Chapter 5

Liquidity and Inflation Premia in Eurozone Bonds

Abstract

In a framework of an equilibrium pricing model, the dynamics of real interest rates and expected inflation are estimated using observations on French bonds of various maturities. Unlike previous models, we allow the existence of a liquidity premium. In the absence of liquidity, the inflation premium runs from 113 basis points to some 250 basis points across the curve. These results are somewhat larger than the premia found for the UK market but comparable with US TIPS. The liquidity premium in real bonds equals some 6 basis points for bonds maturing in 2009 and is slightly humped shaped with a peak at the 10-year bond. The inflation premium is a prominent factor in nominal bonds as it account for more than 50% of the total risk premium across the term structure for nominal interest rates.

5.1 Introduction and Motivation

The recent commitment by the French Treasury to issue inflation-linked bonds almost every month in 2004 and the announcement by the Italian and Greek Treasury
agent to issue more inflation-linked bonds in the coming years reflect the growing importance of these instruments for the Eurozone debt market. Interestingly, most research on inflation-linked bonds is conducted for the UK and US market while little has been said about the inflation-linked bond market in the Eurozone. Given the attention of issuers and investors on the Eurozone inflation-linked market, this is not justified. According to a survey by RISK magazine, the Eurozone “is now the most advanced (in terms of products and market participants) and most liquid (both on the bonds and on the derivatives side) inflation-linked bond market in the world.” Nowadays, a reasonable European real yield curve has emerged, containing maturities varying from 2006 to 2032. Along with this real government curve a relatively liquid and economically significant Eurozone real swap market has evolved. The yield difference between nominal and index-linked bonds includes an inflation premium because index-linked bonds provide a hedge against unexpected inflation. In this paper we analyze the inflation premium contained in French inflation-linked government bonds.

The real interest rate and expected inflation are the key unobservable variables in our analysis. If real interest rates are reflected through index-linked bonds, it is common practice to use a break-even approach to calculate the expected inflation in real bonds. The expected inflation is then the yield difference between a nominal and an index-linked bond with the same maturity. Albeit simple, this method suffers from a number of problems. First, only the maturities of nominal and real bonds are taken into consideration so it does not generate a complete term structure of interest rates. More importantly, the method assumes that the Fisher equation holds and this implies that the inflation premium is set to zero. Ang and Bekaert (2003) and Evans (1998, 2003) show that the classic Fisher equation does not hold, due to the existence of a time-varying inflation premium that is different across maturities. In order to calculate the inflation premium within a break-even framework, expected

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1 See RISK December 2003: special report on index-linked bonds.
inflation extracted from survey data is incorporated. The inflation premium is then the difference between the nominal yield, real yields and the expected inflation. Still, the proposed method assumes that the index-linked curve estimated from inflation-linked price data is equal to the real term structure of interest rates. Barr and Campbell (1997) and Evans (1998) showed that there exist differences due to liquidity and imperfect indexation. As a result, more advanced methods have been proposed to estimate the inflation premium using only nominal price data. Buraschi and Jiltsov (2003) calculates the inflation premium using nominal bonds from the US treasury market within a business cycle framework. They find an average monthly inflation premium of 15 basis points and an inflation premium of 70 basis points in 10-year bonds. Ang and Bekaert (2003) fit nominal data from the US treasury market into a real interest regime-switching framework and find an average inflation premium of 100 basis points in 10-year bonds. McCulloch and Kochin (1998) find an average annual inflation premium between 160 basis points for 10-year bonds up to some 230 basis points for 30-year bonds.\(^2\)

The papers mentioned so far do understand the importance of liquidity but do not explicitly model liquidity as a driving factor of bond prices. It is well known (e.g. Amihud and Mendelson (1986), Warga (1992)) that liquid securities are traded at a premium. Taking liquidity into account allows us to use data on both nominal and inflation-linked debt in markets where the outstanding amount of inflation linked debt is small. In France for example, the fraction of inflation-linked bonds stands at 7% of total debt while the Italian treasury estimates that some 1.3% of its current debt is inflation-linked.\(^3\) The US treasury market is the worlds largest inflation-linked bond market and the outstanding amount of treasury inflation protected securities (or TIPS) equals 150 billion. This is approximately 6% of the total

\(^2\)Figure 4 of McCulloch and Koching (1998) finds an average annual inflation premium between 160 basis points for 10-year bonds up to some 230 basis points for bonds maturing in 30 years.

\(^3\)The amount outstanding in the UK as of May 2004, in France as of June 2004 and in Italy as of May 2004.
outstanding tradable US treasury debt. To some extent, the problem of liquidity can be avoided by analyzing the UK bond market because almost 25% of their tradable debt is inflation linked. In addition, Evans (1998) shows that imperfect indexation due to an indexation lag is of lesser importance than time-varying risk premium. This explains why studies combining inflation-linked and nominal debt focus on the UK debt market. For example, Remolona, Wickens and Gong (1998) finds an inflation premium on the UK market of 100 basis points for 2-year maturities while Shen (1998) reports an inflation premium around 75 basis points for bonds with a 10-year maturity up to 104 basis points for bonds maturing in 25-years. However, even if the outstanding amount is sizeable (like in the UK), liquidity is still important. Trading activity on the secondary market is small compared to conventional debt because most index-linked bonds are bought and held for the remaining of their life (see e.g. Elsasser and Sack (2004) and Shen (1998)).

This objective of this paper is to estimate the inflation premium by taking liquidity into account. This allows us to study the empirical properties of the term structure of real rates in the Eurozone bond market. We use data from French index-linked and nominal bonds and estimate the inflation and liquidity premium in a state space framework using the extended Kalman filter and quasi-maximum likelihood. In particular, we simultaneously derive the nominal and real term structure of interest rates and this enables us to calculate the price of any discount bond. In order to fit coupon bearing bonds into an affine structure, we use the extended Kalman filter to linearize the state equations. Our methodology is closely related to Remolona, Wickens and Gong (1996), who also use data from nominal and real bonds to analyze the dynamics of the real curve and to estimate the inflation premium for the UK bond market. In contrast to their paper, we take liquidity and the coupon effect into account. There are a number of reasons for using a latent variable approach. Not only is the Kalman filter a powerful and efficient algorithm, it also easily applies to the Cox, Ingersoll and Ross framework of unobservable state
variables as expected inflation and liquidity are difficult to observe. For example, Pennacchi (1991), Evans (1998) and Buraschi and Jiltsov (2003) use proxies for expected inflation. In addition, a state space approach is also useful for taking advantage of cross-sectional and time series behavior of nominal and real rates and this should reduce the instability of time series.

Our findings are as follows: in the absence of liquidity, the inflation premium runs from 113 basis points to some 250 basis points across the curve. These numbers implies that the inflation premium in long-term bonds is more than 2 times larger compared with the short-end of the curve. These results are somewhat larger than the findings of Shen (1998) and Remolona, Wickens and Gong (1998) for the UK market but comparable with US TIPS as found by Buraschi and Jiltsov (2003) and McCulloch and Kochin (1998) The liquidity premium in real bonds equals some 6 basis points for bonds maturing in 2009 and is slightly humped shaped with a peak at the 10-year bond. For short-term bonds, the liquidity premium accounts for some 5% of the total real risk premium. For long-term bonds, this approximates some 10%. On the other hand, the inflation premium is a prominent factor in nominal bonds as it account for more than 50% of the total risk premium across the term structure for nominal interest rates. Although the contribution of the liquidity premium is small, it has a large impact on the expected nominal yield spread through the expected liquidity level. We find that this yield spread is upward sloping as it runs from 76 basis points to some 198 basis points for the 2032 bond.

This paper is organized as follows. Section 2 describes the existence of an inflation and liquidity premium in real and nominal bonds. It also introduces the model and provides a discussion about the underlying assumptions and equilibrium conditions. Section 3 introduces the dataset, the estimation method and reports the

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estimation results. Section 4 finally concludes. All proofs are in the appendix.

5.2 Inflation and Liquidity in Index-Linked Bonds

In this section, we focus on the existence of an inflation and liquidity premium in nominal and real bonds. Following closely Fischer's (1975) work on indexed bonds, we analyze the inflation and liquidity premium and stress the importance of the premium when conducting strategies involving nominal and real securities.

5.2.1 Inflation and Liquidity and the Performance of Bonds

Let us assume that the return dynamics of an index-linked bond \( i(t) \), nominal bond \( B(t) \), inflation \( \pi(t) \) and the general price level \( P(t) \) are given by

\[
\begin{align*}
\frac{di}{i} &= (r + \Sigma_A) dt + \sigma_A dW_i \\
\frac{dB}{B} &= R dt + \sigma_B dW_B \\
\frac{d\pi}{\pi} &= -k (\pi - \bar{\pi}) dt + \sigma_\pi dW_\pi \\
\frac{dP}{P} &= \pi dt + \sigma_\pi dW_\pi 
\end{align*}
\]

where \( r \) is the risk-free real return and \( R \) the risk-free nominal return. The parameter \( 0 < k < 1 \) reflects the speed of adjustment of the expected inflation \( \pi \) towards its long run average price level \( \bar{\pi} \) and \( W \) denotes a standard Brownian motion. For simplicity, we assume that the liquidity premium \( \Sigma_A \) contained in real bond is constant and strictly positive in order to induce the investor to keep this security in his portfolio. The inflation \( \pi \) follows an Ornstein-Uhlenbeck process which is a suitable assumption for any issuer whose monetary policy is focused on long and sustainable growth around a certain price level.\(^5\) Using Ito's lemma, we find the real return on

