On a unified description of non-abelian charges, monopoles and dyons
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Chapter 2

Classical monopole solutions

This preliminary chapter serves multiple purposes. First, we want to explain what monopoles are and review some of their properties. Most of these are well known, a few are not. Second, we want to introduce some conventions, concepts and quantities that will be used in the remainder of this thesis. Finally, we want to explain how one can create some order in the monopole jungle by introducing several types of monopoles.

Very roughly speaking a monopole is a solution to equations of motion of a gauge theory with a non-vanishing magnetic charge. The nature of such a charge depends of course on exactly what Yang-Mills theory is considered and specifically what the gauge group is. Nonetheless, in general the magnetic charges constitute a discrete set which can be used to distinguish different monopoles within a given theory. These sets will be discussed in section 2.3. A cruder way to classify monopoles is to distinguish singular monopoles from smooth monopoles and non-BPS monopoles from BPS monopoles. In the first two sections of this chapter we shall review these properties and some related concepts.

2.1 Singular monopoles

Singular monopoles can appear in any gauge theory but the most basic example is a pure Yang-Mills theory. This can be either the abelian theory with gauge group $U(1)$ that arises from the homogeneous Maxwell equations or a generalisation where the gauge group $U(1)$ is replaced by a larger and possibly non-abelian gauge group which we shall denote by $H$. The Lagrangian of such a Yang-Mills theory is completely defined in terms
Chapter 2. Classical monopole solutions

of the field strength tensor $F_{\mu\nu}$:

$$\mathcal{L} = -\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu})$$  \hspace{1cm} (2.1)

The field strength tensor can be further expanded as $F_{\mu\nu} = F_{\mu\nu}^a t_a$, where $t_a$ are the generators of the Lie algebra of $H$. In terms of the gauge field $A_\mu = A_\mu^a t_a$ we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu].$$  \hspace{1cm} (2.2)

Using differential forms one can write $A = A_\mu dx^\mu$ and $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ so that by definition $F = dA - ieA \wedge A$.

The equations of motion derived from the Lagrangian in (2.1) are given by:

$$D \ast F = 0$$  
$$DF = 0.$$  \hspace{1cm} (2.3)

The first of these two equations is the true equation of motion, the second is the Bianchi identity, see e.g. section 10.3 of [41]. The electric and magnetic fields can be expressed in terms of the field strength tensor as

$$E^i = F^{0i} = -F^{i,0} = F_{i,0}$$  \hspace{1cm} (2.4)
$$B^i = \frac{1}{2} \epsilon^{ijk} F_{ij} \iff F^{ij} = \epsilon^{ijk} B^k.$$  \hspace{1cm} (2.5)

If the electric field vanishes we thus have

$$F = \ast B.$$  \hspace{1cm} (2.6)

where $\ast$ corresponds to the Hodge star of the 3-dimensional Euclidean space $\mathbb{R}^3$.

A Dirac monopole [42] is a configuration of the electric-magnetic field with everywhere vanishing electric field and a static magnetic field of the form

$$B = \frac{G_0}{4\pi r^2} dr.$$  \hspace{1cm} (2.7)

Note that for an abelian theory $B$ is gauge invariant. If the gauge group is truly non-abelian the magnetic field transforms as

$$B \mapsto G^{-1} B G$$  \hspace{1cm} (2.8)

under a gauge transformation

$$A \mapsto G^{-1} \left( A + \frac{i}{e} d \right) G.$$  \hspace{1cm} (2.9)
2.1. Singular monopoles

Hence in a non-abelian theory the magnetic field of a Dirac monopole is defined by (2.7) up to gauge transformations.

From equation (2.7) we find for the field strength

\[
F = * \left( \frac{G_0}{4\pi r^2} dr \right) = \frac{G_0}{4\pi} \sin \theta d\theta \wedge d\phi. \tag{2.10}
\]

We shall check that this satisfies the equations motion (2.3) except at the origin where the Bianchi identity is violated. Note that since the field strength transforms in the adjoint representation of the gauge group, its covariant derivatives contain a commutator term with the gauge field. However, there is a gauge in which (2.7) is satisfied and in which the gauge field commutes with the field strength so that effectively the equations of motion reduce to the abelian case where the covariant derivatives of the field strength become ordinary derivatives. If the electric field vanishes, so that \( F = *B \), the equations of motion simplify to:

\[
\begin{align*}
\text{dB} &= 0 \iff \epsilon^{ijk} \partial_j B_k = 0 \\
\text{d} * B &= 0 \iff \partial_i B_i = 0. \tag{2.11}
\end{align*}
\]

While the curl of the magnetic field given in (2.7) obviously vanishes everywhere the divergence vanishes only away from the origin. As a matter fact one finds

\[
\partial_i B_i = G_0 \delta^{(3)}(r). \tag{2.12}
\]

From Gauss’ theorem we now see that the magnetic charge of the monopole equals \( G_0 \). Finally, making a comparison with (2.11) one finds that a monopole with non-vanishing charge \( G_0 \) violates the Bianchi identity at the origin. In that sense the Dirac monopole is singular at the origin.

Another way to view the singularity of the Dirac monopole is to consider the gauge field itself. One possible solution for the gauge field that gives rise to equation 2.7 is given by

\[
A_+ = \frac{G_0}{4\pi} (1 - \cos \theta) d\varphi. \tag{2.13}
\]

On the negative z-axis (including the origin) where \( d\varphi \) diverges \( A_+ \) is singular. This Dirac string, however, is merely a gauge artifact as can be seen by adopting the Wu-Yang formalism [43]. One can introduce a second gauge potential

\[
A_- = -\frac{G_0}{4\pi} (1 + \cos \theta) d\varphi \tag{2.14}
\]

which also gives rise to 2.7 and which is well defined everywhere on \( \mathbb{R}_3 \) except for the positive z-axis and the origin. One could also construct other gauges where the Dirac
string does not coincide with the positive or negative $z$-axis. Nonetheless, in every gauge there is a singularity at the origin $O$ for non-vanishing values of $G_0$. The two gauge potentials $A_+$ and $A_-$ thus give a complete description.

In the region where they are both well-defined $A_+$ and $A_-$ are related by a gauge transformation:

$$A_- = G^{-1}(\varphi) \left( A_+ + \frac{i}{e} d \right) G(\varphi).$$  \hspace{1cm} (2.15)

One can check

$$G(\varphi) = \exp \left( \frac{ie}{2\pi} G_0 \varphi \right).$$  \hspace{1cm} (2.16)

We thus see that a singular monopole in $\mathbb{R}^3$ with non-vanishing magnetic charge defines a non-trivial $H$-bundle on $\mathbb{R}^3 \setminus \{O\}$ and hence a non-trivial bundle on each sphere centred at the origin. We shall discuss this further in section 2.3. Nonetheless, we already note that the non-triviality of the $H$-bundle is closely related to the violation of the Bianchi identity at the origin. In the bundle description one quite literally excises the origin from $\mathbb{R}^3$. One might therefore be tempted to say that such monopoles cannot exist. On the other hand one can simply accept that the magnetic field has certain prescribed singularities.

Still, in some sense singular monopoles seem avoidable if one restricts the fields to be smooth everywhere. This restriction does not rule out the possibility of having classical monopole solutions. If a Higgs field is present in the theory it is also possible to have soliton like monopoles, see e.g. [6, 7]. Such monopoles satisfy the equations of motion, including the Bianchi identity, everywhere on $\mathbb{R}^3$. Since $\mathbb{R}^3$ is contractible a smooth monopole is related to a trivial bundle. Nonetheless, these smooth monopoles behave asymptotically as Dirac monopoles. In section 2.3 we shall explain this relation between singular and smooth monopoles in further detail.

