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On the equivalence of regularity criteria for triangular and tetrahedral finite element partitions

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Abstract

In this note we examine several regularity criteria for families of simplicial finite element partitions in $\mathbf{R}^d$, $d \in \{2, 3\}$. These are usually required in numerical analysis and computer implementations. We prove the equivalence of four different definitions of regularity often proposed in the literature. The first one uses the volume of simplices. The others involve the inscribed and circumscribed ball conditions, and the minimal angle condition.

Keywords: Finite element method; Inscribed ball; Circumscribed ball; Regular family of simplicial partitions; Zlámal's condition

1. Introduction

The finite element method (FEM) is nowadays one of the most powerful and popular numerical techniques widely used in various software packages that solve problems in, for instance, mathematical physics and mechanics. The initial step in FEM implementations is to establish an appropriate partition (also called mesh, grid, triangulation, etc.) on the solution domain. For a number of applications simplicial partitions are preferred over the others due to their flexibility. However, such partitions cannot be constructed arbitrarily from both theoretical and practical points of view. Thus, first of all we must ensure, at least theoretically, that the finite element approximations converge to the exact (weak) solution of the mathematical model under consideration when the associated partitions become finer. Mainly due to this reason the notions of regular families of partitions or nondegenerate partitions or shape-preserving partitions appeared. Second, the regularity is also important for real-life computations because degenerate partitions that contain flat elements may yield ill-conditioned stiffness matrices.

In 1968, Miloš Zlámal [1] introduced the so-called *minimal angle condition* that ensures the convergence of the finite element approximations for solving linear elliptic boundary value problems for $d = 2$. This condition requires that there exists a constant $\alpha_0 > 0$ such that the minimal angle α_S of each triangle S in all triangulations used satisfies

$$\alpha_S \geq \alpha_0.$$

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Zlámal's condition can be generalized into $\mathbf{R}^d$ for any $d \in \{2, 3, \ldots\}$ so that all dihedral angles of simplicies and their lower-dimensional facets are bounded from below by a positive constant. Later, the so-called *inscribed ball condition* was introduced, see, e.g. [2, p. 124], which uses a ball contained in a given element (cf. (2)). Thus, it can also be used for nonsimplicial elements. This condition has an elegant geometrical interpretation: the ratio of the radius of the inscribed ball of any element and the diameter of this element must be bounded from below by a positive constant over all partitions. Roughly speaking, no element of no partitions should degenerate to a hyperplane as the discretization parameter h (i.e. the maximal diameter of all elements in the corresponding partition $\mathcal{T}_h$) tends to zero. This property is called in [2] the *regularity of a family of partitions*. For triangular elements it is, obviously, equivalent to Zlámal's condition.

In 1985, Lin and Xu [3] introduced a somewhat stronger regularity assumption on triangular elements: each triangle $S \in \mathcal{T}_h$ contains a circle of radius $c_1 h$ and is contained in a circle of radius $c_2 h$, where c_1 and c_2 are positive constants that do not depend on S and h. Later, this assumption was modified as follows (see, e.g. [4]): a family of triangulations is called *strongly regular* if there exist two positive constants c_1 and c_2 such that for all $S \in \mathcal{T}_h$

$$c_1 h^2 \leq \text{meas}_2 S \leq c_2 h^2.$$

Notice that in this case no circle or angle conditions appear (cf. (1) and (25) below) and we may obviously take $c_2 = 1$. Here and elsewhere in this paper meas_p stands for the p-dimensional measure.

Recently, in order to prove some superconvergence results, Brandts and Křížek [5] employed another regularity condition based on the circumscribed ball about simplicial elements, which is the unique sphere on which all vertices of the simplex lie (see (3)).

In the present paper we summarize the above proposed conditions into four different definitions of regularity and prove in detail that all these definitions are equivalent for simplicial elements in two and three dimensions.

