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The Euler characteristic of local systems on the moduli of curves and abelian varieties of genus three

Jonas Bergström and Gerard van der Geer

ABSTRACT

We show how to calculate the Euler characteristic of a local system \mathbb{V}_λ associated to an irreducible representation V_λ of the symplectic group of genus 3 on the moduli space \mathcal{M}_3 of curves of genus 3 and the moduli space \mathcal{A}_3 of principally polarised abelian varieties of dimension 3.

1. Introduction

An irreducible representation V_λ with highest weight λ of the symplectic group $\mathrm{GSp}(2g, \mathbb{Q})$ defines a local system \mathbb{V}_λ on the moduli space \mathcal{A}_g of principally polarized abelian varieties. Pullback under the Torelli morphism defines a local system \mathbb{V}'_λ on the moduli space \mathcal{M}_g of curves of genus g . The cohomology of such local systems is intimately connected with the cohomology of the moduli spaces $\mathcal{M}_{g,n}$ of n -pointed curves and also with vector-valued Siegel modular forms of degree g ; cf. [7, 9, 11]. Therefore, it is of some interest to be able to calculate the Euler characteristic of such a local system.

Getzler showed in [12] how to do this for \mathcal{M}_2 and Bini and the second author did this for local systems on the hyperelliptic locus \mathcal{H}_3 of genus 3 in [5] by calculating the Euler numbers of a stratification. In the case of \mathcal{M}_3^0 , the non-hyperelliptic locus of \mathcal{M}_3 , it seems difficult to calculate the Euler characteristics of the individual strata directly. Instead, we use the results of the first author (see [2]) on the motivic Euler characteristics of \mathbb{V}'_λ on \mathcal{M}_3 for λ of small weight, obtained by counting points over finite fields, to calculate linear relations between the Euler numbers of the strata. We were not able to determine the Euler numbers of all strata, but the information obtained suffices for determining the Euler characteristic of all local systems on \mathcal{M}_3 . To go from \mathcal{M}_3 to \mathcal{A}_3 , we need to calculate the Euler characteristics of the pullbacks of our local systems to $\mathcal{M}_2 \times \mathcal{A}_1$ and to the locus $\mathcal{A}_{1,1,1}$ of products of three elliptic curves. We illustrate our results by giving tables of Euler characteristics on \mathcal{M}_3 and \mathcal{A}_3 . We hope that this is a step towards a better understanding of the cohomology of local systems of genus 3.

We would like to thank Torsten Ekedahl for providing us with the argument for the invariance of the Euler characteristic in Section 3. We also thank the Mittag-Leffler Institute for the hospitality enjoyed during the preparation of this paper.

2. The Euler characteristic of a local system

Let \mathcal{M}_3 be the moduli space of smooth genus 3 curves. It is a Deligne–Mumford stack of relative dimension 6 over \mathbb{Z} . The universal curve $\pi' : \mathcal{M}_{3,1} \rightarrow \mathcal{M}_3$ defines, for any prime ℓ , a natural ℓ -adic local system $R^1\pi'_*\mathbb{Q}_\ell$ of rank 6 on $\mathcal{M}_3 \otimes \mathbb{Z}[1/\ell]$. We shall denote this local

system by ${}_{\ell}\mathbb{V}'$, or simply by \mathbb{V}' . It comes equipped with a non-degenerate symplectic pairing $\mathbb{V}' \times \mathbb{V}' \rightarrow \mathbb{Q}_{\ell}(-1)$.

Similarly, we have the moduli space \mathcal{A}_3 of principally polarised abelian varieties of dimension 3, again a Deligne–Mumford stack of relative dimension 6 over \mathbb{Z} . The universal abelian threefold $\pi: \mathcal{X}_3 \rightarrow \mathcal{A}_3$ also defines a natural ℓ -adic local system $R^1\pi_*\mathbb{Q}_{\ell}$ on $\mathcal{A}_3 \otimes \mathbb{Z}[1/\ell]$, which we denote by ${}_{\ell}\mathbb{V}$, or simply by \mathbb{V} . There is the Torelli morphism $t_3: \mathcal{M}_3 \rightarrow \mathcal{A}_3$ of degree 2 between the stacks. The local system \mathbb{V}' on $\mathcal{M}_3 \otimes \mathbb{Z}[1/\ell]$ is a pullback from the local system \mathbb{V} on $\mathcal{A}_3 \otimes \mathbb{Z}[1/\ell]$.

A partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\}$ of weight $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$ determines an irreducible representation V_{λ} of $\mathrm{Sp}(6, \mathbb{Q})$ associated to λ . We lift it to a representation of $\mathrm{GSp}(6, \mathbb{Q})$ with dominant weight $(\lambda_1 - \lambda_2)\gamma_a + (\lambda_2 - \lambda_3)\gamma_b + \lambda_3\gamma_c - |\lambda|\eta$ with γ_a, γ_b and γ_c suitable fundamental roots and η the multiplier representation. Any such representation yields a symplectic local system \mathbb{V}_{λ} of weight $|\lambda|$ on $\mathcal{A}_3 \otimes \mathbb{Z}[1/\ell]$, which appears ‘for the first time’ in the decomposition of

$$\mathrm{Sym}^{\lambda_1 - \lambda_2} \mathbb{V} \otimes \mathrm{Sym}^{\lambda_2 - \lambda_3} (\wedge^2 \mathbb{V}) \otimes \mathrm{Sym}^{\lambda_3} (\wedge^3 \mathbb{V})$$

into irreducibles. If, for example, $\lambda = \{\lambda_1 \geq 0 \geq 0\}$, then $\mathbb{V}_{\lambda} = \mathrm{Sym}^{\lambda_1}(\mathbb{V})$. Similarly, we have a local system \mathbb{V}'_{λ} on $\mathcal{M}_3 \otimes \mathbb{Z}[1/\ell]$.

Our goal is to determine in an algorithmic way, for $k = \mathbb{C}$ or $k = \bar{\mathbb{F}}_p$, the Euler characteristic of the compactly supported cohomology of \mathbb{V}'_{λ} on \mathcal{M}_3 and of \mathbb{V}_{λ} on \mathcal{A}_3 .

The compactly supported cohomology of a local system on a Deligne–Mumford stack presents various subtleties if the order of the automorphism groups is not invertible on the base; cf. the discussion in [17, Remarque 18.3.3]. When the order of the automorphism groups is invertible, one can manage with an *ad hoc* approach and that is what we do here.

Suppose that \mathcal{X} is a Deligne–Mumford stack, $\pi: \mathcal{X} \rightarrow X$ the natural map to its coarse moduli space, $f: \mathcal{X} \rightarrow S$ a map to a scheme S and $g: X \rightarrow S$ the factorisation $f = g\pi$ through the coarse moduli space. For \mathcal{F} a \mathbb{Q}_{ℓ} -sheaf (with ℓ invertible on our base) on \mathcal{X} we have $R\pi_*\mathcal{F} = \pi_*\mathcal{F}$, since π is finite, and hence by the Leray spectral sequence we have $Rf_*\mathcal{F} = Rg_*(\pi_*\mathcal{F})$. We define the direct image with compact support by $Rf_!\mathcal{F} := Rg_!(\pi_*\mathcal{F})$.