## 2.2 BPS monopoles

A very special subtype of monopoles are BPS monopoles which by definition satisfy the BPS equation discussed below. Examples of smooth solutions of the BPS equation for $SU(2)$ are the (BPS) ‘t Hooft-Polyakov monopoles [6, 7, 3, 4]. Precisely these monopoles have been conjectured by Montonen and Olive to correspond to the heavy gauge bosons of the S-dual gauge theory [5]. Though we shall mainly focus on smooth monopoles in this thesis it should be noted that for singular monopoles only BPS monopoles have been shown to transform as representations of the dual gauge group by Kapustin and Witten [18]. This motivates why also for smooth monopoles one should work in the BPS limit to obtain some insight in for example the fusion rules monopoles.
2.2. BPS monopoles

Instead of giving a detailed description of BPS solutions we shall merely try to give an idea of the general context by introducing the BPS limit and by sketching the derivation of the BPS equations and the BPS mass formula [3, 4]. In section 2.3 we shall come back to the asymptotic behaviour of smooth BPS monopoles.

In general smooth monopoles exist in certain Yang-Mills-Higgs theories. Special cases are Grand Unified theories or Yang-Mills-Higgs theories embedded in a larger theory with extra fermionic fields such as for example a super Yang-Mills theory. The Lagrangian for the Yang-Mills-Higgs theory can be written as:

\[ \mathcal{L} = -\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2} \text{Tr}(D_\mu \Phi D^\mu \Phi) - V(\Phi). \] (2.17)

Unless stated otherwise we shall take \( V(\Phi) \) to be the Mexican hat potential given by

\[ V(\Phi) = \frac{\lambda}{4} (|\Phi|^2 - |\Phi_0|^2)^2. \] (2.18)

The energy functional for the Yang-Mills-Higgs theory for this theory is given by:

\[ E[\Phi, A] = \frac{1}{2} |D_0 \Phi|^2 + \frac{1}{2} |D_k \Phi|^2 + \frac{1}{2} |B_k|^2 + \frac{1}{2} |E_k|^2 + V(\Phi) d^3x. \] (2.19)

To get the Bogomolny equations one should restrict the Higgs field \( \Phi \) to transform in the adjoint representation. One can now rewrite the total energy as [3, 44]:

\[ E[\Phi, A] = |\Phi_0| (Q_e \sin \alpha + Q_m \cos \alpha) + \frac{1}{2} |D_0 \Phi|^2 + \frac{1}{2} |B_k - \cos \alpha D_k \Phi|^2 + \frac{1}{2} |E_k - \sin \alpha D_k \Phi|^2 + V(\Phi) d^3x. \] (2.20)

where \( q_e \) and \( q_m \) are the so-called total abelian electric and magnetic charge defined by

\[ Q_e = \frac{1}{|\Phi_0|} \int_{S^2_\infty} dS_i \text{Tr}(E_i \Phi) \] (2.21)

\[ Q_m = \frac{1}{|\Phi_0|} \int_{S^2_\infty} dS_i \text{Tr}(B_i \Phi). \] (2.22)

If we now take the BPS-limit by letting \( \lambda \to 0 \) while keeping \( \Phi_0 \) fixed and set

\[ \sin \alpha = \frac{Q_e}{(Q_e^2 + Q_m^2)^{1/2}} \quad \text{and} \quad \cos \alpha = \frac{Q_m}{(Q_e^2 + Q_m^2)^{1/2}}, \] (2.23)

we find from (2.20) the following inequality for the energy:

\[ E \geq |\Phi_0| (Q_e \sin \alpha + Q_m \cos \alpha) = |\Phi_0| (Q_e^2 + Q_m^2)^{1/2} = |\Phi_0||Q_e + iQ_m|. \] (2.24)

This lower bound for the energy is known as the Bogolmolny bound and is satisfied when the fields satisfy the following field equations:

\[ B_i = \cos \alpha D_i \Phi \]

---

13
\begin{equation}
E_i = \sin \alpha D_i \Phi \tag{2.25}
\end{equation}

\begin{equation}
D_0 \Phi = 0.
\end{equation}

The BPS bound is very natural in supersymmetric Yang-Mills theories in the sense that it is satisfied if the gauge group is broken but the supersymmetry remains unbroken.

In the special case that the electric charge vanishes, i.e. \( Q_e = 0 \), and all fields are static these three equations reduce to the Bogomolny or BPS equation:

\begin{equation}
B_i = D_i \Phi. \tag{2.26}
\end{equation}

A solution to this BPS equation is called a BPS monopole. In general a solution of the equations of motion satisfying the Bogomolny bound is called a BPS dyon. As for ordinary particles the energy of a BPS monopole or dyon is bounded from below by its rest mass. Therefore the right hand side of equation (2.24) is called the BPS mass formula. To obtain a more profound understanding of the BPS limit it is very convenient to re-express the BPS formula as

\begin{equation}
M = |\Phi_0 \cdot \left( e \lambda + \frac{4 \pi i}{e} g \right)|. \tag{2.27}
\end{equation}

The quantities \((\lambda)_{i=1,\ldots,r}\) and \((g)_{i=1,\ldots,r}\) are the electric charge and the magnetic charge. To determine the allowed values of the electric charge \(\lambda\) is somewhat delicate, see [45] and references therein. There exist classical solutions for every value of the electric charge but in the semi-classical theory the electric charge must be quantised. Without going into details we note that in a gauge theory with gauge group \(G\) it is heuristically clear that \(\lambda\) takes value in the weight lattice of \(G\) and that this is at least consistent with the fact that the BPS mass formula reproduces the mass of the massive gauge bosons with charge \(\alpha\) equal to a root of \(G\), see e.g. [46]:

\begin{equation}
M_{\alpha} = e |\Phi_0 \cdot \alpha|. \tag{2.28}
\end{equation}

The magnetic charge is also quantised, we shall discuss this in much more detail in section 2.3.

An interesting modification of the theory is obtained by turning on the \(\theta\) parameter. This means that one adds to the Lagrangian the term:

\begin{equation}
- \frac{\theta e^2}{32 \pi^2} \int \text{Tr}(F \star F). \tag{2.29}
\end{equation}

By introducing the complex coupling parameter \(\tau\) as

\begin{equation}
\tau = \frac{\theta}{2 \pi} + \frac{4 \pi i}{e^2} \tag{2.30}
\end{equation}
the total Lagrangian can now be conveniently rewritten in a commonly used form as:

\[ \mathcal{L} = -\frac{e^2}{32\pi} \text{Im} \left[ \tau \text{Tr}(F_{\mu\nu} + i \star F_{\mu\nu})(F^{\mu\nu} + i \star F^{\mu\nu}) \right] + \frac{1}{2} \text{Tr}(D_\mu D^\mu \Phi) - V(\Phi). \]  

(2.31)

The additional term does not change the equations of motion since (2.29) can be written as a total derivative, see e.g. section 23.5 of [47]. Even though the classical physics is unchanged by turning on the \( \theta \)-parameter, the quantum theory is affected in a subtle way via instanton effects. As shown by Witten [45] these instanton effects give rise to non-integral abelian electric charges in the sense that

\[ |\Phi_0|Q_e = e\Phi_0 \cdot \lambda + \frac{\theta e^2}{8\pi^2} |\Phi_0|Q_m, \]  

(2.32)

with \( \lambda \) taking value on the weight lattice of \( G \). This shift in the abelian electric charge is called the Witten effect. For an arbitrary value of \( \theta \) the BPS mass formula is given by

\[ M = ||\Phi_0|Q_e + i|\Phi_0|Q_m| = \sqrt{\frac{4\pi}{\text{Im} \tau} |\Phi_0 \cdot (\lambda + \tau g)|}. \]  

(2.33)

In section 4.5 we shall review the invariance under S-duality transformations of this BPS mass formula for dyons in a gauge theory with arbitrary gauge group.

