assume to have available composite quadrature rules  $Q_{\Omega_m^{(l',v')}, N}(f)$  of *fixed order* (i.e, the degree of polynomial exactness plus one)  $k$ , and *variable rank*  $N$ . The rank  $N$  of a composite quadrature formula denotes the number of subdomains on which the elementary quadrature formula is applied. Since the order  $k$  of  $Q_{\Omega_m^{(l',v')}, N}$  is fixed, the number of abscissae in the composite rule  $Q_{\Omega_m^{(l',v')}, N}$  is  $\mathcal{O}(N)$ . We assume that the composite quadrature rule  $Q_{\Omega_m^{(l',v')}, N}$  satisfies the error estimate

$$(5.2) \quad \begin{aligned} & \left| \int_{\Omega_m^{(l',v')}} f - Q_{\Omega_m^{(l',v')}, N}(f) \right| \\ & \lesssim \text{vol}(\Omega_m^{(l',v')}) N^{-k/d_m} \text{diam}(\Omega_m^{(l',v')})^k \|f\|_{W_\infty^k(\Omega_m^{(l',v')})} \end{aligned}$$

(cf. [GS04, §2]).

To find an upper bound for the quadrature error when these rules are applied with integrand  $2^{-|\lambda||\alpha_m|} 2^{-|\lambda'||\beta_m|} g_m \partial_m^{\alpha_m} \psi_\lambda^{(m)} \partial_m^{\beta_m} \psi_{\lambda'}^{(m)}$ , we have to bound  $(\partial_m^\rho g_m)(\partial_m^\sigma \partial_m^{\alpha_m} \psi_\lambda^{(m)})(\partial_m^\tau \partial_m^{\beta_m} \psi_{\lambda'}^{(m)})$  for  $|\rho| + |\sigma| + |\tau| \leq k$ .

Since  $g_m$  is assumed to be sufficiently smooth,  $|\lambda'| \geq |\lambda|$ , and  $\partial_m^\tau \partial_m^{\beta_m} \psi_{\lambda'}^{(m)}$  vanishes when  $|\tau| + |\beta_m| \geq e_m$ , by invoking (3.2) we see that the worst case occurs when  $\rho = 0$ ,  $|\tau| + |\beta_m| = q := \min(e_m - 1, k + |\beta_m|)$ , and thus when  $|\sigma| = k - q + |\beta_m|$ , yielding

$$\begin{aligned} & \|2^{-|\lambda||\alpha_m|} 2^{-|\lambda'||\beta_m|} g_m \partial_m^{\alpha_m} \psi_\lambda^{(m)} \partial_m^{\beta_m} \psi_{\lambda'}^{(m)}\|_{W_\infty^k(\Omega_m^{(l',v')})} \\ & \lesssim 2^{(|\lambda|+|\lambda'|)\frac{d_m}{2}} 2^{(k-q+|\beta_m|)|\lambda|} 2^{(q-|\beta_m|)|\lambda'|} \|g_m\|_{W_\infty^k(\Omega_m^{(l',v')})}. \end{aligned}$$

Using the facts that  $\text{diam}(\Omega_m^{(l',v')}) \approx 2^{-|\lambda'|}$ ,  $\text{vol}(\Omega_m^{(l',v')}) \approx 2^{-|\lambda'|d_m}$ , by substituting this result into (5.2), summing over the uniformly bounded number of  $\Omega_m^{(l',v')}$  that make up  $\text{supp } \psi_\lambda^{(m)} \cap \text{supp } \psi_{\lambda'}^{(m)}$ , and by taking  $k = k_m$  satisfying

$$k_m \geq \max(d_m a_m, b_m - |\beta_m| + e_m - 1),$$

we end up with (5.1).