\(^5\)In Europe for example, the European inflation is targeted around a long term price level of \( \bar{\pi} = 2\% \) per year.
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a nominal bond

$$\frac{d(B/P)}{B/P} = (R - \pi + \sigma^2_p) dt + \sigma dW$$  \hspace{1cm} (5.5)$$

where \( \sigma dW = (\sigma_B dW_B - \sigma_p dW_p) \). The expected real yield difference between real and nominal bonds is therefore

$$E \left[ \frac{d(B/P)}{B/P} - \frac{dv}{i} \right] = (R - \pi - r) dt - (\Sigma_A - \sigma^2_p) dt$$  \hspace{1cm} (5.6)$$

and equals the sum of two terms. The first term reflects the difference between the nominal yield, the real yield and the inflation. If the Fisher equation holds, then the expected difference is zero. The second term of equation (5.6) depends on the liquidity premium \( \Sigma_A \) and the uncertainty in future price levels \( \sigma^2_p \). Clearly, the higher the liquidity premium \( \Sigma_A \), the smaller the expected spread because it is more interesting to enter the real bond market due to the larger compensation for liquidity. The same holds for \( \sigma^2_p \) where the spread narrows as more investors will enter the real bond market when future inflation uncertainty is large. Under stochastic inflation, one can alternatively write the real price dynamics of a nominal bond as

$$\frac{d(B/P)}{B/P} = (r + \Sigma_{II}) dt + \sigma_{II} dW$$  \hspace{1cm} (5.7)$$

where \( \Sigma_{II} \) is the inflation premium in nominal bonds. In this case, the expected real yield spread between an nominal and index-linked bond equals

$$E \left[ \frac{d(B/P)}{B/P} - \frac{dv}{i} \right] = (\Sigma_{II} - \Sigma_A) dt$$  \hspace{1cm} (5.8)$$

and is an increasing function of the liquidity premium. Note that a larger liquidity (inflation) premium implies a smaller (larger) yield spread which can be explained using the same rationale as above. Using equation (5.5), one can express the nominal return on a nominal bond as a function of this inflation premium:

$$E \left( \frac{dB}{B} \right) = R dt = (r + \pi + \Sigma_{II} - \sigma^2) dt$$  \hspace{1cm} (5.9)$$
The Fisher equation in this form shows that the return on a nominal bond equals the sum of the real return $r$, the expected inflation $\pi$, an inflation premium $\Sigma_{\Pi}$ and the volatility rate in future inflation $\sigma^2$.

### 5.2.2 Implications of the Model

Let us now consider the impact of the inflation and liquidity on the performance of real and nominal bonds. From equation (5.5), we also know that the real return of a nominal bond follows a Brownian motion with drift $(R - \pi + \frac{1}{2}\sigma^2)$ and volatility parameter $\sigma$. If we apply Ito’s lemma to $Y = \ln \left( \frac{B_t}{B_0} \right)$, we find that

$$dY = (R - \pi + \frac{1}{2}\sigma^2) dt + \sigma dW$$

and the solution is given by

$$Y_T = Y_0 + \int_0^T \left( R - \pi_t + \frac{1}{2}\sigma^2 \right) dt - \int_0^T \pi_s ds + \int_0^T \sigma dW$$

(5.10)

where $\varepsilon_y \sim N(0, \sigma^2(T-t))$. From appendix (point 2) we know that $\pi_{t-\Delta t} = \tilde{\pi} + e^{-k\Delta t}(\pi_t - \tilde{\pi}) + \varepsilon_t$ and hence

$$E_t \left[ \int_t^T \pi_s ds \right] = \tilde{\pi} (T-t) + \frac{1 - e^{-k(T-t)}}{k} (\pi_t - \tilde{\pi})$$

Therefore, the expected log real return on a nominal bond equals

$$E \left( r_t^{\text{nom}} \right) = E_t \ln \left( \frac{B_T}{B_t} \right) - \ln \left( \frac{B_T}{B_0} \right) = A (T-t) - \zeta (\pi_t - \tilde{\pi})$$

(5.11)

where $A \equiv [R - \tilde{\pi} + \frac{1}{2}\sigma^2]$ and $0 < \zeta \equiv \frac{1 - e^{-k(T-t)}}{k} < k^{-1}$. Clearly, equation (5.11) shows that inflation has a lowering effect on nominal bond performance. The same approach can be taken for the real bond where $d\ln (i) = (r + \Sigma_{\Lambda} - \frac{1}{2}\sigma_{\Lambda}^2) dt + \sigma_{\Lambda} dW$ and henceforth

$$r_T^{\text{real}} = E_t \left[ \ln (i_T) \right] - \ln (i_t) = \int_t^T (r + \Sigma_{\Lambda} - \frac{1}{2}\sigma_{\Lambda}^2) dt + \int_t^T \sigma_{\Lambda} dW$$

(5.12)
where \( B \equiv (r + \Sigma_A - \frac{1}{2} \sigma_A^2) \) and \( \varepsilon_A \sim N(0, \sigma_A^2 (T - t)) \). Let us denote the real return difference between a nominal and real bond by \( \Delta = E_t [r_t^\text{Nom}] - E_t [r_t^\text{real}] \) where

\[
\Delta = (A - B) (T - t) - \zeta (\pi_t - \bar{\pi})
\]

\[
\frac{\partial \Delta}{\partial (T - t)} = (A - B) - e^{-k(T-t)} (\pi_t - \bar{\pi})
\]

Equation (5.13) gives us the time \( t \) expectation of the time spread at time \( T > t \) and this allows us to analyze the expected spread between a nominal and real bond over an investment horizon \((T - t)\). The marginal return is given in equation (5.14). The spread \( \Delta \) depends on \((\pi_t - \pi^*)\) and \((A - B)\). First, note that \( \Delta = 0 \) when \((T - t) \to 0\) as the instantaneous real return of a nominal bond equals the instantaneous return of a real bond. This is not a surprise as the devaluation of money due to inflation is zero under an instant payoff.

**Proposition 6** An actual inflation smaller than the long-run inflation is not sufficient to guarantee the outperforming of nominal bonds over real bonds over an investment horizon \((T - t)\). The trend in real bonds must also be lower than the trend in nominal bonds. Conversely, an actual inflation larger than the long-run inflation is not sufficient to guarantee the outperformance of real bonds over nominal bonds over an investment horizon \((T - t)\). The trend in real bonds must also be larger than the trend in nominal bonds. The outperformance however decreases over time.

**Proof.** Let us focus on the first case (the second case is proven in the same way). An actual inflation smaller than the long run inflation level implies that \((\pi_t - \bar{\pi}) < 0\) while a smaller trend in real bond implies that \((A - B) > 0\). Under these restrictions, it follows from (5.13) that \( \Delta > 0 \) for \( T > t \). Hence, nominal bonds are expected to outperform real bonds over the investment horizon \((T - t)\). Moreover, the marginal return is a strict positive but concave function of the investment horizon. \( \blacksquare \)
**Proposition 7** In the situation were \( \pi_t < \bar{\pi} \) and \( A < B \), nominal bonds will only outperform on the short-run. In the situation were \( \pi_t > \bar{\pi} \) and \( A > B \), nominal bonds underperform in the short-run.

**Proof.** Let us denote a function \( \phi(\tau) = k \left( \frac{A-B}{\pi_t - \bar{\pi}} \right) \tau + e^{-k(T-t)} \). An outperformance of nominal bonds \( \Delta > 0 \) is equivalent to \( \phi(\tau) < 1 \) when \( A - B < 0 \) and \( \pi_t - \bar{\pi} < 0 \). On the other hand, an underperformance of nominal bonds \( \Delta < 0 \) is equivalent to \( \phi(\tau) < 1 \) when \( A - B > 0 \) and \( \pi_t - \bar{\pi} > 0 \). In both cases, \( \frac{(A-B)}{(\pi_t - \bar{\pi})} > 0 \) and \( \phi(\tau) \in (0, \infty) \) for \( \tau > 0 \). If we denote \( \tau^* \) such that \( \phi(\tau^*) = 1 \), we have

\[
\Delta_{(A-B)<0 \text{ and } (\pi_t - \bar{\pi})<0} = \begin{cases} 
> 0 & \text{when } \tau < \tau^* \\
< 0 & \text{when } \tau > \tau^* 
\end{cases}
\]

\[
\Delta_{(A-B)>0 \text{ and } (\pi_t - \bar{\pi})>0} = \begin{cases} 
< 0 & \text{when } \tau < \tau^* \\
> 0 & \text{when } \tau > \tau^* 
\end{cases}
\]

In the analysis presented above, the parameter \( k \) plays an important role in determining \( \tau^* \). To see this, we calculate the derivative of the yield spread with respect to the parameter \( k \) and this yields

\[
\frac{\partial \Delta}{\partial k} = -k^{-1} (\pi_t - \bar{\pi}) + k^{-1} e^{-k(T-t)} (\pi_t - \bar{\pi})
\]

\[
= \left[ 1 - \frac{e^{-k(T-t)}}{k^2} - e^{-k(T-t)} \right] (\pi_t - \bar{\pi})
\]

where \( \lim_{k \to 0} \frac{\partial \Delta}{\partial k} \to -(\pi_t - \bar{\pi}) \). Hence, the smaller \( k \), the larger is the change in the yield spread in order to correct for the change of inflation relative to its long-run inflation level. Intuitively, if the inflation is larger than the long-run average, nominal bonds are worse off. However, as long as monetary institutions act accordingly with measures to counter inflation (as gauged by the parameter \( k \)), one would expect nominal bonds to perform better on the longer term. Equation (5.13) is also useful in calculating the expected break-even inflation \( \pi_t^* \). This is the required level of
price change such that the expected real return on an inflation-linked bond equals the return of a nominal bond. We have

\[ \pi_t^* = \bar{\pi} + \zeta^{-1} (A - B) (T - t) \]  

(5.15)

where nominal bonds will underperform inflation-linked bonds when \( \pi_t^{\text{actual}} > \pi_t^* \). Equation (5.15) is important in determining strategies involving real and nominal bonds. For issuers, the break-even inflation reflects the inflation at which the costs of issuing nominal debt equal the costs of issuing real debt. If the actual inflation is higher than the break-even inflation, it is more costly to issue real debt compared to nominal debt. For holders of real debt, having an actual inflation higher than the break-even inflation will result in an outperformance of real bonds compared to their nominal equivalent. Clearly, any strategy involving this break-even inflation may be biased as liquidity must be taken into account. As an example, consider an investor who finds the break-even inflation too high. He may therefore expect the inflation premium \( \Sigma_{\Pi} \) to fall and therefore is willing to finance a long position in real bonds through a short position in nominal bonds. If the investor is correct, the yield spread will narrow, resulting in a profit for the investor. However, the break-even inflation may also fall due to an increase in \( \Sigma_{\Lambda} \). In this case, it becomes much more difficult to conduct this strategy because an increase in the liquidity premium makes it more difficult to establish a long position in real bonds. This means that the repo costs are larger resulting in higher financing costs and therefore a lower profit for the investor. Hence, having a correct understanding of the source of spread movements is important in order to make decisions with respect to the appropriate strategy.