### 2.3 Magnetic charge lattices

In this section we describe and identify the magnetic charges for several classes of monopoles. We shall start with a review for Dirac monopoles, then continue with smooth monopoles in spontaneously broken theories. Specifically for adjoint symmetry breaking we shall explain how the magnetic charge lattice can be understood in terms of the Langlands or GNO dual group of either the full gauge group or the residual gauge group. This will finally culminate in a thorough description of the set of magnetic charges for smooth BPS monopoles.

Dirac monopoles can be described as solutions of the Yang-Mills equations with the property that they are time independent and rotationally invariant. More importantly they are singular at a point as discussed in section 2.1. As a direct generalisation of the Wu-Yang description of \( U(1) \) monopoles [43], singular monopoles in Yang-Mills theory with gauge group \( H \) correspond to a connection on an \( H \)-bundle on a sphere surrounding the singularity. The \( H \)-bundle may be topologically non-trivial, but in addition the monopole connection equips the bundle with a holomorphic structure. The classification of monopoles in terms of their magnetic charge then becomes equivalent to Grothendieck’s classification of \( H \)-bundles on \( \mathbb{C}P^1 \). As a result, the magnetic charge has topological and holomorphic components, both of which play an important role in this thesis.
A different class of monopoles is found from smooth static solutions of a Yang-Mills-Higgs theory on $\mathbb{R}^3$ where the gauge group $G$ is broken to a subgroup $H$. Since $\mathbb{R}^3$ is contractible the $G$-bundle is necessarily trivial. Choosing the boundary conditions so that the total energy is finite while the total magnetic charge is nonzero one finds that smooth monopoles behave asymptotically as Dirac monopoles. Since the long range gauge fields correspond to the residual gauge group this gives a non-trivial $H$-bundle at spatial infinity. The charges of smooth monopoles in a theory with $G$ spontaneously broken to $H$ are thus a subset in the magnetic charge lattice of singular monopoles in a theory with gauge group $H$.

Finally one can restrict solutions to the BPS sector where the energy is minimal. Smooth BPS monopoles are solutions of the BPS equations and therefore automatically solutions of the full equations of motion of the Yang-Mills-Higgs theory. Thus the charges of BPS monopoles are in principle a subset of the charges of smooth monopoles. This subset is determined by the so-called Murray condition which we shall introduce below. We shall also define the fundamental Murray cone which is related to the set of magnetic charge sectors.

### 2.3.1 Quantisation condition for singular monopoles

The magnetic charge of a singular monopole is restricted by the generalised Dirac quantisation condition [1, 2]. This consistency condition can be derived from the bundle description [43]. One can work in a gauge where the magnetic field has the form

$$B = \frac{G_0}{4\pi r^2} dr,$$

(2.34)

with $G_0$ an element in the Lie algebra of the gauge group $H$. This magnetic field corresponds to a gauge potential given by:

$$A_\pm = \pm \frac{G_0}{4\pi} (1 \mp \cos \theta) d\varphi.$$  

(2.35)

The indices of the gauge potential refer to the two hemispheres. On the equator where the two patches overlap the gauge potentials are related by a gauge transformation:

$$A_- = G^{-1}(\varphi) \left( A_+ + \frac{i}{e} d\varphi \right) G(\varphi).$$

(2.36)

One can check

$$G(\varphi) = \exp \left( \frac{i e}{2\pi} G_0 \varphi \right).$$

(2.37)

One obtains similar transition functions for associated vector bundles by substituting appropriate matrices representing $G_0$. All such transition functions must be single-valued.
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In the Dirac picture this means that under parallel transport around the equator electrically charged fields should not detect the Dirac string. Consequently we find for each representation the condition:

\[ \mathcal{G}(2\pi) = \exp(i\epsilon G_0) = \mathbb{I}, \quad (2.38) \]

where \( \mathbb{I} \) is the unit matrix. To cast this condition in slightly more familiar form we note that there is a gauge transformation that maps the magnetic field and hence also \( G_0 \) to a Cartan subalgebra (CSA) of \( H \). Thus without loss of generality we can take \( G_0 \) to be a linear combination of the generators \( (H_a) \) of the CSA in the Cartan-Weyl basis:

\[ G_0 = \frac{4\pi}{e} \sum a g_a \cdot H_a \equiv \frac{4\pi}{e} g \cdot H. \quad (2.39) \]

The generalised Dirac quantisation condition can now be formulated as follows:

\[ 2\lambda \cdot g \in \mathbb{Z}, \quad (2.40) \]

for all charges \( \lambda \) in the weight lattice \( \Lambda(H) \) of \( H \).

We thus see that the magnetic weight lattice \( \Lambda^*(H) \) defined by the Dirac quantisation condition is dual to the electric weight lattice \( \Lambda(H) \). Consider for example the case where \( H \) is semi-simple as well as simply connected so that the weight lattice \( \Lambda(H) \) is generated by the fundamental weights \( \{\lambda_i\} \). Then \( \Lambda^*(H) \) is generated by the simple coroots \( \{\alpha^*_i = \alpha_i/\alpha^2_i\} \) which satisfy:

\[ 2\alpha^*_i \cdot \lambda_j = \frac{2\alpha_i \cdot \lambda_j}{\alpha^2_i} = \delta_{ij}. \quad (2.41) \]

As observed by Englert and Windey and Goddard, Nuyts and Olive, the magnetic weight lattice can be identified with the weight lattice of the GNO dual group \( H^* \). For example if we take \( H = SU(n) \) and define the roots of \( SU(n) \) such that \( \alpha^2 = 1 \), we see that \( \Lambda^*(SU(n)) \) corresponds to the root lattice of \( SU(n) \). The root lattice of \( SU(n) \) on the other hand is precisely the weight lattice of \( SU(n)/\mathbb{Z}_n \). In the general simple case \( \Lambda^*(H) \) resulting from the Dirac quantisation condition is the weight lattice \( \Lambda(H^*) \) of the GNO dual group \( H^* \) whose weight lattice is the dual weight lattice of \( H \) and whose roots are identified with the coroots of \( H \) [1, 2]. In addition the center and the fundamental group of \( H^* \) are isomorphic to respectively the fundamental group and the center of \( H \). Note that for all practical purposes the root system of \( H^* \) can be identified with the root system of \( H \) where the long and short roots are interchanged.