With this definition we have Poincaré duality for a local \mathbb{Q}_{ℓ} -system \mathcal{F} on \mathcal{X} , if \mathcal{X} is assumed to be smooth over some base S (an algebraic space) and purely d -dimensional. That is, we have a natural isomorphism

$$R\mathcal{H}om_S(Rf_!\mathcal{F}, \mathbb{Q}_{\ell}) \cong Rf_*(\mathcal{F}^{\vee}(-d)[-2d]).$$

Indeed, by the duality theorem for g we have

$$R\mathcal{H}om_S(Rf_!\mathcal{F}, \mathbb{Q}_{\ell}) \cong Rg_*(R\mathcal{H}om_X(\pi_*\mathcal{F}, g^!\mathbb{Q}_{\ell})).$$

Now, $g: X \rightarrow S$ is locally (in the étale topology) a quotient $U/G \rightarrow S$ of a smooth morphism $h: U \rightarrow S$ of schemes by the action of a finite group G . Since h is smooth, we have $h^!\mathbb{Q}_{\ell} \cong \mathbb{Q}_{\ell}(-d)[-2d]$ and pushing down this sheaf to U/G and taking G -invariants gives $g^!\mathbb{Q}_{\ell} \cong \mathbb{Q}_{\ell}(-d)[-2d]$. Furthermore, we may choose U to be an étale neighbourhood of an arbitrary point of \mathcal{X} and since $R\mathcal{H}om_{U/G}(\pi_*\mathcal{F}|_{U/G}, \mathbb{Q}_{\ell}) \cong R\mathcal{H}om_U(\pi_*\mathcal{F}|_U, \mathbb{Q}_{\ell})^G$, we see that $R\mathcal{H}om_X(\pi_*\mathcal{F}, \mathbb{Q}_{\ell}) \cong \pi_*(\mathcal{F})^{\vee}$. But since $\pi_*\mathcal{F}|_U$ is a local system, we find by taking G -invariants that $\pi_*(\mathcal{F})^{\vee} = \pi_*(\mathcal{F}^{\vee})$. Putting these statements together gives us the desired duality formula. (Note that the sheaf $\pi_*\mathcal{F}$ need not be a local system on X .)

We apply this to the stack \mathcal{M}_3 by taking the compactly supported cohomology of the direct image $\nu_*\mathbb{V}'_{\lambda}$ on the coarse moduli space M_3 under the natural map $\nu: \mathcal{M}_3 \rightarrow M_3$ and similarly for \mathcal{A}_3 . We thus write

$$e_c(\mathcal{M}_3 \otimes k, \mathbb{V}'_{\lambda}) = \sum_{i=0}^{12} (-1)^i \dim_{\mathbb{Q}_{\ell}} H_c^i(\mathcal{M}_3 \otimes k, \mathbb{V}'_{\lambda})$$

and

$$e_c(\mathcal{A}_3 \otimes k, \mathbb{V}_\lambda) = \sum_{i=0}^{12} (-1)^i \dim_{\mathbb{Q}_\ell} H_c^i(\mathcal{A}_3 \otimes k, \mathbb{V}_\lambda),$$

where H_c^\bullet means compactly supported ℓ -adic étale cohomology, and \mathbb{V}_λ means the local system ${}_\ell\mathbb{V}_\lambda$ with ℓ a prime different from the characteristic of our field. We shall see in the next paragraph that these Euler characteristics are independent of the choice of ℓ , and in the next section that they are independent of k , which justifies our notation.

In characteristic 0 we also have the local system $R^1\pi'_*\mathbb{Q}$ on $\mathcal{M}_3 \otimes \mathbb{Q}$, denoted ${}_0\mathbb{V}'$, and the local system $R^1\pi_*\mathbb{Q}$ on $\mathcal{A}_3 \otimes \mathbb{Q}$ denoted ${}_0\mathbb{V}$, and for each λ , in the same way as in the ℓ -adic case, we have an associated local system, ${}_0\mathbb{V}'_\lambda$ or ${}_0\mathbb{V}_\lambda$, respectively. The comparison theorem [1, Théorème 4.1] tells us that the Euler characteristic $e_c(\mathcal{M}_3 \otimes \mathbb{C}, {}_0\mathbb{V}'_\lambda)$ or $e_c(\mathcal{A}_3 \otimes \mathbb{C}, {}_0\mathbb{V}_\lambda)$ with respect to compactly supported topological cohomology is equal to the ℓ -adic version, $e_c(\mathcal{M}_3 \otimes \mathbb{C}, \mathbb{V}'_\lambda)$ or $e_c(\mathcal{A}_3 \otimes \mathbb{C}, \mathbb{V}_\lambda)$, respectively, defined above.

3. Base change

In this section we show that the ℓ -adic variants of our local systems commute with base change and we conclude that the Euler characteristic $e_c(\mathcal{M}_3 \otimes k, \mathbb{V}'_\lambda)$ is independent of k being equal to \mathbb{C} or $\overline{\mathbb{F}}_p$, for any p different from ℓ .

THEOREM 3.1. *Let $f: X \rightarrow S$ be a smooth, proper, relatively Deligne–Mumford map (cf. [17, 7.3.3]) between algebraic stacks such that the prime ℓ is invertible on S . Suppose given a relative normal crossing divisor $D \subset X$ and a locally constant sheaf \mathcal{E} of $\mathbb{Z}_\ell, \mathbb{Q}_\ell$ or ℓ -torsion modules on $U := X \setminus D$. If \mathcal{E} is tamely ramified along D , then the $R^i g_* \mathcal{E}$ are locally constant and commute with base change where g is the structure map $g: U \rightarrow S$.*

Proof. We start by showing that if $j: U \rightarrow X$ is the open embedding, then $Rj_* \mathcal{E}$ commutes with base change on S and is constructible. Indeed, this is a local statement on X , so we may assume that X is a scheme and then it follows from [15, Théorème finitude, 1.3.3]. For f we have the proper base change theorem [17, Théorème 18.5.1], using [17, Théorème 16.6]. Hence, we conclude that $Rg_* \mathcal{E} = Rf_* Rj_* \mathcal{E}$ commutes with base change and is constructible.

