On a unified description of non-abelian charges, monopoles and dyons
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Appendix D

Skeleton groups for classical Lie groups

A subtle part in constructing the skeleton group is determining the lift of the Weyl group to Lie group. The main part of this appendix is therefore dedicated to describing these lifts for the classical groups. We shall also determine the relevant normal subgroups that by modding out give the Weyl group back.

D.1 Skeleton group for SU(n)

Below we work out the construction of the skeleton group and its irreducible representations in some detail for $G = SU(n)$.

We shall start by identifying the lift $W$ of the Weyl group. For the maximal torus $T$ of $SU(n)$, we take the subgroup of diagonal matrices. The length of the roots is set to $\sqrt{2}$. The raising and lowering operators for the simple roots are the matrices given by $(E_{\alpha_i})_{lm} = \delta_{l,i+1}\delta_{m,i+1}$ and $(E_{-\alpha_i})_{lm} = \delta_{l,i+1}\delta_{m,i}$. From this one finds that $x_{\alpha_i}$, as defined in equation (4.27) is given by:

$$ (x_{\alpha_i})_{lm} = \delta_{lm}(1 - \delta_{l,i} - \delta_{l,i+1} + i(\delta_{l,i}\delta_{m,i+1} + \delta_{l,i+1}\delta_{m,i})) \quad \text{(D.1)} $$

From now on we abbreviate $x_{\alpha_i}$ to $x_i$. One easily shows that

$$ x_i^4 = 1, \quad [x_i, x_j] = 0 \text{ for } |i - j| > 1, \quad x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}. \quad \text{(D.2)} $$
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As it stands, this is not the complete set of relations for \( W \). However, one may show that \( W \) is fully determined if we add the relations

\[
(x_i x_{i+1})^3 = 1. \tag{D.3}
\]

This also makes contact with the presentation of the normaliser of \( T \) obtained by Tits [100, 101].

We shall now determine the group \( D \). Note that the elements \( x_i^2 \in W \) are diagonal and of order 2. In fact we have \((x_i^2)_{lm} = \delta_{lm}(1 - 2\delta_{ii} - 2\delta_{l,i+1})\). One thus sees that the group \( K \) generated by the \( x_i^2 \) is just the group of diagonal matrices with determinant 1 and diagonal entries equal to \( \pm 1 \). Since its elements are diagonal we have \( K \subset T \) and hence \( K \subset D = W \cap T \). As a matter of fact \( K = D \). To prove this one can check that conjugation with the \( x_i \) leaves \( K \) invariant. Hence \( K \) is a normal subgroup of \( W \) and thus the kernel of some homomorphism \( \rho \) on \( W \). The image of \( \rho \) is the Weyl group \( S_n \) of \( SU(n) \). To see this note that \( W/K \) satisfies the relations of the permutation group (these are the same as the relations for the \( x_i \) above, but with \( x_i^2 = 1 \)). An explicit realisation of \( \rho : W \to S_n \) is given by \( \rho(w) : t \in T \mapsto wt w^{-1} \). Obviously \( D \subset \text{Ker}(\rho) = K \), consequently \( D = K \).

Let us work out the \( SU(2) \) case as a small example. \( SU(2) \) has only one simple root and thus \( W \) has only one generator \( x \) which satisfies \( x^4 = 1 \). This gives \( W = \mathbb{Z}_4 \). \( D \) is generated by \( x^2 \) which squares to the identity and hence \( D = \mathbb{Z}_2 \). For higher rank \( W \) is slightly more complicated but \( D \) is simply given by the abelian group \( \mathbb{Z}_2^{n-1} \).