2. Preliminaries

Let $\overline{\Omega} \subset \mathbf{R}^d$, $d \in \{1, 2, 3, \ldots\}$, be a closed domain (i.e. the closure of a domain). If its boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes, we say that $\overline{\Omega}$ is *polytopic*. Moreover, if $\overline{\Omega}$ is bounded, it is called a *polytope*; in particular, $\overline{\Omega}$ is called a *polygon* for $d = 2$ and a *polyhedron* for $d = 3$.

A *simplex* S in $\mathbf{R}^d$ is a convex hull of $d + 1$ points, $A_1, A_2, \ldots, A_{d+1}$, that do not belong to the same hyperplane. We denote by h_S the length of the longest edge of S. Let F_i be the face of a simplex S opposite to the vertex A_i and let v_i be the altitude from the vertex A_i to the face F_i. For $d = 3$ angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

Next we define a *simplicial partition* $\mathcal{T}_h$ over the polytope $\overline{\Omega} \subset \mathbf{R}^d$. We subdivide $\overline{\Omega}$ into a finite number of simplices (called *elements* or *simplicial elements*), so that their union is $\overline{\Omega}$, any two simplices have disjoint interiors and any facet of any simplex is a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$.

The set $\mathcal{F} = \{\mathcal{T}_h\}_{h>0}$ is called a *family of partitions* if for any $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

3. On the equivalence of various regularity conditions

The regularity conditions presented in the introduction can, in fact, be summarized into four conditions for the regularity of simplicial partitions which we will present below. In what follows, all constants C_i are independent of S and h, but can depend on the dimension $d \in \{2, 3\}$.

Condition 1. There exists a constant $C_1 > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$ and any simplex $S \in \mathcal{T}_h$ we have

$$\text{meas}_d\, S \geq C_1 h_S^d. \tag{1}$$

Condition 2. There exists a constant $C_2 > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$ and any simplex $S \in \mathcal{T}_h$ there exists a ball $B \subset S$ with radius r such that

$$r \geq C_2 h_S. \tag{2}$$

Condition 3. There exists a constant $C_3 > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$ and any simplex $S \in \mathcal{T}_h$ we have

$$\operatorname{meas}_d S \geq C_3 \operatorname{meas}_d B^S, \tag{3}$$

where $B^S \supset S$ is the circumscribed ball about S.

Condition 4. There exists a constant $C_4 > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$, any simplex $S \in \mathcal{T}_h$, and any dihedral angle α and for $d = 3$ also any solid angle α of S, we have

$$\alpha \geq C_4. \tag{4}$$

Before we prove that the above four conditions are equivalent, we present three auxiliary lemmas.

Lemma 1. *For any simplex S and any $i \in \{1, \dots, d+1\}$, $d \in \{2, 3\}$, we have*

$$\operatorname{meas}_d S \leq h_S^d, \tag{5}$$

$$\operatorname{meas}_{d-1} F_i \leq h_S^{d-1}, \tag{6}$$

$$v_i \leq h_S. \tag{7}$$

Proof. Relations (5) and (6) follow from the fact that the distance between any two points of a simplex S is not larger than h_S. Thus, S and any of its faces F_i (if $d = 3$), or edges F_i (if $d = 2$), are contained in a cube and a square with edges of length h_S. Inequality (7) is obvious. □

Lemma 2. *For any simplex S we have*

$$2r < h_S \leq 2r^S, \tag{8}$$

where r^S is the radius of the circumscribed ball B^S about S, and r is a radius of any ball $B \subset S$.

Proof. Since $B \subset S \subset B^S$, their diameters are nondecreasing. The sharp inequality in (8) is evident. □

Recall that for any $i \in \{1, \dots, d+1\}$ we have

$$\operatorname{meas}_d S = \frac{1}{d} v_i \operatorname{meas}_{d-1} F_i. \tag{9}$$

Lemma 3. *If condition (1) holds and $d \in \{2, 3\}$, then there exist positive constants C_5, C_6, and C_7 such that for any partition $\mathcal{T}_h \in \mathcal{F}$, any simplex $S \in \mathcal{T}_h$, and any $i \in \{1, \dots, d+1\}$, we have*

$$\operatorname{meas}_{d-1} F_i \geq C_5 h_S^{d-1}, \tag{10}$$

$$v_i \geq C_6 h_S, \tag{11}$$

$$\sin\alpha \geq C_7, \tag{12}$$

where α is any dihedral angle of S and for $d = 3$ also any solid angle of S.