If profits arise mainly due to a fall in the inflation expectation, the investor may continue to hold this strategy. On the other hand, if the source of the profit (or loss) is due to a change in liquidity, then liquidity is the key parameter to monitor.
5.3 An Equilibrium Pricing Model

The previous section showed the importance of the inflation and liquidity premium and its impact on the performance of real versus nominal bonds. Let us now turn our attention to the pricing of these securities using an equilibrium term structure model that takes the liquidity and inflation premium into account. In the past three decades, the literature on the estimation of the term structure has grown tremendously. Many theoretical models have tried to explain movements in the term structure using a small number of factors. Among the most popular are diffusion process models for the (instantaneous) spot interest rate models pioneered by Vasicek (1977). Cox, Ingersoll and Ross (1985) take a related approach. They assume a set of unobservable state variables and derive the equilibrium bond prices as a function of these state variables using no-arbitrage arguments. Instead of using a continuous time notation and a discretization at the end, we start directly with a discrete model following closely Campbell, Lo and MacKinlay (1997, chapter 11). Let the one-period dynamics for nominal and real bonds in an arbitrage-free environment be given by

\[ Q_{k,t}^{(n)} = E_t \left[ M_{t+1}^{(n)} Q_{k-1,t+1}^{(n)} \right] \]  
\[ Q_{k,t}^{(r)} = E_t \left[ M_{t+1}^{(r)} Q_{k-1,t+1}^{(r)} \right] \]  

where \( Q_{k,t}^{(n)} \) and \( Q_{k,t}^{(r)} \) are the nominal and real prices of zero-coupon bond maturing at \( k \) periods from time \( t \).\(^6\) The real stochastic discount factor is denoted by \( M_{t+1}^{(r)} \) and the nominal stochastic discount factor by \( M_{t+1}^{(n)} \). As long as the dynamics of the pricing kernel and price level are specified, we can solve recursively for the set of bond prices as \( b_{k,t}^{(n)} = b_{T,T}^{(r)} = 0 \). The continuously compounded nominal and real yields are given by \( y_{k,t}^{(n)} = -k^{-1} q_{k,t}^{(n)} \) and \( y_{k,t}^{(r)} = -k^{-1} q_{k,t}^{(r)} \) respectively where \( q_{k,t}^{(n)} = \ln Q_{k,t}^{(n)} \)

\(^6\)The indicator \( k \) has a different meaning than in the previous subsection (where it was used to denote the error correction parameter). In here, we use \( k \) to refer to the number of periods until maturity.
and $d_{k,t}^{(r)} = \ln Q_{k,t}^{(r)}$. To keep things tractable, we assume that the distribution of the pricing kernel is conditionally normal and heteroskedastic. We also assume that the factors and pricing kernel are given by the following dynamics:

$$
-m_{t-1}^{(r)} = z_{1,t} + \sqrt{z_{1,t}} \beta_1 \xi_{1,t-1} + z_{2,t} + \sqrt{z_{2,t}} \beta_2 \xi_{2,t-1}
$$

(5.18)

$$
-m_{t-1}^{(n)} = z_{1,t} + \sqrt{z_{1,t}} \beta_1 \xi_{1,t-1} + z_{3,t} + \sqrt{z_{3,t}} \beta_3 \xi_{3,t-1} + \beta_4 \xi_{3,t-1}
$$

(5.19)

$$
z_{t+1} = (I - \Theta) \mu + \Theta z_{t} + \sqrt{S(t)} \xi_{t+1}
$$

(5.20)

In here, we have $m_{t-1}^{(j)} = \ln M_{t-1}^{(j)}$ for $j = r$ (real), $j = n$ (nominal). This model is closely related to Remolona, Wickens and Fong (1996) who use a generalized Cox, Ingersoll and Ross model with 3 states to estimate the inflation premium for UK government bonds. In our model, $I, \Theta$ and $S(t)$ are $(3 \times 3)$ diagonal matrices where $S(t)_{ii} = \alpha_i + \gamma_i z_{i,t}$ and $\xi_{t+1} = \text{diag} (\xi_{1,t+1}, \xi_{2,t+1}, \xi_{3,t+1})$. The exogenous state variables are depicted by a $(3 \times 1)$ vector $z_t = [z_{1,t}, z_{2,t}, z_{3,t}]'$ which follows a square root process. Here $z_{1,t}$ is the real one-period interest rate, $z_{2,t}$ is the liquidity state and $z_{3,t}$ the expected inflation. Clearly, the heteroskedasticity in this model arises from the level dependent volatility. The setup of the model shows the important difference between the real and nominal log-pricing kernel. The real log-pricing kernel does not depend on inflation but only on liquidity. On the other hand, the log-pricing kernel for nominal bonds is depending on inflation (both expected and unexpected). Later we will show that $\beta \neq 0$ implies a risk premium incorporated in real and nominal bonds. We assume that $\xi_{i,t} \sim N(0, \sigma_i^2)$ and independent. The third state $z_{3,t}$ represents expected inflation and the true or total inflation is therefore given by

$$\text{Inflation}_{t+1} = \underbrace{\xi_{3,t-1}}_{\text{expected inflation}} + \underbrace{\xi_{1,t-1}}_{\text{unexpected inflation}}$$

Remolona et al. (1996) do not take the liquidity premia into account. In their model, the nominal curve was driven by a real rate and an inflation premia while the real curve was driven by a real rate.
The model tells us that the inflation premium depends on a (stochastic) expected inflation and the unexpected inflation, which is captured by $\xi_{4,t-1}$. The innovation in the expected inflation $\xi_{3,t-1}$ has an impact on the nominal log-pricing kernel through the parameter $\beta_3$ while the parameter $\beta_4$ can be interpreted as the price of unexpected inflation. As Evans (2003) pointed out, this impact plays an important role in the interpretation of the model. If the pricing kernel and inflation move independently, then the log price of a nominal bond can be priced as the sum of the log price of an equivalent real bond minus the inflation. However, this implies that (1) no inflation premium exists and (2) real yield are uncorrelated with inflation. Penacchi (1991) finds a negative correlation between real interest rates and expected inflation. He also finds that real interest rates follow a slower mean while having a larger volatility compared to expected inflation. Barr and Campbell (1995) also find expected inflation being negative correlated with real interest. This correlation effect weakens when the maturity of bonds increases. They also find the expected inflation being the dominant factor in explaining long bond returns. The term structure in our model is determined by a total of three factors in which the real interest is the common factor. This implication is consistent with Stambaugh (1988) who showed that one-factor CIR models are not able to describe the excess returns in bonds. More importantly, a two-factor model with expected inflation as one of the states is more able to capture the dynamics of nominal bonds. In order to identify the model, some parameter restrictions should be imposed.

Dai and Singleton (2000) suggested $S(t)_{it} = \alpha_i + \gamma_i z_{it}$ in order to test whether the process is homoskedastic ($\alpha \neq 0, \gamma = 0$) or level dependent ($\gamma \neq 0$). In order to limit the number of parameters in the estimation process, we construct a three-factor independent square-root process where $\alpha = 0, \gamma = 1$. In other words, we set the conditional covariance to be a diagonal matrix and let the correlations enter through

\footnote{Penacchi (1991) argues that real interest rates and inflation might be jointly dependent due to technological innovation. This innovation has an impact on output and henceforth an effect on the money demands.}
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the log-pricing kernel. The proposed term structure model can be interpreted as a two-factor discrete version of the Cox, Ingersoll and Ross model (1985) where the log price in affine form is given by

\[ y_{k,t}^{(n)} = -\frac{1}{k} \eta_{k,t}^{(n)} = \frac{1}{k} \left[ A_k^{(n)} + B_{1,k}^{(n)} z_{1,t} + B_{2,k}^{(n)} z_{2,t} \right] \]  
\[ y_{k,t}^{(r)} = -\frac{1}{k} \eta_{k,t}^{(r)} = \frac{1}{k} \left[ A_k^{(r)} + B_{1,k}^{(r)} z_{1,t} + B_{2,k}^{(r)} z_{2,t} \right] \]

and we have to find the structure of \( A_k^{(j)}, B_{1,k}^{(j)}, B_{2,k}^{(r)} \) and \( B_{3,k}^{(n)} \) for real \((j = r)\) and nominal \((j = n)\) bonds. Our model exhibits a total of 4 prices of risk \( \{\beta_1, \beta_2, \beta_3, \beta_4\} \) and theoretically, affine models allows the identification of \( N - 1 \) prices of risk using \( N \) bonds. In our case with one common component and one unexpected inflation component, we need at least 4 nominal bonds (for the identification of \( \beta_1, \beta_3 \) and \( \beta_4 \)) and 2 real bonds (for the identification \( \beta_2 \)). Let us now focus on the equilibrium conditions for both term structures.