In order to determine the representations of \( S \) for \( SU(n) \) we need to solve (4.53) and hence we need to describe how \( D \) is represented on a state \( | \lambda \rangle \) in an arbitrary representation of \( SU(n) \). This turns out to be surprisingly easy. The generating element \( x_i^2 \) of \( D \) acts as the non-trivial central element of the \( SU(2) \) subgroup in \( SU(n) \) that corresponds to \( \alpha_i \). Now let \( (\lambda_1, \ldots, \lambda_{n-1}) \) be the Dynkin labels of the weight \( \lambda \). Note that \( \lambda_i \) is also the weight of \( \lambda \) with respect to the \( SU(2) \) subgroup corresponding to \( \alpha_i \). Recall that the central element of \( SU(2) \) is always trivially represented on states with an even weight while it acts as \(-1\) on states with an odd weight. Hence \( x_i^2 \) leaves \( | \lambda \rangle \) invariant if \( \lambda_i \) is even and sends \( | \lambda \rangle \) to \( \lambda(x_i^2)| \lambda \rangle = - | \lambda \rangle \) if \( \lambda_i \) is odd.

For any given orbit \([\lambda, g]\) we can solve (4.53) by determining \( N_{\lambda, g} \subset W \) and choosing a representation of \( N_{\lambda, g} \) which assures that the elements \( (x_i^2, x_i^2) \) act trivially on the vectors \( | \lambda, v^\gamma \rangle \).

If the centraliser of \([\lambda, g]\) in \( W \) is trivial its centraliser \( N_{(\lambda, g)} \) in \( W \) equals \( D = \mathbb{Z}_2^{n-1} \). An irreducible representations of \( \gamma \) of \( D \) is 1-dimensional and satisfies \( \gamma(x_i^2) = \pm 1 \). The centraliser representations that satisfy the constraint (4.53) are defined by \( \gamma(x_i^2) = \lambda(x_i^2) \).

If \( (\lambda, g) = (0,0) \) the centraliser is \( W \). In this case an allowed centraliser representation
γ satisfies \( \gamma(d)|v\rangle = |v\rangle \), i.e. \( \gamma \) is a representation of \( W/D = W \). The irreducible representations \( \Pi_0^{[0,0]} \) of \( S \) thus correspond to irreducible representations of the permutation group \( S_n \).

If \( N(\lambda, g) \) is neither \( D \) nor \( W \) the situation is more complicated and we will not discuss this any further.

### D.2 Skeleton group for \( \text{Sp}(2n) \)

In this section we shall consider the skeleton group for \( \text{Sp}(2n) \). The skeleton group for the dual group \( \text{SO}(2n+1) \) will be discussed in the next section.

In order to construct the lift \( W \) of the Weyl group to \( \text{Sp}(2n) \) we need to define the Lie algebra, see e.g section 16.1 of [56]. The CSA is generated by \( 2n \times 2n \) matrices \( H_i \) defined by

\[
(H_i)_{kl} = \delta_{ki}\delta_{li} - \delta_{k,n+i}\delta_{n+i,l}.
\]

The short simple roots \( \alpha_i = e_i - e_{i+1} \) with length \( \sqrt{2} \) correspond to \( E_{\alpha_i} \) and \( E_{-\alpha_i} \) which are defined as

\[
(E_{\alpha_i})_{kl} = \delta_{ki}\delta_{i+1,l} - \delta_{k,n+i+1}\delta_{n+i,l},
\]

\[
(E_{-\alpha_i})_{kl} = \delta_{k,i+1}\delta_{i,l} - \delta_{k,n+i}\delta_{n+i+1,l}.
\]

The long simple root \( \alpha_n \) with length 2 is related to the raising and lowering operator \( E_{\alpha_n} \) and \( E_{-\alpha_n} \) given by

\[
(E_{\alpha_n})_{kl} = \delta_{kn}\delta_{2n,l},
\]

\[
(E_{-\alpha_n})_{kl} = \delta_{k,2n}\delta_{n,l}.
\]

From this one finds that \( x_{\alpha_i} \), as defined in equation (4.27) is given by:

\[
(x_{\alpha_i})_{lm} = \delta_{lm}(1 - \delta_{li} - \delta_{l,i+1} - \delta_{l,n+i} - \delta_{l,n+i+1}) + i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{mi} - \delta_{l,n+i}\delta_{m,n+i+1} - \delta_{l,n+i+1}\delta_{m,n+i}).
\]