Proof. From (1), (7) and (9), we obtain

$$C_1 h_S^d \leq \operatorname{meas}_d S = \frac{1}{d} v_i \operatorname{meas}_{d-1} F_i \leq \frac{1}{d} h_S \operatorname{meas}_{d-1} F_i, \tag{13}$$

which implies (10). Further, inequality (11) follows from (13) if we use relation (6) to bound the right-hand side of the equality in (13).

For any angle α of triangular elements or dihedral angle α for tetrahedral elements, we get by (11) that

$$\sin\alpha \geq \frac{v_i}{h_S} \geq C_6,$$

where v_i is a minimal altitude of S. Similar relations hold for the solid angles of the triangular faces of tetrahedron S (i.e. when $d = 3$), since in this case altitudes in the triangular faces are not less than the minimal altitude v_i of the tetrahedron. Thus, $C_7 = \arcsin C_6$. □

Remark 1. Consider a tetrahedron whose base is an equilateral triangle. Let the attitude of the tetrahedron end at the center of the base and let it be very high. Then the three dihedral angles at the base are almost 90° and the remaining three dihedral angles are approximately 60°. However, some solid angles are very small. Therefore, in Condition 4 a positive lower bound on solid angles is prescribed.

Theorem 1. *For the dimension $d \in \{2, 3\}$, Conditions 1–4 are equivalent.*

Proof. We prove that condition (1) is equivalent to each of conditions (2), (3), and (4).

(1) $\Longrightarrow$ (2): Let B_S be the inscribed ball of S with the radius r_S and the center O_S. We decompose S into $d+1$ subsimplicies – conv $\{O_S, F_i\}$, $i \in \{1, \dots, d+1\}$. All of them have the same altitude r_S, i.e. by (9), we get

$$\operatorname{meas}_d S = \frac{1}{d} \sum_{i=1}^{d+1} r_S \operatorname{meas}_{d-1} F_i. \tag{14}$$

Further, for any face of any simplex inequality (6) is valid, i.e. $d \operatorname{meas}_d S \le r_S(d+1)h_S^{d-1}$, and now using (1), we finally observe that

$$r_S(d+1)h_S^{d-1} \ge d \operatorname{meas}_d S \ge C_1 d\, h_S^d, \tag{15}$$

which implies Condition 2 if we take $B = B_S$, $r = r_S$, and $C_2 = \frac{C_1 d}{d+1}$.

(2) $\Longrightarrow$ (1): Obviously, from the fact that $B \subset S$ and (2) we get

$$\operatorname{meas}_d S \ge \operatorname{meas}_d B \ge \pi\, r^d \ge \pi\, C_2^d\, h_S^d, \tag{16}$$

which implies Condition 1.

(1) $\Longrightarrow$ (3): From (1) and (5) we observe that $h_S^d \ge \operatorname{meas}_d S \ge C_1 h_S^d$. Also, $\operatorname{meas}_d B^S = C_8(d)(r^S)^d$, where $C_8(2) = \pi$ and $C_8(3) = \frac{4}{3}\pi$. We prove that under condition (1), there exists a constant $C_9 > 0$ such that for any simplex S from any partition $\mathcal{T}_h \in \mathcal{F}$ we have

$$r^S \le C_9 h_S\,. \tag{17}$$

If (17) holds, then using (1) we immediately prove (3) as follows

$$\operatorname{meas}_d S \ge C_1 h_S^d \ge C_1 \frac{(r^S)^d}{C_9^d} = \frac{C_1}{C_9^d\, C_8(d)} \operatorname{meas}_d B^S. \tag{18}$$