From now on always taking  $b_m := \tilde{p}_m + \min(|\alpha_m|, |\beta_m|) \geq z_m^{(i_m(\lambda, \lambda'))} - \frac{i_m(\lambda, \lambda')}{2}$  ( $\lambda, \lambda' \in \Lambda_m$ ) and the corresponding  $k_m$  (which also depends on  $a_m$ ), using (4.5) and (5.1) we have the following result for the standard tensor product quadrature rule:

**Lemma 5.1.** *For  $g$  being a function on  $\Omega$  with*

$$\partial^\gamma g \in L^\infty(\Omega) \quad \text{for all } |\gamma| \leq \mathbf{max}(\mathbf{k}, \tilde{\mathbf{p}} + \mathbf{min}(|\alpha|, |\beta|)),$$

where  $\mathbf{k} = (k_1, \dots, k_n)$ , we have

$$\begin{aligned} \left| \mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)} - \bigotimes_{m=1}^n Q_{\lambda_m, \lambda'_m, m, N_m}^{(\alpha_m, \beta_m)} \right| &\lesssim \left( \sum_{m=1}^n N_m^{-a_m} \right) 2^{-|\lambda| - |\lambda'|} \cdot \left( \frac{d}{2} + z^{(i(\lambda, \lambda')) - \frac{i(\lambda, \lambda')}{2}} \right) \\ &\times \max_{|\gamma| \leq \mathbf{max}(\mathbf{k}, \mathbf{p} + \mathbf{min}(|\alpha|, |\beta|))} \|\partial^\gamma g\|_{L^\infty(\prod_{m=1}^n \text{supp } \psi_{\lambda_m}^{(m)} \cap \text{supp } \psi_{\lambda'_m}^{(m)})}, \end{aligned}$$

where, assuming each evaluation of  $g$  takes  $\mathcal{O}(1)$  operations, the evaluation of this product rule requires  $\mathcal{O}(\prod_{m=1}^n N_m)$  operations.

With the optimal choice  $N_m \approx N^{\frac{1}{a_m}(\sum_\ell \frac{1}{a_\ell})^{-1}}$ , the number of operations is of order  $N$ , and the error bound is of order  $N^{-(\sum_\ell \frac{1}{a_\ell})^{-1}} 2^{-|\lambda| - |\lambda'|} \cdot \left( \frac{d}{2} + z^{(i(\lambda, \lambda')) - \frac{i(\lambda, \lambda')}{2}} \right)$ , that, in case  $a_1 = \dots = a_n = a$ , reads as  $N^{-a/n} 2^{-|\lambda| - |\lambda'|} \cdot \left( \frac{d}{2} + z^{(i(\lambda, \lambda')) - \frac{i(\lambda, \lambda')}{2}} \right)$ .

*Proof.* Dropping the indices  $\lambda_m, \lambda'_m, N_m$ , and  $(\alpha_m, \beta_m)$ , using induction the proof follows easily by writing

$$\begin{aligned} I_1 \otimes \dots \otimes I_n - Q_1 \otimes \dots \otimes Q_n &= (I_1 \otimes \dots \otimes I_{n-1} - Q_1 \otimes \dots \otimes Q_{n-1}) \otimes I_n \\ &+ I_1 \otimes \dots \otimes I_{n-1} \otimes (I_n - Q_n) - (I_1 \otimes \dots \otimes I_{n-1} - Q_1 \otimes \dots \otimes Q_{n-1}) \otimes (I_n - Q_n). \end{aligned}$$

□

In Lemma 5.1 we see an instance of the ‘‘curse of dimensionality’’: the rate of convergence of the tensor product quadrature formula as a function of the number of quadrature points  $N$  is inversely proportional to  $n$ . If the tensor product quadrature rules are applied for the approximation of the entries  $\mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)}$ , then in order to show  $s^*$ -computability of  $\mathbf{I}^{(\alpha, \beta)}$ , the parameters  $a_m$ , i.e., the orders of the composite rules, and thus at the same time, the orders of the partial derivatives of  $g$  that are required to be bounded, should increase proportionally with  $n$ . As we will now see, this curse of dimensionality can be avoided by applying *sparse* tensor product rules introduced in [Smo63]; see also [GG98].