We shall not repeat the proof of the duality of the center and the fundamental group, but we will sketch the proof of the fact that the root lattice of \( H^* \) is always contained in the magnetic weight lattice. Finally we sketch the generalisation to any connected compact Lie group.
Chapter 2. Classical monopole solutions

If \( H \) is not simply connected we have \( H = \tilde{H}/Z \) where \( \tilde{H} \) is the universal cover of \( H \) and \( Z \subset Z(\tilde{H}) \) a subgroup in the center of \( \tilde{H} \). Since \( \Lambda(H) \subset \Lambda(\tilde{H}) \) with \( Z = \Lambda(\tilde{H})/\Lambda(H) \) the Dirac quantisation condition (2.40) applied on \( H \) is less restrictive than the condition for \( \tilde{H} \). Moreover, one can check [2]:

\[
\Lambda^*(H)/\Lambda^*(\tilde{H}) = \Lambda(\tilde{H})/\Lambda(H).
\] (2.42)

This implies that the coroot lattice \( \Lambda^*(\tilde{H}) \) of \( H \) is always contained in the magnetic weight lattice \( \Lambda^*(H) \) of \( H \) and in particular that any coroot \( \alpha^* = \alpha/\alpha^2 \) with \( \alpha \) a root \( H \), is contained in \( \Lambda^*(H) \).

Without much effort this property can be shown to hold for any compact, connected Lie group. Any such group \( H \) say of rank \( r \) can be expressed as:

\[
H = \frac{U(1)^s \times K}{Z},
\] (2.43)

where \( K \) is a semi-simple, simply connected Lie group of rank \( r - s \) and \( Z \) some finite group. The CSA of \( H \) is spanned by \( \{H_a : a = 1, \ldots, r\} \) where \( H_a \) with \( a \leq s \) are the generators of the \( U(1) \) subgroups and \( \{H_b : s < b \leq r\} \) span the CSA of \( K \). Any weight of \( H \) can be expressed as \( \lambda = (\lambda_1, \lambda_2) \) where \( \lambda_1 \) is a weight of \( U(1)^s \) and \( \lambda_2 \) is a weight of \( K \). Finally one finds that a magnetic charge \( G_0 \) defined by

\[
G_0 = \frac{4\pi}{e} \alpha_j^* \cdot H,
\] (2.44)

where \( \alpha_j \) is any of the \( r - s \) simple roots of \( H \), satisfies the quantisation condition.

<table>
<thead>
<tr>
<th>( H )</th>
<th>( H^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SU(nm)/\mathbb{Z}_m )</td>
<td>( SU(nm)/\mathbb{Z}_n )</td>
</tr>
<tr>
<td>( Sp(2n) )</td>
<td>( SO(2n + 1) )</td>
</tr>
<tr>
<td>( Spin(2n + 1) )</td>
<td>( Sp(2n)/\mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( Spin(4n + 2) )</td>
<td>( SO(4n + 2)/\mathbb{Z}_2 )</td>
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<td>( SO(4n + 2) )</td>
<td>( SO(4n + 2) )</td>
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<tr>
<td>( G_2 )</td>
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<td>( F_4 )</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( E_6/\mathbb{Z}_3 )</td>
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<tr>
<td>( E_7 )</td>
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<td>( E_8 )</td>
<td>( E_8 )</td>
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Table 2.1: Langlands or GNO dual pairs for simple Lie groups.

In this section we have identified the magnetic charge lattice of singular monopoles with
2.3. Magnetic charge lattices

<table>
<thead>
<tr>
<th>$H$</th>
<th>$H^*$</th>
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<tbody>
<tr>
<td>$(U(1) \times SU(n))/\mathbb{Z}_n$</td>
<td>$(U(1) \times SU(n))/\mathbb{Z}_n$</td>
</tr>
<tr>
<td>$U(1) \times Sp(2n)$</td>
<td>$U(1) \times SO(2n + 1)$</td>
</tr>
<tr>
<td>$(U(1) \times Spin(2n + 1))/\mathbb{Z}_2$</td>
<td>$(U(1) \times Sp(2n))/\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$(U(1) \times Spin(2n))/\mathbb{Z}_2$</td>
<td>$(U(1) \times SO(2n))/\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

Table 2.2: Examples of Langlands or GNO dual pairs for some compact Lie groups.

the weight lattice of the dual group $H^*$ of the gauge group $H$. In table 2.1 and 2.2 some examples are given of GNO dual pairs of Lie groups. Table 2.1 is complete up to some dual pairs related to $Spin(4n)$ that are obtained by modding out non-diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroups of the center $\mathbb{Z}_2 \times \mathbb{Z}_2$. The GNO dual groups for these cases can be found in [2]. In section 2.3.3 we shall briefly explain how the dual pairing in table 2.2 is determined.

The magnetic charge lattice contains an important subset which we shall need later on: even if one restricts $G_0$ to the CSA there is some gauge freedom left which corresponds to the action of the Weyl group. Modding out this Weyl action gives a set of equivalence classes of magnetic charges which are naturally labelled by dominant integral weights in the weight lattice of $H^*$.

2.3.2 Quantisation condition for smooth monopoles

Yang-Mills-Higgs theories have solutions that behave at spatial infinity as singular Dirac monopoles but which are nonetheless completely smooth at the origin. This is possible if one starts out with a compact, connected, semi-simple gauge group $G$ which is spontaneously broken to a subgroup $H$. Since all the fields are smooth, the gauge field defines a connection of a principal $G$-bundle over space which we take to be $\mathbb{R}^3$. The Higgs field is a section of the associated adjoint bundle. As $\mathbb{R}^3$ is contractible the principal $G$-bundle is automatically trivial, so $\Phi$ is simply a Lie-algebra valued function. We would like to impose boundary conditions for the Higgs field $\Phi$ and the magnetic field $B$ at spatial infinity which ensure that the total energy carried by a solution of the Yang-Mills-Higgs equations is finite. To our knowledge the question of which conditions are necessary and sufficient has not been answered in general. Below we review some standard arguments, many of them summarised in [48].

We assume an energy functional for static fields of the usual form

$$E[\Phi, A] = \int \frac{1}{2} |D_k \Phi|^2 + \frac{1}{2} |B_k|^2 + V(\Phi) \, d^3 x,$$

(2.45)
where \( D_k = \partial_k - ieA_k \) is the covariant derivative with respect to the \( G \)-connection \( A \), and the magnetic field is given by \(-ieB_k = -\frac{1}{2}ie\epsilon_{klm}F_{lm} = \frac{1}{2}\epsilon_{klm}[D_l,D_m] \). The potential \( V \) is a \( G \)-invariant function on the Lie algebra of \( G \) whose minimum is attained for non-vanishing value of \(|\Phi|\); the set of minima is called the vacuum manifold. The variational equations for this functional are

\[
\epsilon_{klm}D_lB_m = ie[\Phi,D_k\Phi], \quad D_kD_k\Phi = \frac{\partial V}{\partial \Phi}. \tag{2.46}
\]

In order to ensure that solutions of these equations have finite energy we require the fields \( \Phi \) and \( B \) to have the following asymptotic form for large \( r \):

\[
\Phi = \phi(\hat{r}) + \frac{f(\hat{r})}{4\pi r} + \mathcal{O}\left(r^{-1+\delta}\right) \quad r \gg 1
\]

\[
B = \frac{G(\hat{r})}{4\pi r^2} dr + \mathcal{O}\left(r^{-2+\delta}\right) \quad r \gg 1. \tag{2.47}
\]

Here \( \delta > 0 \) is some constant and \( \phi(\hat{r}), f(\hat{r}), \) and \( G(\hat{r}) \) are smooth functions on \( S^2 \) taking values in the Lie algebra of the gauge group \( G \) which have to satisfy various conditions.