**Proposition 8** The difference equations for the parameters can be found recursively. In our affine structure model these are defined as \( A_0^{(r)} = B_{1,0}^{(r)} = A_1^{(r)} = A_2^{(n)} = B_{1,0}^{(n)} = 0 \) and \( B_{i,1}^{(r)} = 1 - \frac{1}{2} \beta_1^2 \sigma_1^2, A_1^{(n)} = \frac{1}{2} \beta_2^2 \sigma_2^2, B_{i,1}^{(n)} = 1 - \frac{1}{2} \beta_3^2 \sigma_3^2 \). For \( k > 1 \) we have

\[ A_k^{(r)} = A_{k-1}^{(r)} + \sum_{i=1,2} B_{i,k-1}^{(r)} (1 - \theta_i) \mu_i \]
\[ B_{i,k}^{(r)} = 1 + \theta_i B_{i,k-1}^{(r)} - \frac{1}{2} \left( \beta_i + B_{i,k-1}^{(r)} \right)^2 \sigma_i^2 \text{ and } i = 1, 2 \]
\[ A_k^{(n)} = A_{k-1}^{(n)} - \frac{1}{2} \beta_3^2 \sigma_3^2 + \sum_{i=1,3} B_{i,k-1}^{(n)} (1 - \theta_i) \mu_i \]
\[ B_{i,k}^{(n)} = 1 + \theta_i B_{i,k-1}^{(n)} - \frac{1}{2} \left( \beta_i + B_{i,k-1}^{(n)} \right)^2 \sigma_i^2 \text{ and } i = 1, 3 \]

**Proof.** A proof of this expression is outlined in the appendix (point 3).
Using these equilibrium conditions, we can analyze the yield spread between the real and nominal bonds. As noted earlier, it is common practice to use a break-even approach to calculate the expected inflation in real bonds. Albeit simple, the break-even method is not innocuous as the following two implications show. First, applying expressions (5.23) and (5.24) gives us also the opportunity to calculate the yield spread between a nominal and real bond (as suggested by the break-even method):

\[ y_{k,t}^{(n)} - y_{k,t}^{(r)} = -\frac{1}{k} q_{k,t}^{(n)} + \frac{1}{k} q_{k,t}^{(r)} = \frac{1}{k} \left[ A_k^{(n)} + B_{3,k}^{(n)} z_{3,t} \right] - \frac{1}{k} \left[ A_k^{(r)} + B_{2,k}^{(r)} z_{2,t} \right] \]  

(5.27)

However, equation (5.27) implies that an increase in the uncertainty of the expected inflation \( \sigma_3 \) results in a decrease in the yield spread. This is incorrect because conventional bonds become less attractive when inflation uncertainty increases. As a result, the yield spread in equation (5.27) must increase. Clearly, the interpretation taken from equation (5.27) is not correct as it reports the difference between a nominal yield and a real yield. We can however express the real yield in nominal terms by adding the appropriate risk premia for the different state variables. The expected log excess return of a \( k \)-period bond over a one-period bond is given by:

**Proposition 9** The log excess return \( \Lambda^{(j)} \) = \( E_t \left( r_{k,t-1}^{(j)} \right) - y_{1,t}^{(j)} \) of a \( k \)-period \( j = \{ r, n \} \) bond over a 1-period bond is given by

\[ \Lambda^{(r)} = -\sum_{i=1,2} B_{i,k-1}^{(r)} \beta_i z_{i,t} \sigma_i^2 - \frac{1}{2} \sum_{i=1,2} \left( B_{i,k-1}^{(r)} \right)^2 z_{i,t} \sigma_i^2 \]

\[ \Lambda^{(n)} = -\sum_{i=1,3} B_{i,k-1}^{(n)} \beta_i z_{i,t} \sigma_i^2 - \frac{1}{2} \sum_{i=1,3} \left( B_{i,k-1}^{(n)} \right)^2 z_{i,t} \sigma_i^2 \]

(5.28)

**Proof.** A proof of this expression is outlined in the appendix (point 4).

Equation (5.28) shows that the log excess return of a \( k \)-period bond over a 1-period bond, depends on the sum of risk premia, which is a function of \( j \) and a
5.3. **THE MODEL**

variance term that arises from Jensen's inequality because we work with the expectation of log return. Using the expressions in (5.28), the risk premium for factor \(i (i = 1, 2, 3)\) in a bond maturing \(k\) periods from now equals

\[
\Lambda_i = - \left[ B_{i,k-1}^{(j)} \beta_i \sigma_i^2 + \frac{1}{2} \left( B_{i,k-1}^{(j)} \sigma_i \right)^2 \right] z_{i,t} \tag{5.29}
\]

By calculating the derivative of \(\Lambda\) with respect to \(z_i\), one can show that the risk-premium is an increasing function of the state variable \(z_i\) as long as \(\beta_i < -\frac{1}{2} B_{i,k-1}^{(j)}\). A sufficient negative beta will therefore create an increasing risk premium on \(k\)-period bonds relative to a one-period bond. If we denote a nominal real yield by \(y_{k,t}^{(R)}\), we can express the expected nominal yield spread between nominal and real bonds as

\[
E \left[ y_{k,t}^{(n)} - y_{k,t}^{(R)} \right] = (\Lambda_1 + \Lambda_3 + \mu_1 + \mu_3) - (\Lambda_1 + \Lambda_2 + \mu_1 + \mu_2 + \mu_3) = \Lambda_3 - \Lambda_2 - \mu_2 \tag{5.30}
\]

Recall that the inflation volatility had a negative impact on yield spread in the break-even approach and we argued that this is incorrect. The correct way, where the different risk premia is taken into account, yields a positive impact on the yield spread. This is easy to see as

\[
\frac{\partial E \left[ y_{k,t}^{(n)} - y_{k,t}^{(R)} \right]}{\partial \sigma_3} = \frac{\partial E \left[ y_{k,t}^{(n)} - y_{k,t}^{(R)} \right]}{\partial \Lambda_3} \cdot \frac{\partial \Lambda_3}{\partial \sigma_3} = - \left[ 2B_{3,k-1}^{(R)} \beta_3 \sigma_3 + \left( B_{3,k-1}^{(R)} \right)^2 \right] z_{3,t} \tag{5.31}
\]

and this is strict positive under \(\beta_i < -\frac{1}{2} B_{i,k-1}^{(j)}\). Another reason why the break-even method is not innocuous lies in the mismatch of yields when the duration between nominal and real bonds is different. When the duration between nominal and real bonds are different, an error will arise as he following proposition shows:

**Proposition 10** The error in yield spread that arises when falsely matching a nominal bond with duration \(k\) to a real bond with duration \(k^* = k + \Delta > k\) is a function of the common state (real interest rate). This error term tends to go to zero if the volatility in the long end of the real curve is small.
Proof. A proof of this expression is outlined in the appendix (point 5).

Brown and Schaeffer (1994), McCulloch and Kochin (1998), Barr and Campbell (1996) and Ang and Bekaert (2003) indeed find that the long real curve incorporates a high degree of persistence compared to nominal interest rates. In other words, the long end of the real curve is less volatile than the long end of the nominal term structure of interest rates and we therefore expect the error in the yield spread being small. However, Pennacchi (1991) finds that real rate volatility is still larger than the volatility in expected inflation. We have to take the dynamics of real interest rates into consideration and therefore estimate yield levels rather than yield spreads.

In summary, our model has 4 driving factors that generate the term structure of interest rates. The first factor is the real interest rate and is a common factor, i.e. it drives both the nominal and real curve. The pricing kernel of nominal bonds is a function of the expected inflation but also of unexpected inflation. On the other hand, because we ignore indexation lags, the pricing kernel of real bonds contains an additional liquidity state. From the theoretical point of view, affine models allow the identification of $N - 1$ prices of risk using $N$ bonds. In our case, we need at least 4 nominal bonds and 2 real bonds. Moreover, it looks appealing to work with yield spreads instead of yield levels because the real rate is a common component. However, working with yield spread may result in a wrong interpretation of state sensitivities. Moreover, the mismatch in duration can play a significant role depending on the difference in duration and the volatility of the real curve. We therefore estimate yield levels rather than yield spreads.

5.4 Empirical Results

Before we start with a description of the estimation procedure, it is useful to introduce the data used to estimate the inflation and risk premium. We use French securities because the French market is the largest inflation-linked bond market with
5.4. EMPIRICAL RESULTS

the longest trading history in the Eurozone. France is also regarded to be a reference in both issue size and credit rating and is used as a pricing tool for other inflation-linked bond markets. Our data reflects the redemption yields of daily closing price of French inflation-linked bonds for the period September 1998 until July 2004. The longest history of observable prices stems from trading in the OAT 2009 and this bond was issued in September 1998 totaling more than 1500 observations. Unfortunately, other bond series do not have the same number of observations because of later issuance.

5.4.1 Data

The issuance of French inflation-linked bonds has increased over 500% in the past years from less than 10 billion Euro in 1999 to almost 60 billion Euro in June 2004. As a comparison, total debt grew in the same period with less than 10%. The first French issuance of an inflation-linked bond occurred in September 1998 and was a 10-year 3% bond followed by a 3.4% 30-year bond the year after.\(^9\) Table 5.1 provides an overview of French index-linked bonds issued since 1998.

The table shows that almost 60 billion Euro is issued since 1998 from which the largest share are linked to the French price level. The table also shows us that a small but growing real curve is emerging in the Eurozone with maturity of bonds ranging from 5 years to 28 years.\(^{10}\) To get an impression of the buy-side allocation in French real bonds, we provide details of the geographical distribution and the type of investors participating in the January 2004 auction in table 5.2.\(^{11}\) The lion’s share of French index-linked debt is located in the Europe followed by the United States.

\(^9\)French sovereign bonds linked to the French CPI are known by their abbreviation OATI. When these securities are linked to the European CPI excluding tobacco, they are known as OATEI.