Furthermore, the smooth base change theorem [1, Corollaire 1.2] for f is true as it is local on X . The fact that $R^i g_* \mathcal{E}$ is locally constant now follows as in the proof of [1, Théorème 2.1]. Indeed, properness is used only through the proper base change theorem (in fact, only to reduce to the case when the base is normal) and, in turn, it is only applied to show that $R^i g_* \mathcal{E}$ commutes with base change. □

COROLLARY 3.2. *Let $\rho: \mathcal{M}_g \rightarrow \text{Spec}(\mathbb{Z}[1/\ell])$ be the structure map and $\pi: \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ the universal curve. If \mathcal{V} is a locally constant system on \mathcal{M}_g , then $R^i \rho_* \mathcal{V}$ is a locally constant system over $\text{Spec}(\mathbb{Z}[1/\ell])$ that commutes with base change.*

Proof. The statement follows from the corresponding statement for $R^i \rho_* \mathcal{V}$ by duality (see Section 2) and the case of $R^i \rho_* \mathcal{V}$ follows from Theorem 3.1 applied to $\overline{\mathcal{M}}_g \rightarrow \text{Spec}(\mathbb{Z}[1/\ell])$ once one has verified tameness. By purity and Abhyankar’s lemma [13, X.3.6], it is enough to verify this at the localisation of $\overline{\mathcal{M}}_g$ at a generic point of a component of the boundary. That generic point is however of characteristic 0. □

In view of this independence of the characteristic, we shall often just write $e_c(\mathcal{M}_3, \mathbb{V}'_\lambda)$ instead of $e_c(\mathcal{M}_3 \otimes k, \mathbb{V}'_\lambda)$ and $e_c(\mathcal{A}_3, \mathbb{V}_\lambda)$ instead of $e_c(\mathcal{A}_3 \otimes k, \mathbb{V}_\lambda)$. The invariance of the

Euler characteristic for \mathcal{A}_3 follows from the stratification of \mathcal{A}_3 (equal to $t_3(\mathcal{M}_3) \sqcup t_2(\mathcal{M}_2) \times \mathcal{A}_1 \sqcup \mathcal{A}_{1,1,1}$) found at the beginning of Section 8.

4. *Calculating the Euler characteristics*

We now turn to the calculation of the Euler characteristics. The first remark is that the Euler characteristics are unchanged if we change the representation λ by a power of the multiplier η (twisting). We can therefore restrict our attention from $\mathrm{GSp}(6, \mathbb{Q})$ to $\mathrm{Sp}(6, \mathbb{Q})$.

In this section, we will work over \mathbb{C} and write \mathcal{M}_3 for $\mathcal{M}_3 \otimes \mathbb{C}$. The local system \mathbb{V}'_λ will stand for ${}_0\mathbb{V}'_\lambda$. The Euler characteristic $e_c(\mathcal{M}_3, \mathbb{V}'_\lambda)$ is calculated by descending from the stack \mathcal{M}_3 to the coarse moduli space M_3 under the natural map $\mu: \mathcal{M}_3 \rightarrow M_3$ and by calculating $e_c(M_3, \mu_*(\mathbb{V}'_\lambda))$ using the stratification of $M_3 \otimes \mathbb{C}$ by the automorphism group of the genus 3 curve. On each stratum $\Sigma(G)$ the automorphism group of every curve is equal to G , and the direct image $\mu_*(\mathbb{V}')$ is a local system. The Euler characteristic is then given by (see, for example, [6, Theorem 5.13], and also [16])

$$e_c(\mathcal{M}_3, \mathbb{V}'_\lambda) = \sum_G e_c(\Sigma(G)) \dim(\mathbb{V}'_\lambda{}^G), \tag{4.1}$$

where $e_c(\Sigma(G))$ is the topological Euler characteristic of $\Sigma(G)$ and $\mathbb{V}'_\lambda{}^G$ is the space of G -invariants.

The dimension of the invariant part $\mathbb{V}'_\lambda{}^G$ is determined as follows. The action of G on the cohomology group $H^1(C, \mathbb{Q})$ of a curve C of genus 3 defines a homomorphism $r: G \rightarrow \mathrm{Sp}(6, \mathbb{Q})$. We denote the eigenvalues of $r(\gamma)$ by

$$\{a_\gamma, b_\gamma, c_\gamma, a_\gamma^{-1}, b_\gamma^{-1}, c_\gamma^{-1}\}.$$

Let h_d be the complete symmetric function of degree d in six variables. For the proof of the following fact, we refer to [10, Proposition 24.22].

PROPOSITION 4.1. *If J_λ is the determinant of the 3×3 matrix whose i th row is*

$$(J_{\lambda_i - i + 2}, J_{\lambda_i - i + 2} + J_{\lambda_i - i}, J_{\lambda_i - i - 1})$$

and

$$J_d(a_\gamma, b_\gamma, c_\gamma) := h_d(a_\gamma, b_\gamma, c_\gamma, a_\gamma^{-1}, b_\gamma^{-1}, c_\gamma^{-1}),$$

then we have

$$\dim(\mathbb{V}'_\lambda{}^G) = \frac{1}{\#G} \sum_{\gamma \in G} J_\lambda(a_\gamma, b_\gamma, c_\gamma).$$

Hence, we can calculate the Euler characteristics of the local systems \mathbb{V}'_λ on \mathcal{M}_3 if we have the strata and the Euler numbers of the strata on M_3 . It seems very difficult to determine these Euler numbers directly, but by using our moduli spaces over finite fields, we shall determine sufficiently many linear relations between them.

5. *The stratification of the moduli of non-hyperelliptic curves*

Also, in this section we write \mathcal{M}_3 for $\mathcal{M}_3 \otimes \mathbb{C}$ and \mathbb{V}'_λ for ${}_0\mathbb{V}'_\lambda$. The calculation of the Euler characteristic of \mathbb{V}'_λ on \mathcal{M}_3 is done by using a stratification on \mathcal{M}_3 (and on M_3). First we have $\mathcal{M}_3 = \mathcal{M}_3^0 \sqcup \mathcal{H}_3$ with \mathcal{H}_3 the hyperelliptic locus and \mathcal{M}_3^0 the locus of non-hyperelliptic curves of genus 3. Since an algorithm for the calculation of the Euler characteristic of \mathbb{V}'_λ on \mathcal{H}_3 is given in [5], we are left with calculating $e_c(\mathcal{M}_3^0, \mathbb{V}'_\lambda)$. For this we use the stratification of \mathcal{M}_3^0 given by the automorphism group in the manner described above.

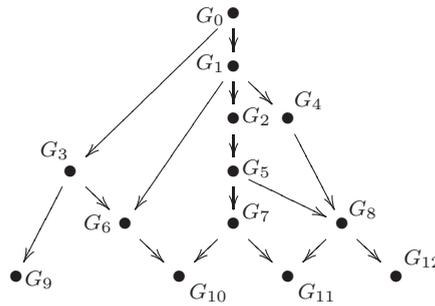


FIGURE 1. *The diagram of strata.*

A non-hyperelliptic curve C of genus 3 over the field of complex numbers \mathbb{C} can be given as a smooth projective curve of degree 4 in the projective plane, the canonical image of C . The automorphism group acts on the vector space $H^0(C, \Omega_C^1)$, and thus induces and is in fact induced by a projective automorphism of the projective space of $H^0(C, \Omega_C^1)$. It is well known which groups occur as an automorphism group of a non-hyperelliptic curve of genus 3. We list the possibilities in Table 1. This list is found in [20]; cf. also [19].

Accordingly, the moduli space \mathcal{M}_3^0 carries a stratification indexed by the automorphism groups of non-hyperelliptic curves of genus 3. There are thirteen (open) strata Σ_{G_i} and the inclusion relations between the closures of the strata are given in Figure 1.