While \( x_{\alpha_n} \) is given by

\[
(x_{\alpha_n})_{lm} = \delta_{lm}(1 - \delta_{ln} - \delta_{l,2n}) + i(\delta_{ln}\delta_{m,2n} + \delta_{l,2n}\delta_{mn}).
\]

To avoid cluttering we abbreviate \( x_{\alpha_i} \) to \( x_i \) and \( x_{\alpha_n} \) to \( y_n \). As follows from the results in section D.1 for \( \text{SU}(n) \) the \( x_i \)'s generate a subgroup \( S_n \) of \( W \) which is completely defined by

\[
x_i^4 = 1, \quad [x_i, x_j] = 0 \text{ for } |i - j| > 1, \quad x_i x_{i+1} x_i x_{i+1} = x_{i+1} x_i x_{i+1}
\]
and
\[(x_i x_{i+1})^3 = 1. \] (D.12)

Another subgroup of $W$ is $\mathbb{Z}_4$ generated by $y_n$, which is nothing but the lift of $\mathbb{Z}_2 \in W$ generated by the Weyl reflection in the plane orthogonal to $\alpha_n$. Note that $\mathcal{W}$ contains a subgroup $\mathbb{Z}_2^2$. One might thus expect that $W$ contains a subgroup $\mathbb{Z}_4^n$. This is indeed the case. We define
\[(y_i)_{lm} = \delta_{lm}(1 - \delta_{li} - \delta_{ln} + i(\delta_{li}\delta_{m,n+i} + \delta_{l,n+i}\delta_{mi})). \] (D.13)

Note that $y_i^4 = 1$ and $[y_i, y_j] = 0$. The $y_i$s thus generate a subgroup $\mathbb{Z}_4^n$ in $W$. Moreover, this subgroup is a normal subgroup of $W$ since it is invariant under conjugation with all the generators of $W$. One can easily check that $y_i^{i+1}$ is related to $y_i$ via conjugation with $x_i$. One might thus expect that the lift of $W = S_n \ltimes \mathbb{Z}_2^n$ is simply $S_n \ltimes \mathbb{Z}_4^n$. Note however that $S_n \cap \mathbb{Z}_2^n \neq \{e\}$. To determine the true value of this intersection one observes that $x_i^2 = y_i^2 y_i^{i+1}$. Consequently $S_n \cap \mathbb{Z}_2^n = \mathbb{Z}_2^{n-1}$ generated by the $x_i^2$s and $W = (S_n \ltimes \mathbb{Z}_4^n)/\mathbb{Z}_2^{n-1}$.

Next we want to compute the intersection of $W$ with the maximal torus $T$ in $Sp(n)$. One immediately sees that $y_i^2$ is a diagonal matrix and thus an element of $T$. Since each $y_i^2$ generates a $\mathbb{Z}_2$ group one finds that $\mathbb{Z}_2^n \subset W \cap T$. Next we want to prove that $W \cap T \subset \mathbb{Z}_2^n$. We use the same approach as in the $SU(n)$ case. $\mathbb{Z}_2^n$ is a normal subgroup of $W$. Hence $\mathbb{Z}_2^n$ is the kernel of some homomorphism $\rho$. The image of this homomorphism is isomorphic to $W/\mathbb{Z}_2^n$. The defining relations of this group can be found from the defining relations of $W$ and the equivalence relations $y_i^2 = 1$. Note that this equivalence relation also implies the relation $x_i^2 = 1$ and we thus retrieve the defining relations for $S_n \ltimes \mathbb{Z}_2^n$.

An explicit realisation of $\rho : W \to S_n \ltimes \mathbb{Z}_2^n$ is given by $\rho(w) : t \in T \mapsto wtwt^{-1}$. Obviously $W \cap T \subset \text{Ker}(\rho) = \mathbb{Z}_2^n$, consequently $W \cap T = \mathbb{Z}_2^n$.

### D.3 Skeleton group for $SO(2n+1)$

Here we shall compute the skeleton group for $SO(2n + 1)$ by determining the lift of the Weyl group $W$ and its intersection with the maximal torus.