\(^{10}\)Next to the French treasury, the Italian and Greece governments have issued inflation-linked bonds since 2003. In 2003 for example, Italy and Greece have issued a 2008 and 2025 bond respectively.

\(^{11}\)Cash-outright bought from primary dealers during the January 2004 auction.
Table 5.1: The French Inflation-Linked Bond Market

Table gives an overview of the amount outstanding in the French inflation-linked Bond market as of June 2004 in billions of Euros. The French OATi are linked to the French index while AOTei are linked to the Eurozone inflation. As a comparison, the total debt (real + nominal) during that period was approximately 820bn Euro’s (from which 529bn Euro’s was long-term debt, i.e. ~2-years).

<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon</th>
<th>Outstanding amount (bn EUR)</th>
<th>Year of issue</th>
</tr>
</thead>
<tbody>
<tr>
<td>OATi 2009</td>
<td>3.00</td>
<td>13.8</td>
<td>1998</td>
</tr>
<tr>
<td>OATi 2011</td>
<td>1.60</td>
<td>3.40</td>
<td>2004</td>
</tr>
<tr>
<td>OATi 2013</td>
<td>2.50</td>
<td>12.4</td>
<td>2003</td>
</tr>
<tr>
<td>OATi 2029</td>
<td>3.40</td>
<td>4.50</td>
<td>1999</td>
</tr>
<tr>
<td>Total OATi</td>
<td></td>
<td>34.1</td>
<td></td>
</tr>
<tr>
<td>OATei 2012</td>
<td>3.00</td>
<td>13.8</td>
<td>2001</td>
</tr>
<tr>
<td>OATei 2020</td>
<td>2.25</td>
<td>5.10</td>
<td>2004</td>
</tr>
<tr>
<td>OATei 2032</td>
<td>3.15</td>
<td>6.80</td>
<td>2002</td>
</tr>
<tr>
<td>Total OATei</td>
<td></td>
<td>25.7</td>
<td></td>
</tr>
</tbody>
</table>

Typical buyers of these securities are pension funds and asset managers although the banking sector also takes a considerable portion of debt into their books. Our data runs from 29 September 1998 until 5 July 2004. The longest time series comes from the July 2009 bond with an annual coupon of 3% and linked to the French consumer price index. In contrast, the bond with the shortest time series in our dataset is a nominal bond (4% April 2014), which started to trade since 11 March 2003. An overview of the bonds and the date of their first observation is reported in table 5.3.

Note first that indexed bonds are not present all over the curve and we cannot construct discount bonds for all maturities. This is especially problematic on the short end of the curve as issuers typically issue indexed bonds at the longer end of the curve. As we can see in table 5.3, a full cross-section sample of bonds starts on 11 March 2003, which yields a total of 345 observations. Moreover, for every
5.4. EMPIRICAL RESULTS

Table 5.2: Buyers and Location of French Real Bonds

Table the geographical distribution of cash outright buyers during the January 2004 auctions. Largest share of French real securities are located in the Eurozone within asset management and pension funds.

<table>
<thead>
<tr>
<th>Geographical allocation</th>
<th>(%)</th>
<th>Investor type</th>
<th>(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eurozone</td>
<td>57</td>
<td>Asset mgmt</td>
<td>31</td>
</tr>
<tr>
<td>Europe (ex Euro)</td>
<td>22</td>
<td>Pension funds</td>
<td>29</td>
</tr>
<tr>
<td>US</td>
<td>13</td>
<td>Banks</td>
<td>20</td>
</tr>
<tr>
<td>Asia</td>
<td>4</td>
<td>Central Banks</td>
<td>10</td>
</tr>
<tr>
<td>Others</td>
<td>4</td>
<td>Hedge funds+others</td>
<td>10</td>
</tr>
</tbody>
</table>

inflation-linked bond, we have a nominal "counterpart". The French treasury agent also uses these nominal bonds to gauge the performance of index-linked bonds. In order to utilize the full sample of our dataset, we assume that all the bonds were available since 29 September 1998 but denote them as missing observations until they start to trade. This gives a total of 11504 observations. In the strictest sense, the time series cannot be regarded as missing observations as they were not issued then. However, if these bonds were traded, the only thing that would change is the maturity of the bond and the dynamics of the term structure for those non-available maturities must still fulfill the equilibrium condition because the states are valid for all maturities. For example, a 10-year nominal bond with 4% coupon in 2003 would behave as a 15-year nominal bond with 4% coupon in 1998. In the Kalman filter, we start with the estimation procedure for the longest observable bond and omit the one-step forecast errors for the bonds which were not issued, i.e. set these to zero and let the state update equals its unconditional expectation. See for example Durbin and Koopman (2001) for the exact estimation procedure in case of missing observations. Table 5.4 reports the mean and standard deviation of nominal and real bonds for every trading day using the cross-section sample of 345 observations. As we can see in this sample, the spread between nominal and real bonds is an
Table 5.3: Observations in dataset

Table gives an overview of the French bonds in our dataset. OATEI's are linked to the Eurozone consumer price index and OATi's are bonds linked to the French consumer Price index. Data spans the period 29 September 1998 until 5 July 2004. Full cross section of bonds available since 11 March 2003 (345 observations).

<table>
<thead>
<tr>
<th>Bond</th>
<th>Maturity</th>
<th>Linkage</th>
<th>1st Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>OATi 3%</td>
<td>07-2009</td>
<td>French CPI</td>
<td>29-09-1998</td>
</tr>
<tr>
<td>OAT 4%</td>
<td>04-2009</td>
<td>Nominal</td>
<td>08-10-1998</td>
</tr>
<tr>
<td>OATi 2.5%</td>
<td>07-2013</td>
<td>French CPI</td>
<td>11-02-2003</td>
</tr>
<tr>
<td>OAT 4%</td>
<td>04-2013</td>
<td>Nominal</td>
<td>11-03-2003</td>
</tr>
<tr>
<td>OATei 3%</td>
<td>07-2012</td>
<td>French CPI</td>
<td>06-11-2001</td>
</tr>
<tr>
<td>OAT 5%</td>
<td>04-2012</td>
<td>Nominal</td>
<td>12-03-2002</td>
</tr>
<tr>
<td>OATi 3.4%</td>
<td>07-2029</td>
<td>French CPI</td>
<td>02-01-2003</td>
</tr>
<tr>
<td>OAT 5.5%</td>
<td>04-2029</td>
<td>Nominal</td>
<td>02-01-2003</td>
</tr>
<tr>
<td>OATei 3.15%</td>
<td>07-2032</td>
<td>EU CPI</td>
<td>02-01-2003</td>
</tr>
<tr>
<td>OAT 5.75%</td>
<td>10-2032</td>
<td>Nominal</td>
<td>02-01-2003</td>
</tr>
</tbody>
</table>

increasing function of the maturity. The nominal curve is also steeper at 140 basis points compared to the 90 basis point spread in the real yield. In addition, the nominal curve is less stable as its volatility is higher than the reported value of the real curve.

5.4.2 Estimation Procedure

In order to estimate the parameters of the model, we use the Kalman filter in combination with Quasi-Maximum Likelihood. The empirical work that follows will use bonds that pay coupons on a regular basis. As a result, the yield of a coupon bond is not a linear function of the state variables and we use the extended Kalman filter to construct an approximate affine structure using a first order Taylor approximation. The reader is referred to Harvey (1993, chapter 4), Hamilton (1994, chapter 13) or Durbin and Koopman (2001) for a detailed description of the Kalman
Table 5.4: Summary Statistics of Real and Nominal French Bonds
Table reports mean and standard deviation of the bonds used in our analysis. Calculations are based Full cross section of bonds available since 11 March 2003 (345 observations).

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Real Mean</th>
<th>Real St.dev</th>
<th>Nominal Mean</th>
<th>Nominal St.dev</th>
<th>Avg.spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>2009</td>
<td>1.657</td>
<td>0.205</td>
<td>3.483</td>
<td>0.228</td>
<td>1.83</td>
</tr>
<tr>
<td>2012</td>
<td>1.995</td>
<td>0.169</td>
<td>4.003</td>
<td>0.198</td>
<td>2.01</td>
</tr>
<tr>
<td>2013</td>
<td>2.116</td>
<td>0.164</td>
<td>4.126</td>
<td>0.190</td>
<td>2.01</td>
</tr>
<tr>
<td>2029</td>
<td>2.573</td>
<td>0.167</td>
<td>4.868</td>
<td>0.174</td>
<td>2.30</td>
</tr>
<tr>
<td>2032</td>
<td>2.557</td>
<td>0.155</td>
<td>4.880</td>
<td>0.171</td>
<td>2.32</td>
</tr>
</tbody>
</table>

filter together with its applications in econometrics. The study of the term structure using state space models have been applied by e.g. Penacchi (1991) and Duan and Simonato (1997).