First of all, the function \( \phi \) has to take values in the vacuum manifold of the potential \( V \). It is thus a smooth map from the two-sphere to that vacuum manifold. The homotopy class of that map defines the monopole’s topological charge \([48]\). Since the vacuum manifold can be identified with the coset space \( G/H \) the topological charge takes value in \( \pi_2(G/H) \). Secondly, writing \( \nabla \) for the induced exterior covariant derivative tangent to the two-sphere “at infinity” it is easy to check that

\[
\nabla\phi = 0, \quad \nabla f = 0 \tag{2.48}
\]

are necessary conditions for the integral defining the energy (2.45) to converge. The first of these equations implies

\[
[\phi(\hat{r}), G(\hat{r})] = 0. \tag{2.49}
\]

The quickest way to see this is to note that the curvature on the two-sphere at infinity is

\[
F^\infty = * \left( \frac{G(\hat{r})}{4\pi r^2} dr \right) = \frac{G(\hat{r})}{4\pi} \sin \theta d\theta \wedge d\varphi. \tag{2.50}
\]

Since \( [\nabla, \nabla] = -ieF^\infty \), it follows that \( \nabla\phi = 0 \) implies \( [F^\infty, \phi] = 0 \). Finally we also require that

\[
\nabla G = 0, \tag{2.51}
\]

and that

\[
[\phi(\hat{r}), f(\hat{r})] = 0. \tag{2.52}
\]

The condition (2.51) is crucial for what follows, and seems to be satisfied for all known finite energy solutions \([48]\). The condition (2.52) is required so that the first of the equations (2.46) is satisfied to lowest order when the expansion (2.47) is inserted. In general...
there will be additional requirements on the functions \( \phi \) and \( f \) that depend on the precise form of the potential \( V \) in (2.45). Since we do not specify \( V \) we will not discuss these further.

The above conditions can be much simplified by changing gauge. The equations (2.48) and (2.51) imply that for each of the Lie-algebra valued functions \( \phi, f \) and \( G \) the values at any two points on the two-sphere at infinity are conjugate to one another (the required conjugating element being the parallel transport along the path connecting the points). We can therefore pick a point \( \hat{r}_0 \), say the north pole, and gauge transform \( \phi \) into \( \Phi_0 = \phi(\hat{r}_0) \), \( f \) into \( \Phi_1 = f(\hat{r}_0) \) and \( G \) into \( G_0 = G(\hat{r}_0) \). However, since \( S^2 \) is not contractible, we will, in general, not be able to do this smoothly everywhere on the two-sphere at infinity. If, instead, we cover the two-sphere with two contractible patches which overlap on the equator, then there are smooth gauge transformations \( g_+ \) and \( g_- \) defined, respectively, on the northern and southern hemisphere, so that the following equations hold where they are defined:

\[
\begin{align*}
\phi(\hat{r}) &= g_+^{-1}(\hat{r}) \Phi_0 g_+ (\hat{r}) \quad \text{(2.53)} \\
f(\hat{r}) &= g_+^{-1}(\hat{r}) \Phi_1 g_+ (\hat{r}) \quad \text{(2.54)} \\
G(\hat{r}) &= g_+^{-1}(\hat{r}) G_0 g_+ (\hat{r}) \quad \text{(2.55)}
\end{align*}
\]

After applying these gauge transformation, our bundle is defined in two patches, with transition function \( G = g_+ g_-^{-1} \) defined near the equator. This transition function leaves \( \Phi_0 \) invariant, and hence lies in the subgroup \( H \) of \( G \) which stabilises \( \Phi_0 \). This, by definition, is the residual or unbroken gauge group referred to in the opening paragraph of this section. It follows from (2.49), that \( [\Phi_0, G_0] = 0 \), so that \( G_0 \) lies in the Lie algebra of \( H \). Similarly, (2.52) implies that \( \Phi_1 \) lies in the Lie algebra of \( H \). After applying the local gauge transformations (2.53), the asymptotic form of the fields is

\[
\begin{align*}
\Phi &= \Phi_0 + \frac{\Phi_1}{4\pi r} + \mathcal{O}(r^{-1+\delta}) \\
B &= \frac{G_0}{4\pi r^2} dr + \mathcal{O}(r^{-2+\delta}). \quad \text{(2.56)}
\end{align*}
\]

Note that “the Higgs field at infinity” is now constant, taking the value \( \Phi_0 \) everywhere. In particular, it therefore belongs to the trivial homotopy class of maps from the two-sphere to the vacuum manifold. The topological charges originally encoded in the map \( \phi \) can no longer be computed from the Higgs field. Instead, they are now encoded in transition function \( G \). Since, in the new gauge, the magnetic field at large \( r \) is that of a Dirac monopole with gauge group \( H \) we can relate the transition function to the magnetic charge as before:

\[
G(\varphi) = \exp \left( \frac{i e}{2\pi} G_0 \varphi \right) \quad \text{(2.57)}
\]
We thus obtain a quantisation condition for the magnetic charge of smooth monopoles, following the same arguments as in the singular case. For each representation of $H$ the gauge transformation must be single-valued if one goes around the equator, so that

$$2\lambda \cdot g \in \mathbb{Z},$$ (2.58)

for all charges $\lambda$ in the weight lattice of $H$.

One observes that the magnetic charge lattice of smooth monopoles lies in the weight lattice of the GNO dual group $H^\ast$. There is, however, another consistency condition [1]. Note that a single-valued gauge transformation on the equator defines a closed curve in $H$ as well as in $G$, starting and ending at the unit element. Since the original $G$-bundle is trivial, this closed curve has to be contractible in $G$. Therefore the monopole’s topological charge is labelled by an element in $\pi_1(H)$ which maps to a trivial element in $\pi_1(G)$. This is consistent with our earlier remark that the topological charge is an element of $\pi_2(G/H)$ because of the isomorphism $\pi_2(G/H) \simeq \text{ker}(\pi_1(H) \to \pi_1(G))$.

To find the appropriate charge lattice we use the fact that a loop in $G$ is trivial if and only if its lift to the universal covering group $\tilde{G}$ is also a loop (closed path). This implies that for smooth monopoles the quantisation condition should not be evaluated in the group $H$ itself but instead in the group $\tilde{H} \subset \tilde{G}$ defined by the Higgs VEV $\Phi_0$. Consequently equation (2.58) must not only hold for all representations of $H$ but in fact for all representations of $\tilde{H}$. Note that if $G$ is simply connected then $\tilde{H} = H$. In the next section we shall work this topological condition out in more detail.

### 2.3.3 Quantisation condition for smooth BPS monopoles

In chapter 3 we will mainly focus on BPS monopoles in spontaneously broken theories. We shall therefore work out some results of the previous section in somewhat more detail for the BPS case. We shall also give an explicit description of the magnetic charge lattice. In addition we introduce terminology that is conveniently used in the remainder of this thesis.

By BPS monopoles we mean static, finite energy solutions of the BPS equations

$$B_i = D_i \Phi$$ (2.59)

in a Yang-Mills-Higgs theory with a compact, connected, semi-simple gauge group $G$. The equations (2.59) imply the second order equations (2.46). In order to obtain finite energy solutions we again impose the boundary conditions (2.47). As in the previous section we can gauge transform these into the form (2.56). There are some differences
2.3. Magnetic charge lattices

with the non-BPS case. The potential $V$ in (2.45) vanishes in the BPS limit, so does not furnish any conditions on the functions $\phi$ and $f$. On the other hand, by substituting (2.56) in the BPS equation and solving order by order one finds that $f = -G$, or, equivalently, $\Phi_1 = -G_0$. As before we have $[\Phi_0, G_0] = 0$, so in the BPS case we automatically have $[\Phi_0, \Phi_1] = 0$. From now on we shall thus define a BPS monopole to be a smooth solution of the BPS equations satisfying the boundary condition (2.47) with $\Phi_1 = -G_0$.