**Lemma 5.2.** *For  $1 \leq m \leq n$ , let  $(N_j^{(m)})_{j \in \mathbb{N}} \subset \mathbb{N}$  be a sequence such that  $C \geq N_{j+1}^{(m)}/N_j^{(m)} \geq c$  for some absolute constants  $C \geq c > 1$ . Then under the assumptions of Lemma 5.1, and with  $Q_{\lambda, \lambda', m, N_{-1}^{(m)}}^{(\alpha, \beta)} := 0$ , for any  $N \in \mathbb{N}$  the sparse tensor product quadrature rule*

$$\mathbf{Q}_{\lambda, \lambda', N}^{(\alpha, \beta)} := \sum_{\{j \in \mathbb{N}^n : \prod_{m=1}^n (N_{j_m}^{(m)})^{a_m} \leq N^{\min_\ell a_\ell}\}} \bigotimes_{m=1}^n (Q_{\lambda_m, \lambda'_m, m, N_{j_m}^{(m)}}^{(\alpha_m, \beta_m)} - Q_{\lambda_m, \lambda'_m, m, N_{j_{m-1}}^{(m)}}^{(\alpha_m, \beta_m)})$$

satisfies

$$\begin{aligned} \left| \mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)} - \mathbf{Q}_{\lambda, \lambda', N}^{(\alpha, \beta)} \right| &\lesssim (\log N)^{n-1} N^{-\min_\ell a_\ell} 2^{-|\lambda| - |\lambda'|} \cdot \left( \frac{d}{2} + z^{(i(\lambda, \lambda')) - \frac{i(\lambda, \lambda')}{2}} \right) \\ &\times \max_{|\gamma| \leq \mathbf{max}(\mathbf{k}, \mathbf{p} + \mathbf{min}(|\alpha|, |\beta|))} \|\partial^\gamma g\|_{L^\infty(\prod_{m=1}^n \text{supp } \psi_{\lambda_m}^{(m)} \cap \text{supp } \psi_{\lambda'_m}^{(m)})}, \end{aligned}$$

where the evaluation of this rule requires  $\mathcal{O}(N(\log N)^{n-1})$  operations.

*Proof.* In this proof, we drop the indices  $\lambda_m, \lambda'_m, \alpha_m$  and  $\beta_m$ . In view of Lemma 5.1, the quadrature error can be written as

$$\sum_{\{j \in \mathbb{N}^n : \prod_{m=1}^n (N_{j_m}^{(m)})^{a_m} > N^{\min_\ell a_\ell}\}} \bigotimes_{m=1}^n (Q_{m, N_{j_m}^{(m)}} - Q_{m, N_{j_{m-1}}^{(m)}}).$$

By the convention  $Q_{m, N_{j_m-1}^{(m)}} = 0$  for  $j_m = 0$ , writing  $Q_{m, N_0^{(m)}} = I_m + Q_{m, N_0^{(m)}} - I_m$ , from (4.5) and (5.1), we infer that the error, in modulus, can be bounded by some absolute multiple of

$$\begin{aligned} & 2^{-|\lambda|-|\lambda'|} \cdot \left(\frac{d}{2} + z^{(i(\lambda, \lambda'))} - \frac{i(\lambda, \lambda')}{2}\right) \sum_{\{j \in \mathbb{N}^n: \prod_{m=1}^n (N_{j_m}^{(m)})^{a_m} > N^{\min_\ell a_\ell}\}} \prod_{m=1}^n (N_{j_m}^{(m)})^{-a_m} \\ & \approx 2^{-|\lambda|-|\lambda'|} \cdot \left(\frac{d}{2} + z^{(i(\lambda, \lambda'))} - \frac{i(\lambda, \lambda')}{2}\right) (\log N)^{n-1} N^{-\min_\ell a_\ell} \end{aligned}$$

times  $\max_{|\gamma| \leq \max(k, \bar{p} + \min(|\alpha|, |\beta|))} \|\partial^\gamma g\|_{L^\infty(\dots)}$ . The work can be bounded by some absolute multiple of

$$\begin{aligned} \sum_{\{j \in \mathbb{N}^n: \prod_{m=1}^n (N_{j_m}^{(m)})^{a_m} \leq N^{\min_\ell a_\ell}\}} \prod_{m=1}^n N_{j_m}^{(m)} & \leq \sum_{\{j \in \mathbb{N}^n: \prod_{m=1}^n N_{j_m}^{(m)} \leq N\}} \prod_{m=1}^n N_{j_m}^{(m)} \\ & \lesssim (\log N)^{n-1} N. \end{aligned}$$

□

## 6. COMPUTABILITY

We now turn to our main result, the  $s^*$ -computability of an approximation of the matrix  $\mathbf{I}(\alpha, \beta)$ .