Note that $\beta_4 \sigma_4$ cannot be identified separately and one can normalize $\beta_4 = 1$ and estimate $\sigma_4$. Unfortunately, this did not work out well as the contribution of $\sigma_4$ to the likelihood function was marginal, i.e. $\beta_4 \sigma_4$ could not be identified separately from expected inflation. We therefore estimate the model by setting $\beta_4 \sigma_4 = 0$ and assume that unexpected inflation is non-stochastic and hence does not require a risk premium. We calculate the factor loadings $(A_1, B_1), \ldots, (A_T, B_{iT})$ using equations (5.23) to (5.26) where $T$ is the maturity date. These values describe the term structure for discount bonds. The coupons and the terms structure enables us to calculate the theoretical yield $y^{(c)}_{k,t}$ of a coupon bond maturing $k$ periods from now. Note that $y^{(c)}_{k,t} = F(z_t)$ where the function $F(\cdot)$ is non-linear in the state variables. If the model is correct, then by no-arbitrage, the true observable yield $y^{(c)}_{k,t-1}$ must
equals its theoretical yield \( \hat{y}_{k,t-1} \) and the state space representation is given by

\[
\begin{align*}
\hat{y}_{k,t+1}^{(c)} &= \hat{y}_{k,t+1}^{(c)} \\
&= F(z_{t+1}) \\
z_{t+1} &= (I - \Theta) \mu + \Theta z_t + Z_t^{1/2} \xi_{t+1} \\
\end{align*}
\]

To deal with the estimation problem, we assume that the observed yield of different maturities \( \hat{y}_{k,t+1}^{(c)} \) are observed with errors \( \varepsilon_{t+1} \) relative to \( \hat{y}_{k,t+1}^{(c)} \). Note that the errors \( \varepsilon_{t+1} \) have an unknown magnitude. If \( F(z_{t-1}) \) is a linear function in the states and the errors are normally distributed with constant variance, our method becomes a maximum likelihood estimation. However, in our case where \( z_{t-1} \) follows a square root process, the method yields an approximate quasi-likelihood function. However, because we are working with coupon bonds, one cannot construct an optimal filter even when \( \varepsilon_{t+1} \) and \( \xi_{t+1} \) are white noise as \( F(z_{t-1}) \) is a non-linear function in the states \( z_{t+1} \). An approximate filter however can be obtained by linearizing the model and applying the usual Kalman filter. If \( F(\cdot) \) is sufficiently smooth, it can be expanded by a first order Taylor expansion around its conditional mean. More specifically, let us denote \( \zeta_t = E(z_t | Y_{t-1}) \) and we have

\[
\begin{align*}
y_t^{(j)} &= F^{(j)}(z_t) \\
&= F^{(j)}(\zeta_t + z_t - \zeta_t) \\
&= F^{(j)}(\zeta_t) + \frac{\partial F^{(j)}}{\partial z'} \bigg|_{z=E(z_t,Y_{t-1})} \times (z_t - \zeta_t) \\
&= \hat{y}_{t,t-1}^{(j)} + B_t^{(j)} \Delta \\
\end{align*}
\]

for \( j = \{ \text{nominal, real} \} \). We dropped the subscript \( k \) for notational convenience, \( \Delta \equiv (z_t - \zeta_t) \) and \( B_t^{(j)} \equiv \frac{\partial F^{(j)}}{\partial z'} \bigg|_{z=E(z_t,Y_{t-1})} \). Note that \( \hat{y}_{t,t-1}^{(j)} \) equals the theoretical yield of the bond evaluated at \( E(z_t | Y_{t-1}) \). The state space representation is finally defined as

\[
\begin{align*}
y_t &= \hat{y}_{t,t-1} + B_t (z_t - \zeta_t) + \varepsilon_t \\
z_{t+1} &= (I - \Theta) \mu + \Theta z_t + Z_t^{1/2} \xi_{t-1} \\
\end{align*}
\]
where $y_t = \left[y_t^{(r)}, y_t^{(n)}\right]'$. $\hat{y}_{t-1} = \left[\hat{y}_{t-1}^{(r)}, \hat{y}_{t-1}^{(n)}\right]'$ and $B_t = \begin{bmatrix} B_t^{(r)} \\ B_t^{(n)} \end{bmatrix}$.

The above procedure is the so-called extended Kalman filter and is an application of the standard Kalman filter to non-linear systems (see Harvey 1990, chapter 3). It is important to recognize that the $\beta, \mu, \sigma$ coefficients are not appearing explicitly. Specifically, these parameters appear in the $A_k^{(r)}, A_k^{(n)}, B_{t,k}^{(r)}$ and $B_{t,k}^{(n)}$ functions for the discount bonds. This implies that the parameters use $B_t$, which is the outcome of a first-order approximation of a non-linear function $F(z_t)$. A description of the Kalman filter procedure for this model can be found in the appendix (point 6) and the optimization procedure is programmed in OX and uses the MaxBFGS algorithm. Several remarks are in order as the proposed econometric method has some important drawbacks. Most importantly, specific assumption must be made with respect to the state dynamics and this will result in an unavoidable increase in the number of parameters to estimate. We therefore assume one inflation premium that drives the observable bond prices although French real bonds are either linked to a French or Eurozone consumer price index. Second, the state variables may be difficult to interpret in a state space framework. Deriving a real curve from index-linked bonds is not as straightforward as with the nominal term structure of interest rates. Evans (1998) however shows that imperfect indexation in UK bonds due to an indexation lag is of lesser importance than time-varying risk premium. Because French bonds have an indexation lag of 3 months (instead of the 8 months for UK bonds), we omit this problem in our approach.

5.4.3 Estimation Results

Because we work with daily data, we have $\Delta z_t = z_t - z_{t-1}$ and henceforth is interpreted as the change in the states from day $t-1$ to day $t$. The estimated parameters are therefore the daily driving factors of the state variables. In order to interpret these values, we also calculate the implied annual value. Before we discuss these val-

\footnote{Strictly speaking this implies the modeling of two inflation premia.}
ues, let us first consider the residuals and the state dynamics. Figure (5.1) reports the annualized state variables. As we see, the implied real interest rate reached a peak around the $1125^{th}$ observation and this corresponds to March 2003 and has been falling since. Moreover, the liquidity factor has been decreasing steadily since the $750^{th}$ observation which corresponds to August 2001. On the other hand, although inflation is relative stable (around a 2% level per annum) it is also slightly increasing since May 2003. Plots of the residuals (in basis points) for the real and nominal bonds are given in figure (5.3) and (5.2) respectively. As we can see in these figures, the average of the residuals is not zero and this implies that the model contains a mis-specification between the observed yields and the theoretical yields (calculated from the model). Overall, the mis-specification is rather small with the mean residual around 0.02 basis points for bonds maturing in 2012 to some 0.2 basis points for bonds maturing in 2029. The same conclusion can be drawn from the residuals of the nominal bonds. In here, the mean residuals runs from 0.03 basis points for the 2013 to some 0.3 basis points for the 2032 bond. Table (5.5) reports the estimation results in basis points and their standard errors using the extended Kalman filter and quasi-maximum likelihood. With respect to the state variables, the annual unconditional expectation of the real interest, the liquidity and expected inflation is calculated as $(1 + \frac{\mu}{100})^{365}$ and approximates 49, 94 and 188 basis points per annum respectively. However, the formula to annualize the unconditional state expectations is valid under the assumption of zero expectation for residuals. We therefore calculate the expectation as the sample average of the state variables. In addition, due to level dependent volatility, the annual variance of the state variables equals

$$var(z_{t+1}) = \sum_{i=0}^{365} \theta^2 var(\sqrt{z_{t-i} \varepsilon_{t-i}})$$

where $var(\sqrt{z_{t-i} \varepsilon_{t-i}})$ is the variance corresponding to day $t-i$. Again, we calculate the annual variance using the sample standard deviation from the state variables.

\[^{13}\text{This result is obtained by multiplying the daily state by 365.}\]
5.4. **EMPIRICAL RESULTS**

Table 5.5: Estimation Results

Table reports the quasi-maximum likelihood estimations of the state space model for daily values. The associated t-statistics are in parenthesis. For convenience, $\mu$ is multiplied by 100. For example, the unconditional expectation for the first state is 0.0013 basis points per day. The reported t-statistics tests $\theta_{praly} = 1$. The optimization procedure is programmed in OX and uses the MaxBFGS algorithm.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value ($\times 100$)</th>
<th>t-statistic</th>
<th>Parameters</th>
<th>Value ($\times 100$)</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.13</td>
<td>8.52</td>
<td>$\sigma_1^2$</td>
<td>0.38</td>
<td>8.89</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.26</td>
<td>5.97</td>
<td>$\sigma_2^2$</td>
<td>0.38</td>
<td>2.37</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0.51</td>
<td>13.97</td>
<td>$\sigma_3^2$</td>
<td>0.27</td>
<td>11.07</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.99996</td>
<td>1.98</td>
<td>$\beta_1$</td>
<td>-3.45</td>
<td>-14.81</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.99994</td>
<td>6.80</td>
<td>$\beta_2$</td>
<td>-3.48</td>
<td>-0.64</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>0.99989</td>
<td>0.47</td>
<td>$\beta_3$</td>
<td>-3.53</td>
<td>-12.11</td>
</tr>
<tr>
<td>$\sigma_1^{(r)}$</td>
<td>0.05</td>
<td>0.21</td>
<td>$\sigma_2^{(a)}$</td>
<td>0.04</td>
<td>1.52</td>
</tr>
</tbody>
</table>

The annual unconditional expectation of real interest rate, liquidity and inflation are 82 basis points, 94 basis points and 203 basis points respectively with a corresponding standard deviation equals 43, 41 and 59 basis points per year. The degree of mean-reversion is a key parameter in determining the patterns of real interest rates and the term structure of yield volatility. The annual mean-reversion term can be calculated using the annual term by $\theta_{praly} = \exp(365 \times \ln(\theta_{dailycl}))$ and equals 0.986, 0.979 and 0.961 respectively. Note that the t-statistic reported in the table test $\theta_{dailycl} = 1$. The parameter $\beta$ enables us to calculate the log-excess return over an one-period bond and can be interpreted as a risk premium. We showed that a smaller parameter implied a larger risk premium. Based on the estimated daily values, we calculate the corresponding risk premium for every bond. In order to compare the premia for each bond, we use the observations for which the complete cross-section is available, i.e. the last 345 observations. The exact procedure goes as follows: given the estimated parameter value and $z_t$ ($t = T - 345, ..., T$), we can
Table 5.6: Risk Premia in French Inflation-Linked Bonds

The real interest ($\lambda_1$), liquidity premium ($\lambda_2$) and inflation ($\lambda_3$) in French bonds in basis points per annum. These premiums are calculated as follows: using the duration values for every available bond and the estimated parameters, we can calculate for every time $t = T - 315, ..., T$ the log excess return using the state variables $z_t$. The average of these excess return is multiplied by 365 to get an annual value.