The groups G_i are listed in [20] as subgroups of PGL_3 acting on the coordinates x, y, z of the normal forms found in Table 1. In view of Proposition 4.1, we need the representations $\rho(G_i)$ on the space of differentials $H^0(C, \Omega_C^1)$, or more precisely, the eigenvalues of all the group elements. These representations are listed in Table 2, where the standard basis is identified (on $z \neq 0$) with $\omega, y_1 \omega, y_2 \omega$, where $y_1 := x/z, y_2 := y/z, G(y_1, y_2) := f(y_1, y_2, 1)$ and ω is the differential found by gluing $dy_1/(\partial G/\partial y_2)$ and $dy_2/(\partial G/\partial y_1)$.

The notation for Tables 1 and 2 is as follows: V_4 is the Klein four-group; D_4 is the dihedral group of order 8; \mathbb{S}_n denotes the symmetric group on n letters; Γ_n denotes a group of order n with $\Gamma_{168} = \text{SL}(3, \mathbb{F}_2)$; ζ_n is a primitive n th root of unity.

6. *Linear relations between the Euler numbers*

In this section we use information obtained by counting points over finite fields to get linear relations between the Euler numbers of the strata.

TABLE 1. *The automorphism groups.*

| i | G | $\#G$ | Normal form: f | $\dim(\Sigma(G))$ |
|-----|--------------------------|-------|--|-------------------|
| 0 | 1 | 1 | f | 6 |
| 1 | $\mathbb{Z}/2\mathbb{Z}$ | 2 | $x^4 + x^2 f(x, y) + g(y, z)$ | 4 |
| 2 | V_4 | 4 | $x^4 + y^4 + z^4 + ax^2 y^2 + by^2 z^2 + cx^2 z^2$ | 3 |
| 3 | $\mathbb{Z}/3\mathbb{Z}$ | 3 | $yz^3 + x(x - y)(x - ay)(x - by)$ | 2 |
| 4 | \mathbb{S}_3 | 6 | $x^3 y + y^3 z + x^2 y^2 + axyz^2 + bz^4$ | 2 |
| 5 | D_4 | 8 | $x^4 + y^4 + z^4 + ax^2 y^2 + bxyz^2$ | 2 |
| 6 | $\mathbb{Z}/6\mathbb{Z}$ | 6 | $z^3 y + x^4 + ax^2 y^2 + y^4$ | 1 |
| 7 | Γ_{16} | 16 | $x^4 + y^4 + z^4 + ax^2 y^2$ | 1 |
| 8 | \mathbb{S}_4 | 24 | $x^4 + y^4 + z^4 + a(x^2 y^2 + x^2 z^2 + y^2 z^2)$ | 1 |
| 9 | $\mathbb{Z}/9\mathbb{Z}$ | 9 | $x^4 + xy^3 + yz^3$ | 0 |
| 10 | Γ_{48} | 48 | $x^4 + y^4 + z^4 + (4\zeta_3 + 2)x^2 y^2$ | 0 |
| 11 | Γ_{96} | 96 | $x^4 + y^4 + z^4$ | 0 |
| 12 | Γ_{168} | 168 | $x^3 y + y^3 z + z^3 x$ | 0 |

TABLE 2. *The eigenvalues.*

| i | $\#G$ | Generators of $\rho(G)$ |
|-----|-------|---|
| 0 | 1 | 1 |
| 1 | 2 | $\text{diag}(-1, 1, -1)$ |
| 2 | 4 | $\text{diag}(-1, 1, -1), \text{diag}(-1, -1, 1)$ |
| 3 | 3 | $\text{diag}(\zeta_3^2, \zeta_3, \zeta_3)$ |
| 4 | 6 | $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \text{diag}(1, \zeta_3, \zeta_3^2)$ |
| 5 | 8 | $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \text{diag}(1, i, -i)$ |
| 6 | 6 | $\text{diag}(-\zeta_3^2, \zeta_3, -\zeta_3)$ |
| 7 | 16 | $\text{diag}(-1, 1, -1), \text{diag}(1, i, -i), \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ |
| 8 | 24 | $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ |
| 9 | 9 | $\text{diag}(\zeta_9^2, \zeta_9^4, \zeta_9)$ |
| 10 | 48 | $\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\zeta_3 & 0 & 0 \\ 0 & \zeta_8 & \zeta_8^3 \\ 0 & \zeta_8 & \zeta_8^7 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{2}\zeta_3^2 & 0 & 0 \\ 0 & \zeta_8^5 & \zeta_8 \\ 0 & \zeta_8^7 & \zeta_8^7 \end{pmatrix}$ |
| 11 | 96 | $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i \\ 0 & i & 0 \\ -1 & 0 & 0 \end{pmatrix}$ |
| 12 | 168 | $\text{diag}(\zeta_7, \zeta_7^4, \zeta_7^2), ((\zeta_7^{2ij} - \zeta_7^{-2ij})/(-\sqrt{-7}))$ |

We consider the moduli space \mathcal{M}_3^0 over $\mathbb{Z}[1/2, 1/\ell]$. By the work of the first author, the Euler characteristics $a_\lambda = e_c(\mathcal{M}_3^0 \otimes \overline{\mathbb{F}}_p, \mathbb{V}'_\lambda)$ are known for low-weight local systems. More specifically, in [2], for $|\lambda| \leq 7$, the trace of Frobenius on $e_c(\mathcal{M}_3^0 \otimes \bar{k}, \mathbb{V}'_\lambda)$ is computed for all finite fields k , and these traces turn out to be given by a polynomial in the cardinality of the field. By Deligne’s proof of the Weil conjectures (see [8]), this implies that the Euler characteristic is obtained by substituting 1 for the cardinality of the field in this polynomial. The results for the traces, in the cases $|\lambda| = 6, 7$, are presented in [2, Theorems 16.2 and 16.3]. For the results in terms of Euler characteristics, see Tables 3 and 4.

Since the Euler characteristics $e_c(\mathcal{M}_3^0 \otimes k, \mathbb{V}'_\lambda)$ for any field k of characteristic different from 2 are the same, this leads, by equation (4.1), to a system of linear equations in the unknowns $e_i := e_c(\Sigma_{G_i})$:

$$\sum_{i=0}^{12} \dim(\mathbb{V}'_\lambda^{G_i}) e_i = a_\lambda.$$

We shall use these equations for all λ with $|\lambda| \leq 4$ and $\lambda = (6, 0, 0)$. We obtain in this way the following result.