**Lemma 6.1.** *As in Lemma 4.2, for some  $\mathbf{w} \in (0, \infty)^n$ ,  $\bar{s} > 0$ , and any  $q \in \mathbb{N}$ , let  $\mathbf{B}_q = ((\mathbf{B}_q)_{\lambda, \lambda'})_{\lambda, \lambda' \in \Lambda}$  be a matrix, partitioned into blocks*

$$(\mathbf{B}_q)_{l, l'} = ((\mathbf{B}_q)_{\lambda, \lambda'})_{|\lambda|=l, |\lambda'|=l'},$$

such that the maximal number of non-zero entries in each row of  $(\mathbf{B}_q)_{l, l'}$  or column of  $(\mathbf{B}_q)_{l', l}$  is bounded by some absolute multiple of  $2^{\max(l'-l, 0) \cdot \mathbf{w}/\bar{s}}$ , and

$$(\mathbf{B}_q)_{l, l'} = 0 \quad \text{when } |l' - l| \cdot \mathbf{w} > q.$$

For some constants  $\sigma, \tau > 0$ ,  $v, w \geq 0$ , and any  $N \in \mathbb{N}$ , suppose we have a numerical approximation  $(\mathbf{B}_q)_{\lambda, \lambda', N}$  to  $(\mathbf{B}_q)_{\lambda, \lambda'}$ , whose computation requires  $\mathcal{O}(N(\log N)^v)$  operations, such that

$$|(\mathbf{B}_q)_{\lambda, \lambda'} - (\mathbf{B}_q)_{\lambda, \lambda', N}| \lesssim N^{-\sigma} (\log N)^w 2^{-\frac{1/2+\tau}{\bar{s}}(|\lambda'| - |\lambda|) \cdot \mathbf{w}}.$$

Then with  $\mathbf{B}_q^*$  being defined by replacing any entry  $(\mathbf{B}_q)_{\lambda, \lambda'}$  of  $\mathbf{B}_q$  with  $|\lambda' - \lambda| \cdot \mathbf{w} \leq q$  by  $(\mathbf{B}_q)_{\lambda, \lambda', N(q)_{\lambda, \lambda'}}$ , where

$$N(q)_{\lambda, \lambda'} \approx 2^{(q - (|\lambda'| - |\lambda|) \cdot \mathbf{w})/\bar{s}},$$

the work for computing each row or column of  $\mathbf{B}_q^*$  is bounded by  $\mathcal{O}(q^{n+v} 2^{q/\bar{s}})$  and  $\|\mathbf{B}_q - \mathbf{B}_q^*\| \lesssim q^{n+w} 2^{-\min(\sigma, \tau)q/\bar{s}}$ .

*Proof.* To prove the statement concerning the amount of work, it is sufficient to bound the number of operations needed for computing the entries  $(\mathbf{B}_q^*)_{\lambda, \lambda'}$  for any fixed  $\lambda$ , and all  $\lambda'$  with  $|\lambda'| \geq |\lambda|$ . This number can be bounded by some absolute multiple of

$$q^v \sum_{\{\lambda': 0 \leq (|\lambda'| - |\lambda|) \cdot \mathbf{w} \leq q\}} 2^{(q - (|\lambda'| - |\lambda|) \cdot \mathbf{w})/\bar{s}} \lesssim q^v 2^{q/\bar{s}} \sum_{\{\lambda': 0 \leq (|\lambda'| - |\lambda|) \cdot \mathbf{w} \leq q\}} 1 \lesssim q^{n+v} 2^{q/\bar{s}}.$$

Consider the block  $\mathbf{E}_{\mathbf{l}, \mathbf{l}'}$  =  $((\mathbf{B}_q)_{\lambda, \lambda'} - (\mathbf{B}_q^*)_{\lambda, \lambda'})_{|\lambda|=\mathbf{l}, |\lambda'|=\mathbf{l}'}$ . Bounding the squared norm of  $\mathbf{E}_{\mathbf{l}, \mathbf{l}'}$  by the product of its maximal absolute row-sum and its maximal absolute column-sum, we find that

$$\begin{aligned} \|\mathbf{E}_{\mathbf{l}, \mathbf{l}'}\| &\lesssim 2^{\frac{1}{2}|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{w}/\bar{s}} q^{\mathbf{w}} 2^{-\frac{\sigma}{\bar{s}}(q-|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{w})} 2^{-\frac{\frac{1}{2}+\tau}{\bar{s}}|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{w}} \\ &= q^{\mathbf{w}} 2^{-\sigma q/\bar{s}} 2^{\frac{\sigma-\tau}{\bar{s}}|\mathbf{l}'-\mathbf{l}|\cdot\mathbf{w}}. \end{aligned}$$