After applying the local gauge transformations discussed in the previous section, these boundary conditions are equivalent to

$$
\Phi = \Phi_0 - \frac{G_0}{4\pi r} + \mathcal{O}\left(r^{-(1+\delta)}\right)
$$

$$
B = \frac{G_0}{4\pi r^2} \hat{r} + \mathcal{O}\left(r^{-(2+\delta)}\right),
$$

where $\Phi_0$ and $G_0$ are commuting elements in the Lie algebra of $G$. These boundary conditions are sufficient to guarantee that the energy of the BPS monopole is finite. It is in general not known what the necessary boundary conditions are to obtain a finite energy configuration. It is expected though [49, 50], and true for $G = SU(2)$ [51], that the boundary conditions above follow from the finite energy condition and the BPS equation.

Before we give an explicit description of the magnetic charge lattice let us summarise some properties of the residual gauge group. Since $[\Phi_0, G_0] = 0$ there is a gauge transformation that maps $\Phi_0$ and $G_0$ to our chosen CSA of $G$. Without loss of generality we can thus express $\Phi_0$ and $G_0$ in terms of the generators $(H_a)$ of that CSA:

$$
\Phi_0 = \mu \cdot H
$$

$$
G_0 = \frac{4\pi}{e^g} \cdot H.
$$

The residual gauge group is generated by generators $L$ in the Lie algebra of $G$ satisfying $[L, \Phi_0] = 0$. Since generators in the CSA by definition commute with the Higgs VEV the residual group $H$ contains at least the maximal torus $U(1)^r \subset G$. For generic values of the Higgs VEV this is the complete residual gauge symmetry. If the Higgs VEV is perpendicular to a root $\alpha$ the residual gauge group becomes non-abelian. This follows from the action of the corresponding ladder operator $E_\alpha$ in the Cartan-Weyl basis on the Higgs VEV: $[E_\alpha, \Phi_0] = -\mu \cdot \alpha E_\alpha = 0$. Accordingly we shall call a root of $G$ broken if it has a non-vanishing inner product with $\mu$ and we shall define it to be unbroken if this inner product vanishes.

The residual gauge group is locally of the form $U(1)^s \times K$, where $K$ is some semi-simple Lie group. The root system of $K$ is derived from the root system of $G$ by removing the broken roots. Similarly, the Dynkin diagram of $K$ is found from the Dynkin diagram of $G$ by removing the nodes related to broken simple roots. For completeness we finally define a fundamental weight to be (un)broken if the corresponding simple root is (un)broken.
The magnetic charge lattice for smooth monopoles lies in the dual weight lattice of $H$, as we saw in the previous chapter. For adjoint symmetry breaking the weight lattice of $H$ is isomorphic to the weight lattice of $G$. Moreover, the isomorphism respects the action of the Weyl group $\mathcal{W}(H) \subset \mathcal{W}(G)$. The existence of an isomorphism between $\Lambda(G)$ and $\Lambda(H)$ is easily understood since the weight lattices of $H$ and $G$ are determined by the irreducible representations of their maximal tori which are isomorphic for adjoint symmetry breaking. A natural choice for the CSA of $H$ is to identify it with the CSA of $G$. In this case $\Lambda(G)$ and $\Lambda(H)$ are not just isomorphic but also isometric. Since the roots of $H$ can be identified with roots of $G$ and since the Weyl group is generated by the reflections in the hyperplanes orthogonal to the roots, this isometry obviously respects the action of $\mathcal{W}(H)$. Often the CSA of $H$ is identified with the CSA of $G$ only up to normalisation factors. This leads to rescalings of the weight lattice of $H$. Of course one can apply an overall rescaling without spoiling the invariance of weight lattice under the Weyl reflections. One can also choose the generators of the $U(1)^s$ factor such that the corresponding charges are either integral or half-integral. Note that these rescalings again respect the action of $\mathcal{W}(H)$. To avoid confusion we shall ignore these possible rescalings in the remainder of this thesis and take $\Lambda(H)$ to be isometric to $\Lambda(G)$.

Since the weight lattices $\Lambda(H)$ and $\Lambda(G)$ are isometric their dual lattices $\Lambda^*(H)$ and $\Lambda^*(G)$ are isometric too. We thus see that the Dirac quantisation condition (2.58) for adjoint symmetry breaking can consistently be evaluated in terms of either $H$ or $G$. Remember that for smooth monopoles there is yet another condition: since one starts out from a trivial $G$ bundle the magnetic charge should define a topologically trivial loop in $G$ as explained in the previous section. For general symmetry breaking this implies that the Dirac quantisation condition must be evaluated with respect to weight lattice of $\tilde{H} \subset \tilde{G}$, where $\tilde{G}$ is the universal covering group of $G$. For adjoint symmetry breaking we can consistently lift the quantisation condition to $\tilde{G}$; the weight lattice of $\tilde{H}$ is isometric to the weight lattice of $\tilde{G}$. The weight lattice of $\tilde{G}$ is generated by the fundamental weights $\{\lambda_i\}$ and hence the magnetic charge lattice for smooth BPS monopoles is given by the solutions of:

$$2\lambda_i \cdot g \in \mathbb{Z},$$

for all fundamental weights $\lambda_i$ of $\tilde{G}$. The most general solution of this equation is easily expressed in terms of the simple coroots of $G$:

$$g = \sum_i m_i \alpha_i^* \quad m_i \in \mathbb{Z},$$

with $\alpha_i^* = \alpha_i/\alpha_2^2$ and $\{\alpha_i\}$ the simple roots of $G$. We thus conclude that the magnetic charge lattice for smooth BPS monopoles is generated by the simple coroots of $G$. The resulting coroot lattice $\Lambda^*(\tilde{G})$ corresponds precisely to the weight lattice $\Lambda(\tilde{G}^*)$ of the GNO dual group $\tilde{G}^*$ as mentioned in section 2.3.1.
Similarly, the dual lattice $\Lambda^*(\widetilde{H})$ can be identified with $\Lambda(\widetilde{H}^*)$. With $\Lambda^*(\widetilde{G})$ being isometric to $\Lambda^*(\widetilde{H})$ we now conclude that the weight lattice of $\widetilde{G}^*$ can be identified with the weight lattice of $\widetilde{H}^*$. For $G$ simply connected we have thus established an isometry between the root lattice of $G^*$ and the weight lattice of $H^*$. We have used this isometry to compute the GNO dual pairs given in table 2.2 which appear in the minimal adjoint symmetry breaking of the classical Lie groups.

Above we have seen that the magnetic charge lattice for smooth BPS monopoles corresponds to the coroot lattice of the gauge group $G$. One can split the set of coroots into broken coroots and unbroken coroots. A coroot is defined to be broken or unbroken if the corresponding root is respectively broken or unbroken. Note that the unbroken coroots are precisely the roots of $H^*$. The distinction between broken and unbroken applies in particular to simple coroots. There is, however, alternative terminology for the components of the magnetic charges that reflects these same properties. Broken simple coroots are identified with topological charges while unbroken simple coroots are related to so-called holomorphic charges.