PROPOSITION 6.1. *The Euler numbers $e_i = e(\Sigma_i)$ of the strata Σ_i on $\mathcal{M}_3^0 \otimes \mathbb{C}$ satisfy the following equations:*

$$e_1 = -3e_0, \quad e_2 = 2e_0 + e_8 + 1, \quad e_4 = -e_8 - 1, \quad e_5 = -e_8,$$

and

$$e_3 = 0, \quad e_6 = e_7 = -1, \quad e_9 = e_{10} = e_{11} = e_{12} = 1.$$

Note that this proposition agrees with the fact that each of the strata Σ_i consists of one point for i equal to 9, 10, 11 and 12. These equations do not determine the Euler numbers e_i , but there are some easy relations between the linear equations in the Euler numbers e_i that reduce the number of unknowns. If we write

$$k_i(\lambda) := \dim(\mathbb{V}_\lambda^{G_i})$$

and substitute the relations of Proposition 6.1 in $\sum_{i=0}^{12} k_i(\lambda)e_i$, then the coefficients of e_0 and e_8 are $c_0 = k_0(\lambda) - 3k_1(\lambda) + 2k_2(\lambda)$ and $c_8 = k_2(\lambda) - k_4(\lambda) - k_5(\lambda) + k_8(\lambda)$. The following lemma shows that $c_0 = 0$ and $-c_0 + 2c_8 = 0$; that is, the sum $\sum_{i=0}^{12} k_i(\lambda)e_i$ does not depend upon e_0 and e_8 .

LEMMA 6.2. *Let $k_i(\lambda) = \dim(\mathbb{V}_\lambda^{G_i})$. Then we have the relations*

$$k_0(\lambda) - 3k_1(\lambda) + 2k_2(\lambda) = 0,$$

and

$$-k_0(\lambda) + 3k_1(\lambda) - 2k_4(\lambda) - 2k_5(\lambda) + 2k_8(\lambda) = 0.$$

Proof. We use the formula

$$\dim(\mathbb{V}_\lambda^G) = \frac{1}{\#G} \sum_{\gamma \in G} J_\lambda(a_\gamma, b_\gamma, c_\gamma),$$

where $\{a_\gamma, b_\gamma, c_\gamma, a_\gamma^{-1}, b_\gamma^{-1}, c_\gamma^{-1}\}$ are the eigenvalues of γ on the standard representation. The lemma now follows from a glance at the table of eigenvalues, from which it follows that the weighted sum of eigenvalues agree. The first relation, for example, follows from the fact that the eigenvalues for the groups G_0, G_1 and G_2 are $\{[1, 1, 1]\}$, $\{[1, 1, 1], [-1, -1, 1]\}$ and $\{[1, 1, 1], [-1, -1, 1], [-1, -1, 1], [-1, -1, 1]\}$, and thus $k_0(\lambda) = J_\lambda(1, 1, 1)$, $k_1(\lambda) = \frac{1}{2}(J_\lambda(1, 1, 1) + J_\lambda(-1, -1, 1))$, $k_2(\lambda) = \frac{1}{4}(J_\lambda(1, 1, 1) + 3J_\lambda(-1, -1, 1))$. \square

Combining the preceding lemma and proposition gives the following result.

PROPOSITION 6.3. *For $i = 0, \dots, 12$, we let $k_i(\lambda) := \dim(\mathbb{V}_\lambda^{G_i})$. Then the Euler characteristic of \mathbb{V}'_λ on the non-hyperelliptic locus \mathcal{M}_3^0 is given by*

$$e_c(\mathcal{M}_3^0, \mathbb{V}'_\lambda) = k_2(\lambda) - k_4(\lambda) - k_6(\lambda) - k_7(\lambda) + \sum_{j=9}^{12} k_j(\lambda).$$

Proof. As in equation (4.1), we have $e_c(\mathcal{M}_3^0, \mathbb{V}'_\lambda) = \sum_{j=0}^{12} k_j(\lambda)e_j$ and by substituting the values of Proposition 6.1 we get $e_c(\mathcal{M}_3^0, \mathbb{V}'_\lambda) = (k_0(\lambda) - 3k_1(\lambda) + 2k_2(\lambda))e_0 + k_2(\lambda) - k_4(\lambda) - k_6(\lambda) - k_7(\lambda) + (k_2(\lambda) - k_4(\lambda) - k_5(\lambda) + k_8(\lambda))e_8 + \sum_{j=9}^{12} k_j(\lambda)$. By Lemma 6.2, the coefficients of e_0 and e_8 vanish. \square

By the additivity of the Euler characteristic, we get the Euler characteristic on \mathcal{M}_3 by adding the contribution from the hyperelliptic locus. An algorithm for determining the values $e_c(\mathcal{H}_3, \mathbb{V}'_\lambda)$ is given in [5].

REMARK 6.4. Note that for local systems \mathbb{V}'_λ of odd weight $|\lambda|$, the Euler characteristic vanishes on the hyperelliptic locus \mathcal{H}_3 in view of the presence of the hyperelliptic involution.

We therefore present results on the Euler characteristic of \mathcal{M}_3 for local systems of even and odd weight separately in Tables 3 and 4. Note that the values that we obtain for the Euler characteristics on the hyperelliptic locus (and its complement) are valid only if the characteristic is either 0 or greater than 2. The closure of \mathcal{H}_3 in $\overline{\mathcal{M}}_3$ is namely singular in characteristic 2, along the closure of the strata consisting of a rational backbone with three elliptic tails; cf. [4, Théorème 2.8]. For examples where the Euler characteristic behaves differently in characteristic 2, see [2] or [3].

6.1. *The hyperelliptic case and the case $g = 2$*

It is interesting to apply the method used here for the non-hyperelliptic case to the hyperelliptic case. We have a stratification \mathcal{H}_3 by strata Σ_{H_i} parametrised by the possibilities H_i , $i = 1, \dots, 11$, for the automorphism group of a hyperelliptic curve; cf. [5]. The values $b_\lambda = e_c(\mathcal{H}_3 \otimes \overline{\mathbb{F}}_p, \mathbb{V}'_\lambda)$ for $|\lambda| \leq 6$, obtained by counting points over finite fields (see [3]) give us a system of linear equations for the Euler numbers $g_i = e(\Sigma_{H_i})$ with a 3-dimensional solution space. Just as for the genus 3 non-hyperelliptic case, the Euler characteristics do not depend on these remaining three parameters.

If the same procedure is applied to \mathcal{M}_2 , you actually find the Euler characteristics of all individual strata from the values $c_\lambda = e_c(\mathcal{H}_2 \otimes \overline{\mathbb{F}}_p, \mathbb{V}'_\lambda)$ for $|\lambda| \leq 6$.

TABLE 3. *Euler characteristics of \mathbb{V}'_λ (even weight) on the parts of the stratification of \mathcal{M}_3 .*

