On a unified description of non-abelian charges, monopoles and dyons
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Chapter 1

Introduction

In the 1970s Goddard, Nuyts and Olive were the first to write down a rough version of what has become one of the most celebrated dualities in high energy physics. Following earlier work of Englert and Windey on the generalised Dirac quantisation condition [1] they showed that the charges of monopoles in a theory with gauge group $G$ take values in the weight lattice of the dual gauge group $G^*$, now known as the GNO or Langlands dual group. Based on this fact they came up with a bold yet attractive conjecture: monopoles transform as representations of the dual group [2].

Within a year Montonen and Olive observed that the Bogomolny Prasad Sommerfield (BPS) mass formula for dyons [3, 4] is invariant under the interchange of electric and magnetic quantum numbers if the coupling constant is inverted as well [5]. This led to the dramatic conjecture that the strong coupling regime of some suitable quantum field theory is described by a weakly coupled theory with a similar Lagrangian but with the gauge group replaced by the GNO dual group and the coupling constant inverted. Moreover, they proposed that in the BPS limit of a gauge theory where the gauge group is spontaneously broken to $U(1)$ the ’t Hooft-Polyakov solutions [6, 7] in the original theory correspond to the heavy gauge bosons of the dual theory. Supporting evidence for the idea of viewing the ’t Hooft-Polyakov monopoles as fundamental particles came from Erick Weinberg’s zero-mode analysis in [8].

Soon after Montonen and Olive proposed their duality, Osborn noted that $\mathcal{N}=4$ Super Yang-Mills theory (SYM) would be a good candidate to possess the duality since BPS monopoles fall into the same BPS supermultiplets as the elementary particles of the theory [9]. $\mathcal{N}=2$ SYM on the other hand has always been considered an unlikely candidate because the BPS monopoles fall into BPS multiplets that do not correspond to the elementary fields of the $\mathcal{N}=2$ Lagrangian. In particular there are no semi-classical monopole states with spin equal to 1 so that the monopoles cannot be identified with
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heavy gauge bosons. Most surprisingly the Montonen-Olive conjecture has never been proven for $\mathcal{N} = 4$ SYM whereas a different version of the duality has explicitly been shown to occur for the $\mathcal{N} = 2$ theory in 1994 by Seiberg and Witten [10]. These authors started out from $\mathcal{N} = 2$ SYM with the $SU(2)$ gauge group broken down to $U(1)$ and computed the exact effective Lagrangian of the theory to find a strong coupling phase described by SQED except that the electrons are actually magnetic monopoles. Moreover, by softly breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry they were able to show that in this strong coupling phase the monopoles condense and thereby demonstrated the 't Hooft-Mandelstam confinement scenario [11, 12]. Similar results hold for higher rank gauge groups broken down to their maximal abelian subgroups [13, 14]. In these cases we indeed have an explicit realisation of a magnetic abelian gauge group at strong coupling.

1.1 Labelling of monopoles in non-abelian phases

A fascinating aspect of Seiberg-Witten theory is that it gives rise to not just one strong coupling phase, but to several. In general only one of these phases contains massless monopoles while the others have an effective description in terms of dyons. A priori it is thus not clear what dynamically the relevant degrees of freedom are. This illustrates the necessity of a proper kinematic description of all possible degrees of freedom contained in the theory. For abelian phases it is not too hard to provide such a kinematic description while for non-abelian phases, that is, phases where the gauge group is broken down to a non-abelian subgroup, this problem has never been solved satisfactorily. The main challenge is to give a proper labelling of monopoles and dyons. In this thesis we tackle this issue.

For monopoles one finds classically that magnetic charges take values in the weight lattice of the dual group. Yet, there is no obvious rule to order these weights into irreducible representations with the appropriate dimensions and degeneracies, let alone that there is a manifest action of the dual group on the classical field configurations. To illustrate this we consider an example with gauge group $U(2)$ embedded as a subgroup in $SU(3)$. The magnetic charge lattice in this case corresponds to the root lattice of $SU(3)$ as depicted in figure 1.1. The GNO dual group of $U(2)$ is again isomorphic to $U(2)$, in other words the magnetic charge lattice can be identified with the weight lattice of $U(2)$. This group is by definition equal to $(U(1) \times SU(2))/\mathbb{Z}_2$. The $SU(2)$ weights can be identified with the components of the charges along the axis defined by one of the simple roots of $SU(3)$, say $\alpha_1$. The $U(1)$ charges then correspond to the components of the charge along the axis perpendicular to $\alpha_1$. 