In order to construct the lift $W$ of the Weyl group to $SO(2n + 1)$ we need to define the Lie algebra, see e.g. section 18.1 of [56]. The CSA is generated by $(2n + 1) \times (2n + 1)$ matrices $H_i$ defined by
\[(H_i)_{kl} = \delta_{ki} \delta_{li} - \delta_{k,n+i} \delta_{n+i,l}. \] (D.14)
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The long simple roots $\alpha_i$ with length $\sqrt{2}$ correspond to $E_{\alpha_i}$ and $E_{-\alpha_i}$ which are defined as

\begin{align}
(E_{\alpha_i})_{kl} &= \delta_{kl}\delta_{i+1,l} - \delta_{k,n+i+1}\delta_{n+i,l} \\
(E_{-\alpha_i})_{kl} &= \delta_{k,i+1}\delta_{i,l} - \delta_{k,n+i}\delta_{n+i+1,l}.
\end{align}

The short simple root $\alpha_n$ with length 1 is related to the raising and lowering operator $E_{\alpha_n}$ and $E_{-\alpha_n}$ which in our conventions are given by

\begin{align}
(E_{\alpha_n})_{kl} &= \sqrt{2}\left(\delta_{k,n}\delta_{2n+1,l} - \delta_{k,2n+1}\delta_{2n,l}\right) \\
(E_{-\alpha_n})_{kl} &= \sqrt{2}\left(\delta_{k,2n+1}\delta_{n,l} - \delta_{k,2n}\delta_{2n,l}\right).
\end{align}

From this one finds that $x_{\alpha_i}$ as defined in equation (4.27) is given by:

\begin{align}
(x_{\alpha_i})_{lm} &= \delta_{lm}(1 - \delta_{li} - \delta_{i,i+1} - \delta_{l,n+i} - \delta_{l,n+i+1}) + \\
&\quad i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{m,i} - \delta_{i,n+i}\delta_{m,n+i+1} - \delta_{l,n+i+1}\delta_{m,n+i}).
\end{align}

While $x_{\alpha_n}$ is given by

\begin{align}
(x_{\alpha_n})_{lm} &= \delta_{lm}(1 - \delta_{l,n} - \delta_{l,2n} - 2\delta_{l,2n+1}) + (\delta_{ln}\delta_{m,2n} + \delta_{l,2n}\delta_{mn}).
\end{align}

To avoid cluttering we abbreviate $x_{\alpha_i}$ to $x_i$ and $x_{\alpha_n}$ to $y_n$. As for $Sp(2n)$ we conclude that the $x_i$s generate a subgroup $S_n$ of $W$. In contrast to the $Sp(2n)$ case we have $y_n^2 = 1$, hence, instead of a $\mathbb{Z}_4$ subgroup $y_n$ simply generates a $\mathbb{Z}_2$ subgroup. In addition we define the generators $y_i = x_i y_{i+1} x_i^{-1}$. Note that $y_i^2 = 1$ and $[y_i, y_j] = 0$. It should be clear that the $y_i$s generate a normal subgroup of $W$ which equals $\mathbb{Z}_2^n$. One can also check that this subgroup does not intersect with $S_n$ except in 1. We thus find $W = S_n \ltimes \mathbb{Z}_2^n$.

Next we want to compute the intersection of $W$ with the maximal torus $T$ in $SO(2n+1)$. Note that $y_i^2$ equals the trivial element in $T$. On the other hand $x_i^2$ is a diagonal matrix not equal to the unit. By adapting our arguments from the $SU(n)$ and $Sp(2n)$ cases it should now be clear that $D = W \cap T$ is generated by the $x_i$s and hence equals $\mathbb{Z}_2^{n-1}$. Moreover, we indeed have that $W/D$ equals the Weyl group $S_n \ltimes \mathbb{Z}_2^n$ of $SO(2n+1)$.