| λ | \mathcal{H}_3 | \mathcal{M}_3^0 | \mathcal{M}_3 | λ | \mathcal{H}_3 | \mathcal{M}_3^0 | \mathcal{M}_3 |
|-----------|-----------------|-------------------|-----------------|-----------|-----------------|-------------------|-----------------|
| (0,0,0) | 1 | 2 | 3 | (5,2,1) | -10 | 6 | -4 |
| (2,0,0) | -1 | 1 | 0 | (4,4,0) | -5 | 7 | 2 |
| (1,1,0) | 0 | -1 | -1 | (4,3,1) | -4 | 4 | 0 |
| (4,0,0) | -1 | 1 | 0 | (4,2,2) | -7 | 8 | 1 |
| (3,1,0) | 0 | 0 | 0 | (3,3,2) | -2 | 0 | -2 |
| (2,2,0) | -1 | 1 | 0 | (10,0,0) | -17 | 13 | -4 |
| (2,1,1) | 0 | 1 | 1 | (9,1,0) | -22 | 20 | -2 |
| (6,0,0) | -5 | 4 | -1 | (8,2,0) | -43 | 37 | -6 |
| (5,1,0) | -2 | 1 | -1 | (8,1,1) | -8 | 7 | -1 |
| (4,2,0) | -5 | 3 | -2 | (7,3,0) | -34 | 26 | -8 |
| (4,1,1) | 0 | -1 | -1 | (7,2,1) | -32 | 28 | -4 |
| (3,3,0) | 0 | 2 | 2 | (6,4,0) | -37 | 31 | -6 |
| (3,2,1) | 0 | 0 | 0 | (6,3,1) | -26 | 17 | -9 |
| (2,2,2) | -3 | 4 | 1 | (6,2,2) | -27 | 25 | -2 |
| (8,0,0) | -7 | 9 | 2 | (5,5,0) | -6 | 8 | 2 |
| (7,1,0) | -8 | 7 | -1 | (5,4,1) | -22 | 22 | 0 |
| (6,2,0) | -13 | 14 | 1 | (5,3,2) | -12 | 9 | -3 |
| (6,1,1) | -2 | 0 | -2 | (4,4,2) | -15 | 13 | -2 |
| (5,3,0) | -10 | 8 | -2 | (4,3,3) | 0 | -2 | -2 |

TABLE 4. *Euler characteristics of \mathbb{V}'_λ (odd weight) on \mathcal{M}_3 .*

| λ | \mathcal{M}_3 | λ | \mathcal{M}_3 | λ | \mathcal{M}_3 | λ | \mathcal{M}_3 |
|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|
| (1,0,0) | 0 | (2,2,1) | 0 | (3,2,2) | 0 | (5,3,1) | 20 |
| (3,0,0) | 0 | (7,0,0) | -2 | (9,0,0) | 4 | (5,2,2) | 2 |
| (2,1,0) | 0 | (6,1,0) | 4 | (8,1,0) | 8 | (4,4,1) | 2 |
| (1,1,1) | 0 | (5,2,0) | 4 | (7,2,0) | 10 | (4,3,2) | 4 |
| (5,0,0) | 0 | (5,1,1) | 10 | (7,1,1) | 18 | (3,3,3) | 8 |
| (4,1,0) | 2 | (4,3,0) | 2 | (6,3,0) | 20 | (11,0,0) | 4 |
| (3,2,0) | -2 | (4,2,1) | 2 | (6,2,1) | 12 | (10,1,0) | 30 |
| (3,1,1) | 2 | (3,3,1) | 4 | (5,4,0) | 6 | (9,2,0) | 36 |

7. *Branching*

In order to calculate the Euler characteristic on \mathcal{A}_3 , we need branching formulas for representations of the symplectic group. Since the Euler characteristic of \mathbb{V}_λ is unchanged by twisting with the multiplier representation, we may and do work with $\mathrm{Sp}(6) := \mathrm{Sp}(6, \mathbb{Q})$ instead of $\mathrm{GSp}(6, \mathbb{Q})$.

Let U be a finite-dimensional irreducible complex representation of the group $G := \mathrm{Sp}(2)^3 \rtimes \mathbb{S}_3$, where \mathbb{S}_3 denotes the symmetric group of the three factors $\mathrm{Sp}(2) := \mathrm{SL}(2, \mathbb{Q})$. We will need branching formulas from $\mathrm{Sp}(6)$ to G . In the literature, we found formulas for branching from $\mathrm{Sp}(6)$ to $\mathrm{Sp}(2)^3$ (see, for example, [18]), but none to G . The restriction of U to $\mathrm{Sp}(2)^3$ decomposes as a direct sum $U = \bigoplus_{i \in I} W_i$ of irreducible representations of $\mathrm{Sp}(2)^3$. The group \mathbb{S}_3 acts transitively on the index set. Let $\Sigma = \{\sigma \in \mathbb{S}_3 : \sigma(W) = W\}$ be the stabiliser of a component $W = W_i$. Then W is an irreducible representation of the semi-direct product $\mathrm{Sp}(2)^3 \rtimes \Sigma$, and hence of the form $V_\alpha \boxtimes V_\beta \boxtimes V_\gamma$ with a Σ -action, where V_n for $n \geq 0$ denotes the irreducible representation of rank $n + 1$ of $\mathrm{Sp}(2)$. If Σ contains a 3-cycle, then $\alpha = \beta = \gamma$ and if Σ contains a 2-cycle, then $\#\{\alpha, \beta, \gamma\} \leq 2$.

If Σ is trivial, then $\#I = 6$ and U consists of six copies of $V_\alpha \boxtimes V_\beta \boxtimes V_\gamma$. In this case, U is obtained by inducing the representation $V_\alpha \boxtimes V_\beta \boxtimes V_\gamma$ from $\mathrm{Sp}(2)^3$ to G . We denote it by $R_{\alpha, \beta, \gamma}$. It is independent of the ordering of α, β, γ , and therefore we shall assume that $\alpha \geq \beta \geq \gamma$.

Next, suppose that Σ is of order 2. Then W is of the form $V_\alpha \boxtimes V_\beta \boxtimes V_\beta$ and Σ acts on it, making it a representation of $\mathrm{Sp}(2)^3 \rtimes \Sigma$. There are two possibilities for the action of a generator of Σ :

$$u \boxtimes v \boxtimes w \mapsto u \boxtimes w \boxtimes v \quad \text{or} \quad -u \boxtimes w \boxtimes v,$$

and U is then the induced representation from $\mathrm{Sp}(2)^3 \rtimes \Sigma$ to G . We denote the two possibilities by $R_{\alpha, \beta}^+$ and $R_{\alpha, \beta}^-$. Note that α and β need not be different.

If Σ is of order 3, then $\alpha = \beta = \gamma$ and a generator σ of Σ acts on $V_\alpha \boxtimes V_\alpha \boxtimes V_\alpha$ via

$$u \boxtimes v \boxtimes w \mapsto \rho^\epsilon w \boxtimes u \boxtimes v,$$

with ρ a primitive third root of 1 and ϵ equal to 0, 1 or 2. The representation U is then the induced representation from $\mathrm{Sp}(2)^3 \rtimes \Sigma$ to G . If $\epsilon = 0$ we denote it by T_α , while for $\epsilon = 1$ and $\epsilon = 2$, the result is the same and is denoted by T'_α .

Finally, if $\Sigma = \mathbb{S}_3$ then U is $V_\alpha \boxtimes V_\alpha \boxtimes V_\alpha$, with the action of \mathbb{S}_3 given by

$$u_1 \boxtimes u_2 \boxtimes u_3 \mapsto u_{\sigma^{-1}(1)} \boxtimes u_{\sigma^{-1}(2)} \boxtimes u_{\sigma^{-1}(3)} \quad \text{or} \quad \mapsto \mathrm{sgn}(\sigma) u_{\sigma^{-1}(1)} \boxtimes u_{\sigma^{-1}(2)} \boxtimes u_{\sigma^{-1}(3)}.$$

We denote the two representations by R_α^+ and R_α^- .