<table>
<thead>
<tr>
<th>Real Bonds</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1 + \lambda_2$</th>
<th>Contribution $\lambda_2$ to total risk premium (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OATi 3% 2009</td>
<td>105</td>
<td>6</td>
<td>113</td>
<td>5%</td>
</tr>
<tr>
<td>OATi 3% 2012</td>
<td>111</td>
<td>13</td>
<td>124</td>
<td>11%</td>
</tr>
<tr>
<td>OATi 2.5% 2013</td>
<td>121</td>
<td>24</td>
<td>145</td>
<td>17%</td>
</tr>
<tr>
<td>OATi 3.4% 2029</td>
<td>219</td>
<td>22</td>
<td>241</td>
<td>9%</td>
</tr>
<tr>
<td>OATi 3.15% 2032</td>
<td>219</td>
<td>22</td>
<td>241</td>
<td>9%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nominal Bonds</th>
<th>$\lambda_1$</th>
<th>$\lambda_3$</th>
<th>$\lambda_1 + \lambda_3$</th>
<th>$\lambda_3 - \lambda_2 - z_2$</th>
<th>Contribution $\lambda_3$ to total risk premium (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>OAT 4% 2009</td>
<td>106</td>
<td>113</td>
<td>219</td>
<td>76</td>
<td>52%</td>
</tr>
<tr>
<td>OAT 5% 2012</td>
<td>112</td>
<td>133</td>
<td>245</td>
<td>89</td>
<td>54%</td>
</tr>
<tr>
<td>OAT 4% 2013</td>
<td>122</td>
<td>158</td>
<td>280</td>
<td>103</td>
<td>56%</td>
</tr>
<tr>
<td>OAT 5.5% 2029</td>
<td>219</td>
<td>250</td>
<td>469</td>
<td>197</td>
<td>53%</td>
</tr>
<tr>
<td>OAT 5.75% 2032</td>
<td>219</td>
<td>251</td>
<td>470</td>
<td>198</td>
<td>53%</td>
</tr>
</tbody>
</table>

Apply equations (5.23) to (5.29) and calculate the log-excess returns for every day. The average of these excess returns is the daily risk premium for each bond and we multiply this value with 365 to get an annual value. The annualized outcome is reported in table (5.6). In the absence of a liquidity premium, we find an average inflation premium of 113 basis points for bonds maturing in 2009 to some 251 basis points for bonds maturing in 2032. In other words, the price of inflation risk for long-term bonds is more than 2 times higher compared to bonds at the short end of the curve. The finding that long-end bonds have a larger inflation premium does not come as a surprise. Long-term bonds are more sensitive to inflation than short-end bond, which are more sensitive to interest rates. The liquidity premium
in real bonds equals some 6 basis points for bonds maturing in 2009 and is slightly humped shaped with a peak at the 10-year bond. The second last column in table (5.6) reports the contribution of liquidity and inflation on the total log-excess return. The contribution of liquidity on real bonds is relative small and stands at 5% for short-term bonds and 10% for long-term bonds with a peak of 17% in the 10-year bond. On the other hand, the impact of inflation on bond return is much larger for nominal bonds as more than 50% of nominal bond return depends on the inflation premium.

Finally, using the estimated value and equation (5.30), we can calculate expected nominal yield spread $\Delta$ between nominal and real bonds. Because we assumed that unexpected inflation is non-stochastic, we find $\Delta = \Delta_3 - \Delta_2 - \hat{z}_2$ where $\hat{z}_2 = 31$ basis points is the average state value for the last 345 observations. Note that $\Delta$ is depicted in the last column of the table. Specifically, $\Delta$ runs from 76 basis points in the 2009 bonds to some 198 basis points for bonds maturing in 2032. To compare this result with previous studies for the US and UK government bonds, Buraschi and Jiltsov find an inflation risk premium of 70 basis points in 10-year bonds and up to 240 basis points at the long-end of the curve for the US TIPS market. McCulloch and Kochin (1998) found comparable results. Ang and Bekaert (2003) fit nominal data from the US treasury market into a real interest regime-switching framework and find an average inflation premium of 100 basis points in 10-year bonds. For the UK market, Remolona, Wickens and Gong (1998) finds an inflation premium on the UK market of 100 basis points for 2-year maturities while Shen (1998) reports an inflation premium around 75 basis points for bonds with a 10-year maturity up to 104 basis points for bond maturing in 25 years. Our results for $\Delta_3$ suggest that the inflation premium in the long end of the term structure is larger than the premium

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11Note that this number is not totally correct as we neglect the maturity difference between real and nominal bonds.

12Figure 4 of McCulloch and Koching (1998) finds an average annual inflation premium between 160 basis points for 10-year bonds up to some 230 basis points for bonds maturing in 30 years.
for the UK bond market but comparable with US TIPS.

5.5 Conclusions

Although inflation-linked bonds offer a hedge against price movements, they are lacking in liquidity for two reasons. First, although the activity on the primary market has grown over the past number of years, the total size outstanding is still small compared to conventional bonds. Second, and more importantly, inflation-linked bonds are typical buy-and-hold securities as they are kept for the remaining of their life. Hence, yields on nominal bonds contain an inflation premium while yields on inflation-linked bonds contain a liquidity premium. Having a correct understanding of the source of (yield) spread movements is important in order to make decisions with respect to the appropriate trading strategy. We therefore estimate the liquidity and inflation premium in European sovereign bonds using data from nominal and inflation-linked French bonds. The estimation method is set within a state space framework where the state variables follow a discrete Cox, Ingersoll and Ross process. In this framework, the yields of the coupons are a function of the state variables. In order to calculate the yields of coupon-bearing bonds, we linearize the state equations around its one-step conditional forecast and use the Kalman filter for updating the level of the state. This is the so-called extended Kalman filter approach. Our findings are as follows: in the absence of liquidity, the inflation premium runs from 113 basis points to some 250 basis points across the curve. These numbers implies that the inflation premium in long-term bonds is more than 2 times larger compared with the short-end of the curve. These results are somewhat larger than the findings of Shen (1998) and Remolona, Wickens and Gong (1998) for the UK market but comparable with US TIPS. The liquidity premium in real bonds equals some 6 basis points for bonds maturing in 2009 and is slightly humped shaped with a peak at the 10-year bond. If liquidity is taken into consideration, the expected nominal yield spread between nominal and real bonds equals some 15 basis points.
in the short end of the curve but increases to 135 basis points in the long end of the curve. For short-term (long-term) bonds, the liquidity premium accounts for some 5% (10%) of the total real risk premium. On the other hand, the inflation premium is a prominent factor in nominal bonds as it account for more than 50% of the total risk premium across the term structure for nominal interest rates. As a final remark, although the contribution of the liquidity premium to total premium is small, it has a large impact on the expected nominal yield spread through the expected liquidity level. We find that this yield spread is upward sloping as it runs from 76 basis points to some 198 basis points for the 2032 bond.
5.A Appendix to Chapter 5

5.A.1 Appendix: The Real Return on Nominal Bonds

Define the real return on equity by \( Y = f(X) \) where \( X \) are Ito processes. By Ito’s lemma we must have

\[
dY = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i + \frac{\partial f}{\partial t} dt + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j} dX_i dX_j \]

and applying this to \( Y = \frac{B}{P} \) yields

\[
\frac{dY}{Y} = \frac{1}{Y} \left[ \frac{\partial Y}{\partial B} dB + \frac{\partial Y}{\partial P} dP + \frac{1}{2} \left( 2 \frac{\partial^2 Y}{\partial B \partial P} dB dP + \frac{\partial^2 Y}{\partial P^2} dP^2 \right) \right]
\]

\[
= \frac{1}{P} dB - \frac{B}{P^2} dP - \frac{1}{2} dB dP + \frac{B}{P^3} dP^2
\]

\[
= \frac{1}{B} dB - \frac{1}{P} dP - \frac{1}{BP} dB dP + \frac{1}{P^2} dP^2
\]

\[
= (R dt + \sigma_B dW) - (\pi dt + \sigma_P dW) + (\pi dt + \sigma_P dW) (\pi dt + \sigma_P dW)
\]

\[
= (R - \pi + \sigma_P^2) dt + (\sigma_B dW - \sigma_P dW)
\]

as \( dBdP = 0, dWdt = 0 \) and \( dW_W dW_j = \rho_{ij} dt \)

5.A.2 Appendix: Inflation Dynamics

Define a functional form \( Y_t = e^{kt} \pi_t \) and use Ito’s lemma to find

\[
dY_t = e^{kt} d\pi_t + k e^{kt} \pi_t dt
\]

\[
= ke^{kt} \pi_t dt + e^{kt} \sigma_\pi dW_\pi
\]

To solve this stochastic differential equation, one can use

\[
Y_{t+\Delta t} = Y_t + \pi k \int_t^{t+\Delta t} e^{ks} ds + \sigma_\pi \int_t^{t+\Delta t} e^{ks} dW_{\pi,s}
\]

where the latter term is \( N \left( 0, \sigma_\pi \int_t^{t+\Delta t} e^{2ks} ds \right) \). Solving this integral yields

\[
Y_{t+\Delta t} = Y_t + \pi \left( e^{kt+\Delta t} - e^{kt} \right) + \epsilon_t \tag{5.36}
\]
with \( v_t \sim N \left( 0, \frac{\sigma^2}{2k} \left( e^{2kt(t-\Delta t)} - e^{2kt} \right) \right) \). In order to transform this back to its original parameter, we need to post multiply equation (5.36) with \( e^{-k(t-\Delta t)} \) as \( Y_{t-\Delta t} e^{-k(t-\Delta t)} = \pi_{t-\Delta t} \) and this implies

\[
\pi_{t-\Delta t} = Y_{t-\Delta t} e^{-k(t-\Delta t)} \\
= [Y_t + \pi \left( e^{k(t+\Delta t)} - e^{kt} \right) + \varepsilon_t] e^{-k(t+\Delta t)} \\
= \pi_t e^{-k\Delta t} + \pi \left( e^{k(t+\Delta t)} e^{-k\Delta t} - e^{kt} e^{-k\Delta t} \right) + e^{-k(t+\Delta t)} \varepsilon_t \\
= \pi_t e^{-k\Delta t} + \pi \left( 1 - e^{-k\Delta t} \right) + e^{-k(t+\Delta t)} \varepsilon_t \\
= \pi + e^{-k\Delta t} \left( \pi_t - \pi \right) + \varepsilon_t
\]

(5.37)

with \( \varepsilon_t \sim N \left( 0, \frac{\sigma^2}{2k} \left( 1 - e^{-2k\Delta t} \right) \right) .