Bounding  $\|\mathbf{B}_q - \mathbf{B}_q^*\|^2$  on  $\max_{\mathbf{l}} \sum_{\{\mathbf{l}': |\mathbf{l}'-\mathbf{l}|\cdot\mathbf{w} \leq q\}} \|\mathbf{E}_{\mathbf{l}, \mathbf{l}'}\| \times \max_{\mathbf{l}'} \sum_{\{\mathbf{l}: |\mathbf{l}'-\mathbf{l}|\cdot\mathbf{w} \leq q\}} \|\mathbf{E}_{\mathbf{l}, \mathbf{l}'}\|$ , which is  $\mathcal{O}((q^{\mathbf{w}} 2^{-\sigma q/\bar{s}})^2)$ ,  $\mathcal{O}((q^{\mathbf{w}} q^n 2^{-\sigma q/\bar{s}})^2)$ , or  $\mathcal{O}((q^{\mathbf{w}} q^{n-1} 2^{-\tau q/\bar{s}})^2)$ , for  $\sigma < \tau$ ,  $\sigma = \tau$ , or  $\sigma > \tau$ , respectively, the proof is completed.  $\square$

**Theorem 6.2.** *Let  $\alpha, \beta, \mathbf{I}^{(\alpha, \beta)}, \mathbf{I}_q^{(\alpha, \beta)}, \mathbf{z}^{(i)}$ , and  $s^*$  be as in Theorem 4.1. Assume that for  $1 \leq m \leq n$ , (5.1) is valid with  $a_m = s^*$  and  $b_m = \tilde{p}_m + \min(|\alpha_m|, |\beta_m|)$ , and that for the corresponding  $k_m$  and with  $\mathbf{k} := (k_1, \dots, k_n)$ ,*

$$\partial^\gamma g \in L^\infty(\Omega), \quad |\gamma| \leq \max(\mathbf{k}, \tilde{\mathbf{p}} + \min(|\alpha|, |\beta|)).$$

Denote for  $\mathbf{i} \in \{0, 1\}^n$ ,

$$\mathbf{w}^{(i)} := \frac{s^*}{s^*+1/2} \left( \frac{\mathbf{d}}{2} + \mathbf{z}^{(i)} - \frac{\mathbf{i}}{2} \right).$$

We define for  $q \in \mathbb{N}$  the approximation  $\mathbf{I}_q^{*(\alpha, \beta)}$  for  $\mathbf{I}_q^{(\alpha, \beta)}$  approximating  $\mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)}$  for  $|\lambda'| - |\lambda| \cdot \mathbf{z}^{(i(\lambda, \lambda'))} \leq q$  by the sparse product rule  $\mathbf{Q}_{\lambda, \lambda', N(q)_{\lambda, \lambda'}}^{(\alpha, \beta)}$  with

$$N(q)_{\lambda, \lambda'} \approx 2^{(q - |\lambda'| - |\lambda|) \cdot \mathbf{w}^{(i(\lambda, \lambda'))} / s^*}.$$

Then the work for computing each row or column of  $\mathbf{I}_q^{*(\alpha, \beta)}$  subject to the global accuracy constraint  $\|\mathbf{I}^{(\alpha, \beta)} - \mathbf{I}_q^{*(\alpha, \beta)}\| \lesssim q^{2n-1} 2^{-q}$ , can be bounded by  $\mathcal{O}(q^{2n-1} 2q/s^*)$ .

In particular,  $\mathbf{I}^{(\alpha, \beta)}$  is  $s^*$ -computable.