We have the following two relations:

$$T_\alpha = R_\alpha^+ \oplus R_\alpha^- \quad \text{and} \quad R_{\alpha, \alpha}^\pm = T'_\alpha \oplus R_\alpha^\pm.$$

Thus we obtain the following result.

LEMMA 7.1. *Every irreducible representation of $\mathrm{Sp}(2)^3 \rtimes \mathbb{S}_3$ is a virtual sum of representations of the form $R_{\alpha, \beta, \gamma}$, $R_{\alpha, \beta}^\pm$ and R_α^\pm .*

The exterior products $\wedge^i V_{1,0,0}$ for $i = 1, 2, 3$ of the standard representation of $\mathrm{Sp}(6)$ form a basis for the representation ring of $\mathrm{Sp}(6)$. Their restrictions to the subgroup $\mathrm{Sp}(2)^3 \rtimes \mathbb{S}_3$ are given by

$$V_{1,0,0}|G = R_{1,0}, \quad \wedge^2 V_{1,0,0}|G = R_{0,1}^- \oplus R_{0,0}^+, \quad \wedge^3 V_{1,0,0}|G = R_{1,1}^+ \oplus R_{1,0,0}.$$

Explicitly, the representation V_λ corresponds to the symmetric function J_λ and by writing J_λ in terms of the elementary symmetric functions e_1, e_2 and e_3 , one finds the expression of V_λ in terms of the three exterior products. Thus we can determine the branching completely. We have implemented this in a computer program.

8. *The Euler characteristics on \mathcal{A}_3*

In this section we will use the stratification

$$\mathcal{A}_3 = t_3(\mathcal{M}_3^0) \sqcup t_3(\mathcal{H}_3) \sqcup t_2(\mathcal{M}_2) \times \mathcal{A}_1 \sqcup \mathcal{A}_{1,1,1}$$

to compute $e_c(\mathcal{A}_3, \mathbb{V}_\lambda)$ for any partition λ , where $t_g: \mathcal{M}_g \rightarrow \mathcal{A}_g$ is the Torelli morphism. For hyperelliptic curves, the Torelli morphism has degree 1, and hence $e_c(t_3(\mathcal{H}_3), \mathbb{V}_\lambda)$ and $e_c(t_2(\mathcal{M}_2), \mathbb{V}_\lambda)$ are equal to $e_c(\mathcal{H}_3, \mathbb{V}'_\lambda)$ and $e_c(\mathcal{M}_2, \mathbb{V}'_\lambda)$, respectively. On the locus \mathcal{M}_3^0 in \mathcal{M}_3 of non-hyperelliptic curves, the Torelli morphism has degree 2, due to the existence of the automorphism ‘ -1 ’ on the Jacobians. That is, if G is the automorphism group of a non-hyperelliptic curve C , then $G' := \{\pm g : g \in G\}$ is the automorphism group of its Jacobian. The canonical isomorphism between $H^1(C)$ and $H^1(J(C))$ and the fact that $-1 \in G'$ acts as -1 on $H^1(J(C))$ then shows how the elements of G' act on $H^1(J(C))$. The Torelli morphism is a homeomorphism of the coarse moduli spaces, and thus the Euler characteristics of the strata of $t_3(\mathcal{M}_3^0)$ are the same as the corresponding strata of \mathcal{M}_3^0 . Since the polynomials J_λ are even or odd according to whether λ is even or odd, we find by applying formula (4.1), that $e_c(t_3(\mathcal{M}_3^0), \mathbb{V}_\lambda)$ is equal to $e_c(\mathcal{M}_3^0, \mathbb{V}'_\lambda)$ if λ is even and zero if λ is odd.

We will apply a Künneth formula to compute $e_c(t_2(\mathcal{M}_2) \times \mathcal{A}_1, \mathbb{V}_\lambda)$. For this we need to know the branching, that is, the restriction of the representation V_λ of $\mathrm{Sp}(6, \mathbb{Q})$ to $\mathrm{Sp}_{4,2} := \mathrm{Sp}(4, \mathbb{Q}) \times \mathrm{Sp}(2, \mathbb{Q})$. If $V_\lambda|_{\mathrm{Sp}_{4,2}} = \bigoplus_{\mu, \nu} (V_\mu \boxtimes V_\nu)^{\oplus m_{\mu, \nu}}$, it follows that

$$e_c(t_2(\mathcal{M}_2) \times \mathcal{A}_1, \mathbb{V}_\lambda) = \sum_{\mu, \nu} m_{\mu, \nu} e_c(\mathcal{M}_2, t^* \mathbb{V}_\mu) e_c(\mathcal{A}_1, \mathbb{V}_\nu).$$

A formula for the branching from $\mathrm{Sp}(6, \mathbb{Q})$ to $\mathrm{Sp}_{4,2}$ (that is, for the numbers $m_{\mu, \nu}$) can, for instance, be found in [18].

Finally, to compute $e_c(\mathcal{A}_{1,1,1}, \mathbb{V}_\lambda)$ we will also use branching, in this case from $\mathrm{Sp}(6, \mathbb{Q})$ to $G = \mathrm{Sp}(2, \mathbb{Q})^3 \rtimes \mathbb{S}_3$, where the symmetric group \mathbb{S}_3 acts by permuting the three factors $\mathrm{Sp}(2, \mathbb{Q})$. This branching was treated in the preceding section. Since $\mathcal{A}_{1,1,1} = (\mathcal{A}_1)^3 / \mathbb{S}_3$, we have to compute the invariant part of the cohomology of the local system \mathbb{V}_λ restricted to $(\mathcal{A}_1)^3$.

By Lemma 7.1 we can write $V_\lambda|_G$, as a virtual representation,

$$\sum_{\alpha \geq \beta \geq \gamma} m_{\alpha, \beta, \gamma} R_{\alpha, \beta, \gamma} + \sum_{\alpha, \beta} (m_{\alpha, \beta}^+ R_{\alpha, \beta}^+ + m_{\alpha, \beta}^- R_{\alpha, \beta}^-) + \sum_{\alpha} (m_{\alpha}^+ R_{\alpha}^+ + m_{\alpha}^- R_{\alpha}^-).$$

We then have

$$\begin{aligned} e_c(\mathcal{A}_{1,1,1}, \mathbb{V}_\lambda) &= \sum_{\alpha \geq \beta \geq \gamma} m_{\alpha, \beta, \gamma} e_c(\mathcal{A}_1, \mathbb{V}_\alpha) e_c(\mathcal{A}_1, \mathbb{V}_\beta) e_c(\mathcal{A}_1, \mathbb{V}_\gamma) \\ &+ \sum_{\alpha, \beta} \left(m_{\alpha, \beta}^+ \binom{e_c(\mathcal{A}_1, \mathbb{V}_\beta) + 1}{2} + m_{\alpha, \beta}^- \binom{e_c(\mathcal{A}_1, \mathbb{V}_\beta)}{2} \right) e_c(\mathcal{A}_1, \mathbb{V}_\alpha) \\ &+ \sum_{\alpha} \left(m_{\alpha}^+ \binom{e_c(\mathcal{A}_1, \mathbb{V}_\alpha) + 2}{3} + m_{\alpha}^- \binom{e_c(\mathcal{A}_1, \mathbb{V}_\alpha)}{3} \right). \end{aligned}$$

Thus our knowledge of the Euler characteristics of local systems for $g = 1$ and $g = 2$, together with those of local systems on \mathcal{M}_3 , suffices to calculate algorithmically the Euler characteristics on \mathcal{A}_3 .