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Figure 1.1: The magnetic charge lattice for $G = U(2)$ corresponds to the root lattice of $SU(3)$ but also to the weight lattice of $U(2)$, hence $G^*$ equals $U(2)$ in this case. One of the simple roots of $SU(3)$, say $\alpha_1$ is identified with the root of $SU(2) \subset U(2)$.

As a next step one would want to use the magnetic charge lattice to characterise the magnetic $U(2)$ multiplets in accordance with the GNO conjecture. The origin of the charge lattice, i.e. the vacuum, can consistently be identified with the trivial representation of $U(2)$. Naively one would simply associate the doublet representation with unit $U(1)$ charge with the pair of weights $g_1 = \alpha_1 + \alpha_2$ and $g_2 = \alpha_2$. This relation, however, raises some questions about the action of the dual group which become even more pressing as soon as one takes fusion of monopoles into account. There is an action that maps $g_1$ to $g_2$ and vice versa, suggesting that we are indeed dealing with a doublet. This action corresponds precisely to the action of the Weyl group $Z_2$ of $U(2)$ generated by the reflection in the line perpendicular to $\alpha_1$. If we now consider the product of two monopoles in the doublet representation of the dual group then we expect from the GNO conjecture that one would obtain a singlet and a triplet. On the other hand in the classical theory the charge of a combined monopole equals the sum of charges of the constituents, and in this particular case thus equals $2g_1, g_1 + g_2$ or $2g_2$. The charges $2g_1$ and $2g_2$ are again related by the action of the Weyl group, but this Weyl action does not relate these two charges to $g_1 + g_2$. Moreover, it is not clear if a combined classical monopole solution with charge $g_1 + g_2$, i.e. with an $SU(2)$ weight equal to zero, corresponds to a triplet or a singlet state. One possible argument to resolve this issue comes from the fact that the action of the Weyl group is nothing but a large gauge transformation which suggests that charges on a single Weyl orbit should not be distinguished as different weights of a dual representation. Instead, such magnetic charges should be identified in the sense that they constitute a single gauge invariant charge sector. Pushing this argument a bit further one may conclude that a monopole should be labelled by an integral dominant weight, i.e. by the highest weight of an irreducible representation of the GNO dual group. The drawback of this interpre-
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tation is that it does neither manifestly show the dimension of the dual representations in the magnetic charge lattice nor does it directly explain the degeneracies implied by the fusion rules of $G^\ast$. From this example we conclude that solving this labelling problem for monopoles is closely related to proving the GNO conjecture. It is also important to note that the heuristic arguments above do not disprove the GNO conjecture since, as we have learned from Montonen and Olive, one only expects the dual symmetry to be manifest at strong coupling. From that perspective it is not very surprising that the dual symmetry is to a certain extent hidden in the classical regime. Nonetheless, it should be clear that that the charge labels of monopoles are related to weights or dominant integral weights of the GNO dual group.

For dyons with non-vanishing electric and magnetic charges the situation is worse since, as we shall explain below, it is not known what the relevant algebraic object is that will give rise to a proper labelling.

Before we continue one should wonder whether Yang-Mills theories can have non-abelian phases at strong coupling. Both the classical $\mathcal{N} = 4$ and $\mathcal{N} = 2$ pure SYM theories have a continuous space of ground states corresponding to the vacuum expectation value of the adjoint Higgs field. A non-abelian phase corresponds to the Higgs VEV having degenerate eigenvalues. In the $\mathcal{N} = 4$ theory the supersymmetry is sufficient to protect the classical vacuum structure even non-perturbatively [15]. So the non-abelian phases manifestly realised in the classical regime must survive at strong coupling as well. In the $\mathcal{N} = 2$ theory the vacuum structure is changed in quite a subtle way by non-perturbative effects. In those subspaces of the quantum moduli space where a non-abelian phase might be expected there are no massless W-bosons. Instead, the perturbative degrees of freedom correspond to photons and massless monopoles carrying abelian charges. In the best case there are some indications that a non-abelian phase may exist at strong coupling in certain $\mathcal{N} = 2$ theories with a sufficient number of hyper multiplets [16, 17].

Although a kinematic description of monopoles and dyons in non-abelian phases is probably not very relevant for $\mathcal{N} = 2$ SYM it may be important in understanding strongly coupled non-abelian phases of $\mathcal{N} = 4$ SYM or non-abelian phases of other theories. We shall therefore discuss these issues in a general context. In a very specific theory, however, progress has already been made.

Quite recently Witten and Kapustin have found extraordinary new evidence to support the non-abelian Montonen-Olive conjecture. This evidence was constructed in an effort to show that the mathematical concept of the geometric Langlands correspondence arises naturally from electric-magnetic duality in physics [18].