Remember that the magnetic charge $g = m_i \alpha_i^*$ defines an element in $\ker(\pi_1(H) \to \pi_1(G))$. One might hope that every single magnetic charge $g$, i.e. every point in the coroot lattice, defines a unique topological charge. If in that case a static monopole solution does indeed exist even its stability under smooth deformations is guaranteed. Such a picture does hold for maximally broken theories where the residual gauge group equals the maximal torus $U(1)^r \subset G$. If $H$ contains a non-abelian factor the situation is slightly more complicated because these factors are not detected by the fundamental group. For $G$ equal to $SU(3)$ for instance the magnetic charge lattice is 2-dimensional and $\pi_1(SU(3)) = 0$. In the maximally broken theory we have $\pi_1(U(1) \times U(1)) = \mathbb{Z} \times \mathbb{Z}$, while for minimal symmetry breaking $\pi_1(U(2)) = \pi_1(U(1)) = \mathbb{Z}$. As a rule of thumb one can say that the components of the magnetic charges related to the $U(1)$-factors in $H$ are topological charges. It should be clear that these components correspond to the broken simple coroots. We therefore call the coefficients $m_i = 2\lambda_i \cdot g$ with $\lambda_i$ a broken fundamental weight the topological charges of $g$. The remaining components of $g$ are often called holomorphic charges.

### 2.3.4 Murray condition

We have found that magnetic charges of smooth monopoles in a Yang-Mills-Higgs theory lie on the coroot lattice of the gauge group. In the BPS limit there is yet another consistency condition which was first discovered by Murray for $SU(n)$ [52]. We refer to this condition as the Murray condition even though its final formulation for general gauge groups stems from a paper by Murray and Singer [50]. For a derivation of the Murray condition we refer to these original papers. We shall only briefly review some properties
Chapter 2. Classical monopole solutions

of roots which are crucial for the Murray condition. Next we shall formulate the results of Murray and Singer in such a way that the set of magnetic charges for BPS monopoles can easily be identified. Finally we show that our formulation is equivalent to the condition as stated in [50]. Both formulations of the Murray condition will show up in later sections. The set of magnetic charges satisfying the Murray condition shall be called the Murray cone. At the end of this section we shall also introduce the fundamental Murray cone.

The Murray condition hinges on the fact that one can split the root system of $G$ into positive and negative roots with respect to the Higgs VEV. If for a root $\alpha$ we have $\alpha \cdot \mu > 0$ it is by definition positive and if $\alpha \cdot \mu < 0$ it is negative. The set of roots is now partitioned into two mutually exclusive sets, at least if the residual gauge group is abelian. In that case we can as usual define a simple root to be a positive root that cannot be expressed as a sum of two other positive roots and it turns out that the Higgs VEV defines a unique set of simple roots. These form a basis of the root diagram is such a way that every positive root is a linear combination of simple roots with positive coefficients and similarly every negative root is a linear combination with negative coefficients. In the non-abelian case there exist roots such that $\alpha \cdot \mu = 0$. Hence there are several choices for a set of simple roots which are consistent with the Higgs VEV. Again for a fixed choice such simple roots must by definition have the property that all roots are a linear combination of simple roots with either only positive or only negative coefficients. In addition the simple roots must have either a strictly positive or a vanishing inner product with the Higgs VEV:

$$\alpha_i \cdot \mu \geq 0.$$  \hfill (2.64)

This condition implies that $\mu$ must lie in the closure of the fundamental Weyl chamber. In the remainder of this thesis we shall always choose simple roots so that the inequality in (2.64) is satisfied.

All choices for a set of simple roots respecting the Higgs VEV are related by the residual Weyl group $W(H)$. This is seen as follows. In general all choices of simple roots in the root system of $G$ are related by the Weyl group $W(G)$ of $G$. Since Weyl transformations are orthogonal we have for all $w \in W(G)$, $w(\alpha_i) \cdot \mu = \alpha_i \cdot w^{-1}(\mu)$. Given a set of positive roots satisfying (2.64) the action of $w \in W(G)$ gives another set of simple roots satisfying the same condition if and only if $\mu$ and $w(\mu)$ lie in the closure of same Weyl chamber. This is only possible if $\mu$ is actually invariant under $w$, implying that $w \in W(H) \subset W(G)$.

Above we have defined a positivity condition for the roots of $G$ that is consistent with the Higgs VEV. This same definition is applicable for coroots since these differ from the roots by a scaling. We now also extend this definition of positivity in a consistent way to the complete (co)root lattice. We call an element on the (co)root lattice positive if it is a linear combination of simple (co)roots with positive integer coefficients. Note that the intersection of the set of positive elements in the (co)root lattice with the set of (co)roots
2.3. Magnetic charge lattices

is precisely the set of positive (co)roots. Finally we see that if the Higgs VEV lies in the fundamental Weyl chamber then the inner product of any positive element in the (co)root lattice with $\mu$ is non-negative.

Murray and Singer have found that the magnetic charge must be positive with respect to all possible choices of simple roots consistent with the Higgs VEV. This means that in the expansion $g = \sum_i m_i \alpha_i^*$ the coefficients $m_i$ should be positive for all possible choices of simple roots $(\alpha_i)$ that satisfy $\alpha_i \cdot \mu \geq 0$. The Murray condition can be summarised as follows:

$$2w(\lambda_i) \cdot g \geq 0 \quad \forall w \in \mathcal{W}(H), \forall \lambda_i. \quad (2.65)$$

This is seen from the fact that the fundamental weights and simple roots satisfy $2\lambda_i \cdot \alpha_j^* = \delta_{ij}$ and that all allowed choices of positive simple roots and fundamental weights are related by the residual Weyl group $\mathcal{W}(H) \subset \mathcal{W}(G)$.

The Murray condition defines a solid cone in the CSA. In combination with the Dirac quantisation condition this results in a discrete cone of magnetic charges. We shall call this cone the Murray cone. As an example one can consider $SU(3)$ broken to either $U(1) \times U(1)$ or $U(2)$ as depicted in figure 2.1. In the first case the Weyl group of the residual gauge group is trivial and the Murray condition simply implies that the topological charges must be positive. In the second case the residual Weyl group is $\mathbb{Z}_2$, the reflections in the line perpendicular to $\alpha_1$. Consequently there are two possible choices of positive simple roots which makes the Murray condition more restrictive. The topological charge still has to be positive, just like the holomorphic charge, but the holomorphic charge is bounded by the topological charge.

We shall finish this section with yet another formulation of the Murray condition originating from proposition 4.1 in the paper of Murray and Singer [50]. It relies on the fact that the holomorphic charges can be minimised under the action of the residual Weyl group. For any element $g$ in the coroot lattice there exists a uniquely determined reduced magnetic charge $\tilde{g}$ in the Weyl orbit of $g$ such that $\alpha_j \cdot \tilde{g} \leq 0$ for all unbroken simple roots $\alpha_j$. The Murray condition can be expressed in terms of this minimised charge. A magnetic charge $g$ is positive with respect to any chosen set of simple roots if and only if for a fixed choice of simple roots its reduced magnetic charge is positive. The reduced magnetic charge should thus satisfy:

$$2\lambda_i \cdot \tilde{g} \geq 0 \quad \forall \lambda_i. \quad (2.66)$$

We shall shortly show that $\tilde{g}$ does indeed exist and is unique. But already we can see that this last condition easily follows from (2.65). Since $\tilde{g} = \tilde{w}(g)$ for some $\tilde{w} \in \mathcal{W}(H)$ we have $w(\lambda_i) \cdot \tilde{g} = w(\lambda_i) \cdot \tilde{w}(g) = \tilde{w}^{-1} (w(\lambda_i)) \cdot g = w'(\lambda_i) \cdot g \geq 0$, where $w' = \tilde{w}^{-1} w \in \mathcal{W}(H)$. To show equivalence, however, we also have to show that (2.65) follows from (2.66), which boils down to proving the following proposition:
Proposition 2.1 If the reduced magnetic charge $\tilde{g}$ is positive then $w(\tilde{g})$ is positive for all $w \in W(H)$.