*Proof.* As shown in Lemma 5.2, we have

$$\begin{aligned} |\mathbf{I}_{\lambda, \lambda'}^{(\alpha, \beta)} - \mathbf{Q}_{\lambda, \lambda', N}^{(\alpha, \beta)}| &\lesssim (\log N)^{n-1} N^{-s^*} 2^{-|\lambda| - |\lambda'|} \cdot \left( \frac{\mathbf{d}}{2} + \mathbf{z}^{(i(\lambda, \lambda'))} - \frac{\mathbf{i}(\lambda, \lambda')}{2} \right) \\ &= (\log N)^{n-1} N^{-s^*} 2^{-\frac{1/2+s^*}{s^*}|\lambda| - |\lambda'|} \cdot \mathbf{w}^{(i(\lambda, \lambda'))}, \end{aligned}$$

with the evaluation of  $\mathbf{Q}_{\lambda, \lambda', N}^{(\alpha, \beta)}$  taking  $\mathcal{O}(N(\log N)^{n-1})$  operations. Suppressing the dependence on  $(\alpha, \beta)$  in the notations, we write  $\mathbf{I} = \sum_{\mathbf{i} \in \{0, 1\}^n} \mathbf{I}^{(i)}$ ,  $\mathbf{I}_q = \sum_{\mathbf{i} \in \{0, 1\}^n} \mathbf{I}_q^{(i)}$ , and  $\mathbf{I}_q^* = \sum_{\mathbf{i} \in \{0, 1\}^n} \mathbf{I}_q^{*(i)}$ , where analogously to  $\mathbf{I}^{(i)}$  and  $\mathbf{I}_q^{(i)}$ ,  $\mathbf{I}_q^{*(i)}$  contains all non-zero entries of  $\mathbf{I}_q^*$  with indices  $\lambda, \lambda'$  with  $\mathbf{i}(\lambda, \lambda') = \mathbf{i}$ . Recall that the non-zero part of  $\mathbf{I}_q^{(i)}$  consists of those blocks  $\mathbf{I}_{\mathbf{l}, \mathbf{l}'}^{(i)} = (\mathbf{I}_{\lambda, \lambda'}^{(i)})_{|\lambda|=\mathbf{l}, |\lambda'|=\mathbf{l}'}$  with  $|\mathbf{l}' - \mathbf{l}| \cdot \mathbf{z}^{(i)} \leq q$ , and that the number of non-zero entries in each row of  $\mathbf{I}_{\mathbf{l}, \mathbf{l}'}^{(i)}$  or column of  $\mathbf{I}_{\mathbf{l}', \mathbf{l}}^{(i)}$  can be bounded on some absolute multiple of  $2^{\max(\mathbf{l}' - \mathbf{l}, 0) \cdot (\mathbf{d} - \mathbf{i})}$ .

By definition of  $s^*$ , we have  $\mathbf{z}^{(i)} \geq s^*(\mathbf{d} - \mathbf{i})$ , which inequality is equivalent to both  $(\mathbf{d} - \mathbf{i}) \leq \mathbf{w}^{(i)}/s^*$  and  $\mathbf{z}^{(i)} \geq \mathbf{w}^{(i)}$ . Now the proof is completed by an application, for each  $\mathbf{i} \in \{0, 1\}^n$ , of Lemma 6.1 with  $(\mathbf{B}_q, \mathbf{B}_q^*)$  reading as  $(\mathbf{I}_q^{(i)}, \mathbf{I}_q^{*(i)})$ .  $\square$

## 7. CONCLUSION

With  $s^*$  from (4.4), we have shown  $s^*$ -computability of the infinite stiffness matrix of  $\partial^\beta g \partial^\alpha$  with respect to suitable tensor product wavelet bases. As exposed in Sect. 1.3, 2.2, and 2.3, as a consequence the stiffness matrices resulting from Galerkin discretizations of scalar PDEs using the tensorized wavelet bases (1.4), and that of a system of PDEs resulting from an  $n$ -scale homogenization problem are  $s^*$ -computable, and the stiffness matrix of an anisotropic problem is uniformly  $s^*$ -computable, all for sufficiently large values of  $s^*$ . As a result, adaptive wavelet algorithms for solving these problems converge with the dimension independent rates of best  $N$ -term approximations in linear complexity. The dependence on  $n$  of the constant involved in the error bound for the approximations produced by the adaptive wavelet algorithm, and actually the reduction of this dependence, will be the topic of our future investigations in this field.

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