The starting point for Kapustin and Witten is a twisted version of $\mathcal{N} = 4$ gauge theory. They identify ’t Hooft operators, which create the flux of Dirac monopoles, with Hecke operators. The labels of these operators are given by the generalised Dirac quantisation rule and can up to a Weyl transformation be identified with dominant integral weights
of the dual gauge group. Since a dominant integral weight is the highest weight of a
unique irreducible representation, magnetic charges thus correspond to irreducible repre-
sentations of the dual gauge group. The moduli spaces of the singular BPS monopoles
are identified with the spaces of Hecke modifications. The operation of bringing two
separated monopoles together defines a non-trivial product of the corresponding moduli
spaces. The resulting space can be stratified according to its singularities. Each singular
subspace is again the compactified moduli space of a monopole related to an irreducible
representation in the tensor product. The multiplicity of the BPS saturated states for each
magnetic weight is found by analysing the ground states of the quantum mechanics on
the moduli space. The number of ground states given by the De Rham cohomology of the
moduli space agrees with the dimension of the irreducible representation labelled by the
magnetic weight. Moreover, Kapustin and Witten exploited existing mathematical results
on the singular cohomology of the moduli spaces to show that the products of 't Hooft op-
erators mimic the fusion rules of the dual group. The operator product expansion (OPE)
algebra of the 't Hooft operators thereby reveals the dual representations in which the
monopoles transform.

There is an enormous amount of evidence to support the Montonen-Olive conjecture for
the ordinary $\mathcal{N} = 4$ SYM theory, see for example [19, 20, 21]. These results which mainly
concern the invariance of the spectrum do not leave much room to doubt that the strongly
coupled theory can be described in terms of monopoles. However, they do not say much
about the fusion rules of these monopoles. If the original GNO conjecture does indeed ap-
ply for $\mathcal{N} = 4$ SYM theory with residual non-abelian gauge symmetry, smooth monopoles
should have properties similar to those of the singular BPS monopoles in the Kapustin-
Witten setting. By the same token we claim that one can exploit these properties to find
new evidence for the GNO duality in spontaneously broken theories. In chapter 3 we aim
to set a first step in this direction by generalising the classical fusion rules found by Erick
Weinberg for abelian BPS monopoles [22] to the non-abelian case. Our results indicate
that smooth BPS monopoles are naturally labelled by dominant integral weights of the
residual dual gauge group.

1.2 Dyonic complications and the skeleton group

A stronger version of the GNO conjecture is that a gauge theory has a hidden electric-
magnetic symmetry of the type $G \times G^*$. The problem with this proposal is that the dy-
onic sectors do not respect this symmetry in phases where one has a residual non-abelian
gauge symmetry. In such phases it may be that in a given magnetic sector there is an
obstruction to the implementation of the full electric group. In a monopole background the global electric symmetry is restricted to the centraliser in $G$ of the magnetic charge $[23, 24, 25, 26, 27, 28]$. Dyonic charge sectors are thus not labelled by a $G \times G^*$ representation but instead (up to gauge transformations) by a magnetic charge and an electric centraliser representation. For example in the case of $G = U(2)$, the centraliser for the magnetic charge $\alpha_2$, see figure 1.1, equals the abelian subgroup $U(1) \times U(1)$. Hence, a dyon with this magnetic charge has an electric label corresponding to a representation of this abelian centraliser $[29]$. For a dyon with magnetic charge equal to $\alpha_1 + 2\alpha_2$ the electric charge corresponds to a representation of the non-abelian centraliser group $U(2)$. This interplay of electric and magnetic degrees of freedom is not captured by the $G \times G^*$ structure. Therefore one would like to find a novel algebraic structure reflecting the complicated pattern of the different electric-magnetic sectors in such a non-abelian phase. We see that one does not need this algebraic structure just to find a labelling for dyons but actually, first, to prove the consistency of the labelling already proposed, and second, to retrieve the fusion rules of non-abelian dyons which are not known at present. In terms of centraliser representations one seems to run into trouble as soon as one considers fusion of dyons. On the electric side it is not clear how to define a tensor product involving the representations of distinct centraliser groups such as for example $U(1) \times U(1)$ and $U(2)$, even though the fusion rules for each of the centraliser groups are known. The algebraic structure we seek would thus have to generate the complete set of fusion rules for all the different sectors and in particular it would have to combine the different centraliser groups that may occur in such phases within one framework. It also has to be consistent with the fact that in the pure electric sector charges are labelled by the the full electric gauge group $G$, while in the purely magnetic sector, at least for the twisted $\mathcal{N} = 4$ theory considered by Kapustin and Witten in $[18]$, monopoles form representations of the magnetic gauge group $G^*$.