Proof. We take the gauge group $G$ broken to $H$. The magnetic charges of BPS monopoles lie on the coroot lattice of $G$ or equivalently the root lattice of $G^*$. We can assume $G$ to be simply connected since this does not affect the magnetic charge lattice. Under this assumption there is an isomorphism $\lambda$ from the coroot lattice $\Lambda^*(G)$ to the weight lattice $\Lambda(H^*)$ of $H^*$ as discussed in section 2.3.3. Up to discrete factors $H^*$ is of the form $U(1)^s \times K^*$, where $K^*$ is some semi-simple Lie group. Similarly, the set of simple roots of $G$ is split up into $s$ broken roots $\{\alpha_i\}$ with $0 < i \leq s$ and $r - s$ unbroken roots $\{\alpha_j\}$ with $s < j \leq r$. The magnetic charges are thus expanded as $g = \sum_i m_i \alpha_i^* + \sum_j h_j \alpha_j^*$. The linear map $\lambda$ is defined by the images of the simple coroots. For the unbroken simple coroots this is particularly simple. We have $\lambda(\alpha_j^*) = \alpha_j^*$. More generally the image is given in terms of the abelian charges and a weight of $K^*$. While the abelian charges are identified with the topological charges $\{m_i\}$ the non-abelian charge can be expanded in terms of the fundamental weights $\lambda_j$ of $K^*$. The coefficients, i.e. the Dynkin labels, are given by the projection on the roots of $K^*$: $k_j = 2 \alpha_j^* \cdot g/\alpha_j^*$. Being sums of multiples of the entries of the Cartan matrix of $G^*$ these labels are indeed integers.

Figure 2.1: The Murray cone for $SU(3)$ as a subset of the Cartan subalgebra. If the residual gauge group equals $U(1) \times U(1)$ (left) the Higgs VEV determines a unique set of simple roots. The static BPS monopoles have magnetic charges equal to a positive linear combination of these roots. These charges are in one-to-one correspondence with the positive topological charges. If the residual gauge group is $U(2)$ (right) there are two choices of simple roots. Only those charges that have a positive expansion for both these choices correspond to non-empty moduli spaces of static BPS monopoles. There is only a single topological charge which is proportional to the inner product of the magnetic charge with the Higgs VEV $\mu$. As can be seen from the picture the total magnetic charge is not uniquely determined by the topological charge alone: non-abelian monopoles may carry non-trivial holomorphic charges.
We can now easily prove that the reduced magnetic charge $\tilde{g}$ exists and is unique. Let $h := \lambda(g)$. Any weight $h \in \Lambda(H^*)$ can be mapped to a unique weight $\tilde{h}$ in the anti-fundamental Weyl chamber via a Weyl transformation. We thus have $\tilde{h} \cdot \alpha \leq 0$. The reduced magnetic charge $\tilde{g}$ is fixed by $\lambda(\tilde{g}) = \tilde{h}$. Since $2\lambda(g) \cdot \alpha_j^* / \alpha_j^{*2} = 2g \cdot \alpha_j^* / \alpha_j^{*2}$ we have $\alpha_j^* \cdot g \leq 0$ for all unbroken roots of $G^*$. The same inequality holds for the unbroken roots of $G$ itself.

We now return to the proof of the proposition. First we shall use the fact that $\lambda$ respects the residual Weyl group is the sense that $\lambda(w(g)) = w(\lambda(g))$ for all $w \in W(H)$. This can be proved using the fact that any Weyl transformation is a sequence of Weyl reflections $w_j$ in the hyperplanes perpendicular to the simple coroots $\alpha_j^*$. It is thus sufficient to prove that $\lambda$ commutes with $w_j$ for all unbroken simple roots. We have

$$\lambda(w_j(g)) = \lambda(g - \frac{2g \cdot \alpha_j^*}{\alpha_j^{*2} \cdot \alpha_j^*}) = \lambda(g) - \frac{2g \cdot \alpha_j^*}{\alpha_j^{*2} \cdot \alpha_j^*} \lambda(\alpha_j^*) = \lambda(g) - \frac{2\lambda(g) \cdot \alpha_j^*}{\alpha_j^{*2} \cdot \alpha_j^*} = w_j(\lambda(g)). \quad (2.67)$$

Note that for the unbroken roots $\lambda(\alpha_j) = \alpha_j$ and that $\lambda$ is an isometry as discussed in section 2.3.3 and thus leaves the inner product invariant.

Secondly for the proof of the proposition we use the fact that for a lowest weight $\tilde{h}$ we have $w(\tilde{h}) = \tilde{h} + n_j \alpha_j^*$ with $n_j \geq 0$ for any $w \in W(H^*)$, see for example chapter 10 to 13 of [53]. For $\tilde{g}$ and any $w \in W(H^*) = W(H)$ we now get:

$$\lambda(w(\tilde{g})) = \lambda(w(\tilde{g})) = w(\tilde{h}) = \tilde{h} + n_j \alpha_j^* = \lambda(\tilde{g}) + n_j \lambda(\alpha_j^*) = \lambda(\tilde{g} + n_j \alpha_j^*). \quad (2.68)$$

Consequently in terms of the unbroken simple coroots of $G$ we find $w(\tilde{g}) = \tilde{g} + n_j \alpha_j^*$ where $n_j \geq 0$. Thus for the all fundamental weights of $G$ we have $2\lambda_i \cdot w(\tilde{g}) \geq 0$ if $2\lambda_i \cdot \tilde{g} \geq 0$.

Note that the set of positive reduced magnetic charges is a subset of the Murray cone and can be obtained by modding out the residual Weyl group. The set of Weyl orbits in the Murray cone is a physically important object; it corresponds to the magnetic charge sectors of the theory. This follows from the fact that a magnetic charge $g$ is defined only modulo the action of the residual Weyl group. For this reason we shall introduce a set called the fundamental Murray cone which is bijective to the set of of Weyl orbits in the Murray cone. The set of positive reduced magnetic charges can of course be identified with the fundamental Murray cone. However, it would be more appropriate to call this set the anti-fundamental Murray cone. We recall that a reduced magnetic charge $\tilde{g}$ satisfies $\alpha_j \cdot \tilde{g} \leq 0$ for all unbroken simple roots $\alpha_j$. It follows from this condition that $\tilde{g}$ can be
identified with a lowest weight of $H^\ast$. Similarly, we can define the subset of the Murray cone $\{ g : \alpha_j \cdot \tilde{g} \geq 0 \}$. These magnetic charges now map to the fundamental Weyl chamber of $H^\ast$, hence we call this set the fundamental Murray cone. We thus find that the magnetic charge sectors are labelled by dominant integral weights of the residual gauge group. A similar conclusion was drawn for singular monopoles by Kapustin [54].