Consider first the case $d = 2$. Let S denote the triangular element $A_1A_2A_3$. It is well known that

$$r^S = \frac{|A_1A_2| \cdot |A_2A_3| \cdot |A_1A_3|}{4 \operatorname{meas}_2 S}. \tag{19}$$

Then in view of (1) (for $d = 2$) and the fact that any edge of S is of a length not greater than h_S we have

$$r^S \le \frac{h_S^3}{4C_1 h_S^2} = \frac{1}{4C_1}\, h_S = C_9 h_S.$$

For the case $d = 3$ we use the following formula for the calculation of the circumradius presented in [6, p. 316] (cf. [7, p. 212]) for the tetrahedral element $S = A_1A_2A_3A_4$

$$r^S = \frac{\sqrt{Q_S}}{24 \operatorname{meas}_3 S}, \tag{20}$$

where

$$Q_S = 2\, l_1^2 l_2^2 l_4^2 l_5^2 + 2\, l_1^2 l_3^2 l_4^2 l_6^2 + 2\, l_2^2 l_3^2 l_5^2 l_6^2 - l_1^4 l_4^4 - l_2^4 l_5^4 - l_3^4 l_6^4. \tag{21}$$

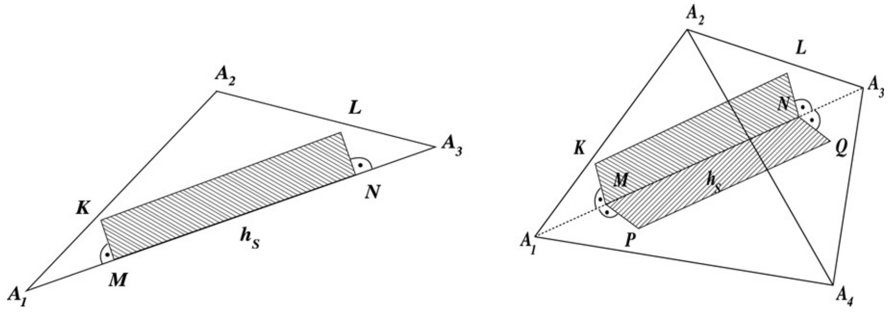

Fig. 1. Illustration for the proof (4) $\Longrightarrow$ (1).

In the above l_p and l_{p+3} are the lengths of opposite edges of S, $p = 1, 2, 3$. Obviously, using again the fact that $l_j \le h_S,\ j = 1, \ldots, 6$, we have $V_S \le 6h_S^8$. Thus, from (1) (for $d = 3$) and (20) we get

$$r^S \le \frac{\sqrt{6\,h_S^8}}{24\,C_1\,h_S^3} = \frac{\sqrt{6}}{24\,C_1}\,h_S = C_9 h_S.$$

(3) $\Longrightarrow$ (1): In view of (3) and (8) we observe that

$$\operatorname{meas}_d S \ge C_3 \operatorname{meas}_d B^S = C_3 C_8(d)(r^S)^d \ge \frac{C_3 C_8(d)}{2^d} h_S^d, \tag{22}$$

which implies Condition 1.

(1) $\Longrightarrow$ (4): See (12).

(4) $\Longrightarrow$ (1): First we consider the case $d = 2$. Let S be again the triangular element $A_1A_2A_3$ and let $h_S = |A_1A_3|$. Now, we cut out of the edge A_1A_3 the segment $|MN|$ of the length $\frac{h_S}{2}$ with the endpoints M and N be at the distance $\frac{h_S}{4}$ from the vertices A_1 and A_3, respectively (see the left of Fig. 1).

Thus, $|A_1M| = |NA_3| = \frac{h_S}{4}$. Since the angles $\angle A_2A_1A_3$ and $\angle A_2A_3A_1$ are bounded from above and below due to (4), we can form a rectangle $KLNM$ inside of S (see the shadowed area in Fig. 1 (left)) so that $|MK| = |LN| = \frac{h_S}{4}\tan C_4$. Then, it is clear that $\operatorname{meas}_2 S \ge \operatorname{meas}_2 KLNM = \frac{h_S}{2}\frac{h_S}{4}\tan C_4 = C_1 h_S^2$, where $C_1 = \frac{1}{8}\tan C_4$.

