On a unified description of non-abelian charges, monopoles and dyons
Kampmeijer, L.

Citation for published version (APA):
Kampmeijer, L. (2009). On a unified description of non-abelian charges, monopoles and dyons

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Appendix C

Proto skeleton groups for classical Lie groups

Below the representation theory is considered for the proto skeleton groups associated to the groups $SU(2)$, $SU(3)$ and $Sp(4)$.

C.1 Proto skeleton group for SU(2)

In this section we compute the irreducible representations and the characters of the proto skeleton group $\mathcal{W} \ltimes (T \times T^*)$ starting out from a Yang-Mills theory with $G = SU(2)$. The Weyl group of $SU(2)$ is of course $\mathbb{Z}_2$. The maximal torus $T$ of $SU(2)$ is simply the subgroup $U(1) \subset SU(2)$ generated by $H_\alpha = \sigma_3$. The dual group of $SU(2)$ is $SO(3) = SU(2)/\mathbb{Z}_2$. We thus find that $T^*$ can be identified with $U(1)/\mathbb{Z}_2$ where $U(1)$ is generated by $\sigma_3$ and $\mathbb{Z}_2$ is generated by $-\mathbb{I}$. We thus see that if we identify the electric weight lattice with the integer numbers the magnetic weight lattice is given by the even integers.

An irreducible representation of $T \times T^*$ is labelled by a pair $(2\lambda, 2g) \equiv (n, n^*) \in \mathbb{Z} \times \mathbb{Z}$ with $n^*$ even. The Weyl group $\mathbb{Z}_2$ acts on these pairs via the reflection $(n, n^*) \mapsto (-n, -n^*)$. Hence only $(n, n^*) = (0, 0)$ has a non-trivial centraliser in $\mathbb{Z}_2$. In this case the centraliser actually equals $\mathbb{Z}_2$, and therefore the trivial charges together with the irreducible centraliser representation gives us a one-dimensional irreducible representation of $\mathbb{Z}_2 \ltimes (T \times T^*)$:

$$\Pi_{(i,[0,0])} : (w, g, \tilde{g}) \mapsto \Pi_i(w).$$

(C.1)
Appendix C. Proto skeleton groups for classical Lie groups

If either the electric or the magnetic charge is non-trivial we will obtain an irreducible representation of the proto skeleton group which is induced from the \( \Pi_{(n,n^*)} \) representation of \( T \times T^* \). As described in section 4.4.1 such an induced representation is constructed by using the action of the group on a coset space. In this case the coset space is the proto skeleton group modded out by \( T \times T^* \), which is isomorphic to the Weyl group \( \mathbb{Z}_2 \). Since there are two cosets the induced representation is two dimensional. We shall denote these representations by \( \Pi_{[n,n^*]} \).

Before we continue to discuss the tensor products of these irreducible representations one last remark should be made: irreducible representations with charges related by the action of the Weyl group are equivalent, i.e. \( \Pi_{[n,n^*]} \cong \Pi_{[-n,-n^*]} \). For the pure electric representations with \( n^* = 0 \) this means that the representation is defined unambiguously by the absolute value \( |n| \).

The fusion rules for the proto skeleton group can be computed by evaluating equation (4.43). This gives the following results. If the charges are zero one simply retrieves the \( \mathbb{Z}_2 \) fusion rules. For nonzero charges one finds

\[
(i, [0, 0]) \otimes [n, n^*] = [n, n^*] \tag{C.2}
\]

\[
[n_1, n_1^*] \otimes [n_2, n_2^*] = \left[ n_1 + n_2, n_1^* + n_2^* \right] \oplus \left[ n_1 - n_2, n_1^* - n_2^* \right]. \tag{C.3}
\]

If the charges are equal (up to a Weyl transformation) this fusion rule is slightly different

\[
[n, n^*] \otimes [n, n^*] = (0, [0, 0]) \oplus (1, [0, 0]) \oplus [2n, 2n^*]. \tag{C.4}
\]

These fusion rules can be understood from two perspectives. First, if we take either the magnetic or the electric charges zero we obtain the fusion rules of \( O(2) \). This should not be surprising at all since \( \mathbb{Z}_2 \rtimes U(1) \cong O(2) \). Second, if the centraliser charges are ignored the fusion rules above give the product in \( \mathbb{Z}[\Lambda \times \Lambda^*]^W \) as discussed in section 4.4.4.

C.2 Proto skeleton group for SU(3)

The next example we are going to work out corresponds to an \( SU(3) \) theory. All that we shall do here is repeating the recipe from the previous section. Nonetheless, the Weyl group of \( SU(3) \) is truly non abelian and therefore the representation theory of the related proto skeleton group is potentially much more interesting. By the same token it is also much more complicated. We shall still be able to work out all irreducible representations. We shall deal with the fusion rules on the other hand on a case by case basis.

To avoid too much cluttering we shall use a compact notation and use \( T \) to denote \( T \times T^* \).
Thus for the proto skeleton group we write $S_3 \ltimes T$. As before we should start by choosing the charges corresponding to the $U(1)$ factors in $T$, and determine the centraliser subgroup. The $T$ charge which we denote by $\mu$ has 2 components, one related to the electric charge and one to the magnetic charge. Each of these components correspond to a point in the weight lattice of $SU(3)$, the magnetic charge however is restricted to the root lattice. The Weyl group acts on the charge $\mu$. To visualise this action one take 2 copies of the $SU(3)$ weight lattice. The Weyl group acts on these charges simultaneously. It follows that the centraliser of $\mu$ is simply the intersection of the electric and the magnetic centralisers.