D.4 Skeleton group for $SO(2n)$

Finally we shall determine the lift of the Weyl group for $SO(2n+1)$ and its intersection with the maximal torus. Together with the maximal torus itself and the dual torus this fixes the skeleton group.

As can be read off from table B.1 and as illustrated by figure B.1 the root diagram of $SO(2n)$ looks like the root diagrams of $SO(2n+1)$ and $Sp(2n)$ but with the exceptional roots removed. This similarity is directly reflected upon the matrix representations, see
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e.g. section 18.1 of [56]. The CSA of \( SO(2n) \) is generated by \( 2n \times 2n \) matrices \( H_i \) defined by
\[
(H_i)_{kl} = \delta_{ki} \delta_{li} - \delta_{k,2i} \delta_{2i,l}.
\] (D.21)
The first \( n-1 \) simple roots \( \alpha_i \) with length \( \sqrt{2} \) correspond to \( E_{\alpha_i} \) and \( E_{-\alpha_i} \) which are defined as
\[
(E_{\alpha_i})_{kl} = \delta_{ki} \delta_{l,i+1} - \delta_{k,n+i+1} \delta_{n+i,l} \] (D.22)
\[
(E_{-\alpha_i})_{kl} = \delta_{k,i+1} \delta_{i,l} - \delta_{k,n+i} \delta_{n+i+1,l}. \] (D.23)
The \( n \)th simple root \( \alpha_n \), whose \( SO(2n+1) \) counterpart is actually not simple, is related to the raising and lowering operator \( E_{\alpha_n} \) and \( E_{-\alpha_n} \) given by
\[
(E_{\alpha_n})_{kl} = \delta_{k,n-1} \delta_{2n,l} - \delta_{k,n} \delta_{2n-1,l} \] (D.24)
\[
(E_{-\alpha_n})_{kl} = \delta_{k,2n} \delta_{n-1,l} - \delta_{k,2n-1} \delta_{n,l}. \] (D.25)
From this one finds that \( x_{\alpha_i} \) as defined in equation (4.27) is given by:
\[
(x_{\alpha_i})_{lm} = \delta_{lm} (1 - \delta_{li} - \delta_{l,i+1} - \delta_{l,n+i} - \delta_{l,n+i+1}) +
\]
\[
i(\delta_{li} \delta_{m,i+1} + \delta_{l,i+1} \delta_{mi} - \delta_{l,n+i} \delta_{m,n+i+1} - \delta_{l,n+i+1} \delta_{m,n+i}). \] (D.26)
While \( x_{\alpha_n} \) is given by
\[
(x_{\alpha_n})_{lm} = \delta_{lm} (1 - \delta_{l,n-1} - \delta_{l,n} - \delta_{l,2n-1} - \delta_{l,2n}) +
\]
\[
i(\delta_{l,n-1} \delta_{2n,m} + \delta_{l,2n} \delta_{n-1,m} - \delta_{l,n} \delta_{2n-1,m} - \delta_{l,2n-1} \delta_{n,m}). \] (D.27)
We abbreviate \( x_{\alpha_i} \) to \( x_i \) with \( i = 1, \ldots, n-1 \). The \( x_i \)s generate the group \( S_n \). We define \( y_{n-1} \) as \( x_{n-1} x_{\alpha_n} \) and finally \( y_i = x_i y_{i+1} x_i^{-1} \). One can check that \( y_i^2 = 1 \) and \( [y_i, y_j] = 0 \) and in particular that the group \( \mathbb{Z}_2^{n-1} \) generated by the \( y_i \)s correspond to the double sign flips of the Weyl group action as discussed in appendix B.1. It is important to note that \( \mathbb{Z}_2^{n-1} \) is invariant under the action of \( S_n \). It is also not hard to see that \( S_n \cap \mathbb{Z}_2^{n-1} = e \). The lift of the Weyl group \( W \) is thus given by \( S_n \ltimes \mathbb{Z}_2^{n-1} \). Finally we note that \( D = W \cap T \) equals \( \mathbb{Z}_2^{n-1} \) generated by the set \( \{x_i^2\} \).