Generalising an earlier proposal by Schroers and Bais $[30]$ we suggest in chapter 4 a formulation of a gauge theory based on the so-called skeleton group $S$. This is in general a non-abelian group that allows one to manifestly include non-abelian electric and magnetic degrees of freedom. The skeleton group therefore implements (at least part of) the hidden electric-magnetic symmetry explicitly and the representation theory of $S$ provides us with a consistent set of fusion rules for the dyonic sectors for an arbitrary gauge group. Nonetheless, it does not quite fulfill our original objective. The skeleton group has roughly the product structure $S = \mathcal{W} \ltimes (T \times T^*)$ where $T$ and $T^*$ are the maximal tori of $G$ and $G^*$ and $\mathcal{W}$ the Weyl group. Therefore $S$ contains neither the full electric gauge group $G$ nor the magnetic group $G^*$, and this of course implies that its representation theory will not contain the representation theories of either $G$ or $G^*$. We show, however, that in the purely electric sector the representation theory of the skeleton group is consistent with the representation theory of $G$. 
1.2. Dyonic complications and the skeleton group

The appearance of the skeleton group can be understood from gauge fixing and in that sense our approach matches an interesting proposal of ’t Hooft [31]. In order to get a handle on non-perturbative effects in gauge theories, like chiral symmetry breaking and confinement, ’t Hooft introduced the notion of non-propagating gauges. An important example of such a non-propagating gauge is the so-called abelian gauge. In this gauge a non-abelian theory can be interpreted as an abelian gauge theory with monopoles in it. This has led to a host of interesting approximation schemes to tackle the aforementioned non-perturbative phenomena which remain elusive from a first principle point of view, see e.g. [32, 33, 34, 35].

We present a generalisation of ’t Hooft’s proposal from an abelian to a minimally non-abelian scheme. That is where the skeleton group comes in: it plays the role of the residual symmetry in a gauge which we call the skeleton gauge. The attractive feature is that our generalisation does not affect the continuous part of the residual gauge symmetry after fixing. It is still abelian, but our generalisation adds (non-abelian) discrete components. This implies that the non-abelian features of the effective theory manifest themselves through topological interactions only, and that makes them manageable. The effective theories we end up with are actually generalisations of Alice electrodynamics [36, 37, 38]. In this sense the effective description of the non-abelian theory with gauge group $G$ in the skeleton gauge is an intricate merger of an abelian gauge theory and a (non-abelian) discrete gauge theory [39, 40]. Moreover, the skeleton gauge incorporates configurations which are not accessible in the abelian gauge. Hence, compared to the abelian gauge, the skeleton gauge and thereby the skeleton group may yield a much wider scope on certain non-perturbative features of the original gauge theory.

The motivation for exploring non-propagating gauges is to obtain a formulation of the theory as much as possible in terms of the physically relevant degrees of freedom. In that sense ’t Hooft’s approach looks like studying the Higgs phase in a unitary gauge, but it goes beyond that because one does not start out from a given phase determined by a suitable (gauge invariant) order parameter. Instead, the effective theory in the abelian gauge is obtained after integrating out the non-abelian gauge field components. Nonetheless, the resulting theory is particularly suitable for describing the Coulomb phase where the residual gauge symmetry is indeed abelian. Similarly, the skeleton group is related to a generalised Alice phase.

Once this gauge-phase relation is understood our skeleton formulation not only allows us to obtain the precise fusion rules for the mixed and neutral sectors of the theory, but as a bonus allows us to analyse the phase structure of gauge theories. Yang-Mills theories give rise to confining phases, Coulomb phases, Higgs phases, discrete topological phases, Alice phases etc. These phases differ not only in their particle spectra but also in their topological structure. It is therefore crucial to have a formulation that highlights the relevant degrees of freedom, allowing one to understand what the physics of such phases is.

Starting from the skeleton gauge we are in a position to answer kinematic questions con-
Concerning different phases and possible transitions between them. For this purpose it is of the utmost importance to work in a scheme where one can compute the fusion rules involving electric, magnetic and dyonic sectors. This is deduced from some common wisdom concerning the abelian case where the fusion rules are very simple: if there is a condensate corresponding to a particle with a certain electric or magnetic charge then any particle with a multiple of this charge can consistently be thought of as absorbed by the vacuum. In other words, the condensation of a particle leads to an identification of charge sectors. For confinement we know that if two electric-magnetic charges do not confine then the sum of these charges will also not confine. Given the fusion rules predicted by the skeleton group we can in principle analyse all phases that emerge from generalised Alice phases by condensation or confinement.