We calculated these for all λ with $|\lambda| \leq 60$; see Table 6 for the results for $|\lambda| \leq 18$. A first check is that the value $e_c(\mathcal{A}_3, \mathbb{V}_\lambda) = 5$ for the trivial local system given by $\lambda = (0, 0, 0)$ agrees with a result of Hain, who calculated the rational cohomology of \mathcal{A}_3 ; cf. [14]. A further indication of the correctness, besides the fact that while we were summing rational numbers we always found integer values, is that the absolute value of the Euler characteristic of \mathbb{V}_λ on \mathcal{M}_3 is in general much smaller than the Euler characteristic of \mathbb{V}_λ on the hyperelliptic locus and its complement. This is illustrated in Table 3. A similar phenomenon was observed for the genus $g = 2$ case (cf. [9]) and can be observed for \mathcal{A}_3 too, as illustrated in Table 5.

TABLE 5. Euler characteristics of \mathbb{V}_λ (of high weight) on the parts of the stratification of \mathcal{A}_3 .

| λ | \mathcal{H}_3 | \mathcal{M}_3^0 | $\mathcal{M}_2 \times \mathcal{A}_1$ | $\mathcal{A}_{1,1,1}$ | \mathcal{A}_3 |
|------------|-----------------|-------------------|--------------------------------------|-----------------------|-----------------|
| (40,0,0) | -3825 | 3257 | 731 | -161 | 2 |
| (32,5,3) | -188587 | 156651 | 44843 | -12615 | 292 |
| (24,12,4) | -502970 | 419733 | 116703 | -32415 | 1051 |
| (21,15,4) | -351374 | 292508 | 82372 | -22910 | 596 |
| (14,13,13) | -2262 | 1795 | 649 | -191 | -9 |

TABLE 6. Euler characteristics of \mathbb{V}_λ on \mathcal{A}_3 .

| λ | \mathcal{A}_3 | λ | \mathcal{A}_3 | λ | \mathcal{A}_3 | λ | \mathcal{A}_3 |
|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|
| (0,0,0) | 5 | (4,3,3) | 0 | (7,6,1) | 0 | (18,0,0) | -5 |
| (2,0,0) | -2 | (12,0,0) | -1 | (7,5,2) | 1 | (17,1,0) | -7 |
| (1,1,0) | -1 | (11,1,0) | -3 | (7,4,3) | 0 | (16,2,0) | -4 |
| (4,0,0) | -2 | (10,2,0) | 0 | (6,6,2) | 4 | (16,1,1) | -1 |
| (3,1,0) | 0 | (10,1,1) | -1 | (6,5,3) | 1 | (15,3,0) | 0 |
| (2,2,0) | 0 | (9,3,0) | 0 | (6,4,4) | 0 | (15,2,1) | 0 |
| (2,1,1) | 1 | (9,2,1) | 0 | (5,5,4) | 0 | (14,4,0) | -4 |
| (6,0,0) | -3 | (8,4,0) | 0 | (16,0,0) | -2 | (14,3,1) | -4 |
| (5,1,0) | -1 | (8,3,1) | -2 | (15,1,0) | -4 | (14,2,2) | 7 |
| (4,2,0) | 0 | (8,2,2) | 3 | (14,2,0) | 4 | (13,5,0) | 0 |
| (4,1,1) | 1 | (7,5,0) | -2 | (14,1,1) | -1 | (13,4,1) | 0 |
| (3,3,0) | 0 | (7,4,1) | 0 | (13,3,0) | -4 | (13,3,2) | 1 |
| (3,2,1) | 0 | (7,3,2) | 1 | (13,2,1) | 0 | (12,6,0) | 3 |
| (2,2,2) | 1 | (6,6,0) | 5 | (12,4,0) | 4 | (12,5,1) | -3 |
| (8,0,0) | 0 | (6,5,1) | -1 | (12,3,1) | -3 | (12,4,2) | 4 |
| (7,1,0) | -1 | (6,4,2) | 0 | (12,2,2) | 4 | (12,3,3) | -1 |
| (6,2,0) | 1 | (6,3,3) | 1 | (11,5,0) | -2 | (11,7,0) | -3 |
| (6,1,1) | 0 | (5,5,2) | -1 | (11,4,1) | 0 | (11,6,1) | 0 |
| (5,3,0) | 0 | (5,4,3) | 0 | (11,3,2) | -1 | (11,5,2) | 7 |
| (5,2,1) | 0 | (4,4,4) | 0 | (10,6,0) | 4 | (11,4,3) | 0 |
| (4,4,0) | 0 | (14,0,0) | -2 | (10,5,1) | -2 | (10,8,0) | 0 |
| (4,3,1) | 0 | (13,1,0) | -3 | (10,4,2) | 4 | (10,7,1) | -1 |
| (4,2,2) | 1 | (12,2,0) | -1 | (10,3,3) | 0 | (10,6,2) | 5 |
| (3,3,2) | 0 | (12,1,1) | 0 | (9,7,0) | 0 | (10,5,3) | 0 |
| (10,0,0) | -4 | (11,3,0) | -2 | (9,6,1) | 0 | (10,4,4) | 4 |
| (9,1,0) | 0 | (11,2,1) | 0 | (9,5,2) | 0 | (9,9,0) | 4 |
| (8,2,0) | 0 | (10,4,0) | 4 | (9,4,3) | 0 | (9,8,1) | 0 |
| (8,1,1) | 1 | (10,3,1) | -2 | (8,8,0) | 8 | (9,7,2) | 0 |
| (7,3,0) | -2 | (10,2,2) | 1 | (8,7,1) | 1 | (9,6,3) | 8 |
| (7,2,1) | 0 | (9,5,0) | 0 | (8,6,2) | 3 | (9,5,4) | 0 |
| (6,4,0) | 0 | (9,4,1) | 0 | (8,5,3) | 0 | (8,8,2) | 3 |
| (6,3,1) | -1 | (9,3,2) | 0 | (8,4,4) | 8 | (8,7,3) | -3 |
| (6,2,2) | 0 | (8,6,0) | 0 | (7,7,2) | 0 | (8,6,4) | 0 |
| (5,5,0) | 0 | (8,5,1) | -2 | (7,6,3) | 0 | (8,5,5) | 0 |
| (5,4,1) | 0 | (8,4,2) | 0 | (7,5,4) | 0 | (7,7,4) | -1 |
| (5,3,2) | -1 | (8,3,3) | 0 | (6,6,4) | 3 | (7,6,5) | 0 |
| (4,4,2) | 0 | (7,7,0) | -1 | (6,5,5) | 0 | (6,6,6) | 3 |

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