Consider now the case $d = 3$ and let S denote a tetrahedron $A_1A_2A_3A_4$ (see Fig. 1 (right)). Using the above argumentation for the triangular faces $A_1A_2A_3$ and $A_1A_3A_4$ we can form two rectangles $KLNM$ and $PQNM$ with areas equal to $\frac{1}{8}h_S^2\tan C_4$. Further, we consider the triangular prism $KLMNPQ$, which is inside of the tetrahedron S. Thus, $\operatorname{meas}_3 S \ge C_1 h_S^3$, where $C_1 = \frac{1}{2}\frac{1}{64}\tan^2 C_4 \sin C_4$, due to boundedness of the dihedral angle between faces $A_1A_2A_3$ and $A_1A_3A_4$, see (4). $\square$

Definition 1. A family of simplicial partitions is called *regular* if Condition 1 or 2 or 3 or 4 holds.

Remark 2. Condition (1) seems to be simpler than the ball conditions or the angle condition, and therefore, it may be preferred in theoretical finite element analysis. On the other hand, the angle conditions are often used in finite element codes to keep simplices nondegenerating. For this purpose condition (1) can be useful as well.

4. Final remarks

In 1957, Synge [8, p. 211] proved that linear triangular elements have optimal interpolation properties in the C^1-norm provided there exists a positive constant $\gamma_0 < \pi$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any triangle $S \in \mathcal{T}_h$ we have

$$\gamma_S \le \gamma_0, \tag{23}$$

where γ_S is the maximal angle of S. We observe that in this case the minimal angles α_S may tend to zero as $h \to 0$. On the other hand, if Zlámal's condition holds, then the maximal angle condition (23) holds as well. In 1974, several authors [9–12] independently proved the convergence of the finite element method under Synge's condition (23). If this condition is not valid, linear triangular elements lose their optimal interpolation properties (see e.g. [9, p. 223]).

According to [13], the maximal angle condition (23) is equivalent to the following circumscribed ball condition for $d = 2$: there exists a constant $C_{10} > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$ and any triangle $S \in \mathcal{T}_h$ we have (cf. (17))

$$r^S \leq C_{10} h_S. \tag{24}$$

The associated families of partitions are called *semiregular*. Each regular family is semiregular, but the converse implication does not hold. Therefore, (3) implies (24), but (24) does not imply (3).

Synge's condition (23) is generalized to the case of tetrahedra in [14] and [15]. However, an extension of (23) or (24) to $\mathbf{R}^d$ so that simplicial finite elements preserve their optimal interpolation properties in Sobolev norms is still an open problem.

It is easy to verify that conditions (1) and (2) are equivalent also for nonsimplicial finite elements.

Replacing (1) and (2) by

$$\operatorname{meas}_d S \geq C_1' h^d, \tag{25}$$

and

$$r \geq C_2' h,$$

respectively, we can show that these conditions are also equivalent. The associated family of such partitions is called *strongly regular* (cf. [2, p. 147]).

Let us point out that for a strongly regular family of partitions the well-known inverse inequalities hold (see [2, p. 142]), e.g.

$$\|v_h\|_1 \leq \frac{C}{h} \|v_h\|_0 \quad \forall v_h \in V_h, \tag{26}$$

where V_h are finite element subspaces of the Sobolev space $H^1(\Omega)$, the symbol C stands for a constant independent of h and $\|\cdot\|_k$ is the standard Sobolev norm. Inverse inequalities play an important role in proving convergence of the finite element method of various problems (see [15]).

In [16–18] we show how to generate partitions with nonobtuse dihedral angles, i.e. all simplices satisfy the maximal angle condition in $\mathbf{R}^d$. Such partitions guarantee the discrete maximum principle for a class of nonlinear elliptic problems solved by linear simplicial elements (see [19]).

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