We distinguish 3 different classes of Weyl orbits, with either 1, 3 or 6 elements, corresponding to 3 different classes of representations of the proto skeleton group. The simplest case is when the Weyl orbit is trivial, that is when the charges are zero. All elements in the Weyl group leave this weight fixed which means that the centraliser subgroup is $S_3$ itself. Thus by choosing an $S_3$ representation $\Pi_i$ we define a representation of the complete proto skeleton group:

$$\Pi_{(i,[0])} : (w, t) \mapsto \Pi_i(w).$$

(C.5)

If the charge is a multiple of either $(\lambda_1, \lambda_1)$ or $(\lambda_2, \lambda_2)$ the centraliser subgroup is isomorphic to $Z_2 \subset S_3$. Choosing a charge corresponding to an irreducible representation of the $Z_2$ centraliser group gives us a one dimensional representation of $Z_2 \ltimes T$. From the fact that in these cases the Weyl orbits of the charges have 3 elements we conclude that the induced representations $\Pi_{(i,[\mu_k])}$ are 3 dimensional.

The set of charges with trivial centralisers lead to six dimensional representations of the proto skeleton group which we denote by $\Pi_{[\mu]}$.

We shall finally compute some fusion rules for the proto skeleton group of $SU(3)$. If we restrict to either pure electric case charges we have $\mu = n\lambda_1 + m\lambda_2$ for some positive integers $n$ and $m$. Here we use the fundamental weights with $2\lambda_i \cdot \alpha_j / \alpha_j^2 = \delta_{ij}$. We shall first compute the fusion rule related to $3 \otimes 3$ in $SU(3)$.

$$(i, [\lambda_1]) \otimes (i, [\lambda_1]) = (0, [\lambda_2]) \oplus (1, [\lambda_2]) \oplus (0, [2\lambda_1])$$

(C.6)

$$(i, [\lambda_1]) \otimes (j, [\lambda_1]) = (0, [\lambda_2]) \oplus (1, [\lambda_2]) \oplus (1, [2\lambda_1]) \quad i \neq j.$$  

(C.7)

As far as it concern the electric charges it is clear that this agrees with $3 \otimes 3 = 6 \oplus \overline{3}$ for $SU(3)$.

Some more fusion rules related to $3 \otimes \overline{3}$ are given by:

$$(i, [\lambda_1]) \otimes (i, [\lambda_2]) = [\lambda_1 + \lambda_2] \oplus (0, [0]) \oplus (2, [0])$$

(C.8)

$$(i, [\lambda_1]) \otimes (j, [\lambda_2]) = [\lambda_1 + \lambda_2] \oplus (1, [0]) \oplus (2, [0]) \quad i \neq j.$$  

(C.9)

If we ignore the centraliser charges this corresponds to $3 \otimes \overline{3} = 8 \oplus 1$. 

125
C.3 Proto skeleton group for $Sp(4)$

In the last case we work out we take the residual gauge symmetry to contain a factor $Sp(4)$. This example is not much different from the $SU(3)$ case, except for the fact that $Sp(4)$ is not selfdual. Consequently the magnetic lattice is not directly embedded in the electric weight lattice but corresponds to the weight lattice of the dual group. Once we have taken this into account we can follow exactly the same procedure as before to compute the irreducible representations and the fusion rules.

As in the previous sections we shall denote the proto skeleton group by $\mathcal{W} \rtimes T$. Where $T$ contains the maximal tori of both the electric group $Sp(4)$ and its magnetic dual $SO(5)$. The Weyl group in this particular case is $D_4 = S_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ as discussed in appendix B. To construct the irreducible representations we first choose the $T$ charge $\mu = (\lambda, g)$. The centraliser of $\mu$ is either $D_4 \times T$, $\mathbb{Z}_2 \ltimes T$ or $T$. We shall discuss these cases separately. Only if $\mu$ is zero it is invariant under the whole Weyl group. This will lead to irreducible representations $\Pi_{(i,[0])}$ which correspond to representations $\Pi_i$ of $D_4$ reviewed in appendix B.2.

There are several possibilities to realise a $\mathbb{Z}_2$ centraliser. In each case the subgroup $\mathbb{Z}_2 \subset \mathcal{W}$ is generated by the reflection $w_\alpha$ that leaves the $T$ charge fixed. Taking $\alpha$ to be a simple root is sufficient to capture all the isomorphism classes of irreducible representations. This leaves us with two possibilities corresponding to $\alpha_1$ and $\alpha_2$ as depicted in figure B.2. In the first case the centraliser corresponds to a sign flip in the second case the $\mathbb{Z}_2$ subgroup comes from the permutation of the coordinates of the weight space. Either way the resulting Weyl orbits have four elements and since there are two irreducible $\mathbb{Z}_2$ representations one obtains two inequivalent 4-dimensional proto skeleton group representations for each such Weyl orbit.

Finally one can have an orbits represented by $\mu \in \Lambda \times \Lambda^*$ such that $\mu$ has only a trivial centraliser. Such orbits thus have 8 elements and corresponding irreducible representation $\Pi_{[\mu]}$ which are 8 dimensional.

Fusion rules for the proto skeleton group of $Sp(4)$ can be computed from formula (4.43) using the characters of $D_4$ listed in table B.2.