5.A.3 Appendix: Difference Equations for \( A(\cdot) \) and \( B(\cdot) \) Functions

For convenience, rewrite the equations for the real term structure dynamics

\[
-m_{t+1}^{(r)} = z_{t+1} + \sqrt{z_{t+1} \beta_1 \xi_{t+1}} + \sqrt{z_{t+2} \beta_2 \xi_{t+2}} \\
z_{t+1} = (1 - \theta_1) \mu_1 + \theta_1 z_{t+2} + \sqrt{z_{t+2} \xi_{t+2}} \\
z_{t+2} = (1 - \theta_2) \mu_2 + \theta_2 z_{t+3} + \sqrt{z_{t+3} \xi_{t+3}}
\]

Because we assume an affine structure, we can use a recursive procedure to write the log price of a real discount bond as:

\[
p_{k,t}^{(r)} = E_t \left[ m_{t+1}^{(r)} + p_{k-1,t+1}^{(r)} \right] + \frac{1}{2} \text{var}_t \left[ m_{t+1}^{(r)} + p_{k-1,t+1}^{(r)} \right] \]  

(5.38)

because \( P_{0,t}^{(r)} = 1 \) we must have \( p_{0,t}^{(r)} = 0 \) and hence

\[
p_{1,t}^{(r)} = E_t \left[ m_{t+1}^{(r)} \right] + \frac{1}{2} \text{var}_t \left[ m_{t+1}^{(r)} \right] \\
= -z_{t+1} - z_{t+2} + \frac{1}{2} \left( \beta_1^2 \xi_{t+1}^2 + \xi_{t+2}^2 \beta_2^2 \right) \\
= - (1 - \frac{1}{2} \beta_1^2 \sigma_1^2) z_{1,t} - (1 - \frac{1}{2} \beta_2^2 \sigma_2^2) z_{2,t}
\]
as the moments are given by \( E_t \left[ m_{t-1}^{(r)} \right] = -z_{1,t} - z_{2,t} \) and \( \beta_1^2 \sigma_1^2 z_{1,t} + \beta_2^2 \sigma_2^2 z_{2,t} \). We therefore have

\[
B_{t-1}^{(r)} = 1 - \frac{1}{2} \beta_1^2 \sigma_1^2
\]

\[
A_t^{(r)} = 0
\]

where \( (i = 1, 2) \). For \( k > 1 \), we have

\[
p_{k,t}^{(r)} = E_t \left[ m_{t-1}^{(r)} + p_{k-1,t-1}^{(r)} \right] + \frac{1}{2} \text{var}_t \left[ m_{t+1}^{(r)} + p_{k-1,t-1}^{(r)} \right]
\]

where the expectations are given by

\[
E_t \left[ m_{t-1}^{(r)} \right] = -z_{1,t} - z_{2,t}
\]

\[
E_t \left[ p_{k,t-1}^{(r)} \right] = E_t \left[ -A_k^{(r)} - B_{1,k-1}^{(r)} z_{1,t} - B_{2,k-1}^{(r)} z_{2,t} \right]
\]

\[
= -A_1^{(r)} - B_{1,k-1}^{(r)} \mu_1 + \theta_1 z_{1,t} - B_{2,k-1}^{(r)} \mu_2 + \theta_2 z_{2,t}
\]

and the variances by

\[
\text{var}_t \left[ m_{t-1}^{(r)} \right] = \text{var}_t \left[ z_{1,t} + \sqrt{z_{1,t}^2 + z_{2,t}^2} \right]
\]

\[
= z_{1,t} \beta_1^2 \sigma_1^2 + z_{2,t} \beta_2^2 \sigma_2^2
\]

\[
\text{var}_t \left[ p_{k-1,t-1}^{(r)} \right] = \text{var}_t \left[ -A_k^{(r)} - B_{1,k-1}^{(r)} z_{1,t-1} - B_{2,k-1}^{(r)} z_{2,t-1} \right]
\]

\[
= \left( B_{1,k-1}^{(r)} \right)^2 z_{1,t} \sigma_1^2 + \left( B_{2,k-1}^{(r)} \right)^2 z_{2,t} \sigma_2^2
\]

\[
\text{cov}_t \left[ m_{t-1}^{(r)}, p_{k-1,t-1}^{(r)} \right] = \text{cov}_t \left[ -A_k^{(r)} - B_{1,k-1}^{(r)} z_{1,t-1} - B_{2,k-1}^{(r)} z_{2,t-1} \right]
\]

\[
= z_{1,t} B_{1,k-1}^{(r)} \beta_1 \sigma_1^2 + z_{2,t} \beta_2 \sigma_2^2 B_{2,k-1}^{(r)}
\]

Applying these results to the recursive equation (5.38) yields

\[
p_{k,t}^{(r)} = -A_{k-1}^{(r)} - \sum_{i=1,2} B_{i,k-1}^{(r)} (1 - \theta_i) \mu_i - \sum_{i=1,2} \left( 1 + \theta_i B_{i,k-1}^{(r)} - \frac{1}{2} \left( \beta_i + B_{i,k-1}^{(r)} \right)^2 \sigma_i^2 \right) z_{i,t}
\]
and therefore

\[
A_k^{(r)} = A_{k-1}^{(r)} + \sum_{i=1,2} B_{i,k-1}^{(r)}(1 - \theta_i)\mu_i
\]

\[
B_{i,k}^{(r)} = 1 + \theta_i B_{i,k-1}^{(r)} - \frac{1}{2} \left( \beta_i + B_{i,k-1}^{(r)} \right)^2 \sigma_i^2
\]

The same approach can be applied to nominal bonds. However, due to an additional element in the pricing kernel (unexpected inflation) of the nominal bonds, we have

\[
P_{1,t}^{(n)} = E_t \left[ m_{t+1}^{(n)} \right] + \frac{1}{2} \text{var}_t \left[ m_{t+1}^{(n)} \right]
\]

\[
= - \left( 1 - \frac{1}{2} \beta_1^2 \sigma_1^2 \right) z_{1,t} - \left( 1 - \frac{1}{2} \beta_3^2 \sigma_3^2 \right) z_{3,t} + \frac{1}{2} \beta_4^2 \sigma_4^2
\]

Again, the affine structure permits us to write the log price as

\[
P_{n,t}^{(n)} = -A_{n,t} - B_{1,n}z_{1,t} - B_{2,n}z_{2,t} \text{ or}
\]

\[
B_{i,1}^{(n)} = 1 - \frac{1}{2} \beta_i^2 \sigma_i^2
\]

\[
A_1^{(n)} = \frac{1}{2} \beta_4^2 \sigma_4^2
\]
For $k > 1$ we have

$$E_t \left[ m_{t-1}^{(n)} \right] = -z_{1,t} - z_{3,t}$$

$$E_t \left[ p_{k,t-1}^{(n)} \right] = E_t \left[ -A_1^{(n)} - B_{1,k-1}^{(n)} z_{1,t} - B_{3,k-1}^{(n)} z_{3,t} \right]$$

$$= -A_1 - B_{1,k-1} \left[ (1 - \theta_1) \mu_1 + \theta_1 z_{1,t} \right] - B_{3,k-1} \left[ (1 - \theta_3) \mu_3 + \theta_3 z_{3,t} \right]$$

$$\text{var}_t \left[ m_{t-1}^{(n)} \right] = \text{var}_t \left[ z_{1,t} + \sqrt{z_{1,t}} \beta_1 \xi_{1,t-1} + z_{3,t} + \sqrt{z_{3,t}} \beta_3 \xi_{3,t-1} + \beta_4 \xi_{4,t-1} \right]$$

$$= z_{1,t} \beta_1^2 \sigma_1^2 + z_{3,t} \beta_3^2 \sigma_3^2 + \beta_4^2 \sigma_4^2$$

$$\text{var}_t \left[ p_{k,t-1}^{(n)} \right] = \text{var}_t \left[ -A_{k-1}^{(n)} - B_{1,k-1}^{(n)} z_{1,t+1} - B_{3,k-1}^{(n)} z_{3,t+1} \right]$$

$$= \left( B_{1,k-1}^{(n)} \right)^2 z_{1,t} \sigma_1^2 + \left( B_{3,k-1}^{(n)} \right)^2 z_{3,t} \sigma_3^2$$

$$\text{cov}_t \left[ m_{t-1}^{(n)}, p_{k,t-1}^{(n)} \right] = \text{cov}_t \left[ -z_{1,t} - \sqrt{z_{1,t}} \beta_1 \xi_{1,t+1} - z_{3,t} - \sqrt{z_{3,t}} \beta_3 \xi_{3,t+1} - \beta_4 \xi_{4,t+1} \right]$$

$$= z_{1,t} \beta_1 B_{1,k-1} \sigma_1^2 + z_{3,t} \beta_3 B_{3,k-1} \sigma_3^2$$

Applying these results yields

$$p_{k,t}^{(n)} = -A_{k-1}^{(n)} + \frac{1}{2} \beta_3^2 \sigma_3^2 - \sum_{i=1,3} B_{i,k-1}^{(n)} (1 - \theta_i) \mu_i$$

$$- \sum_{i=1,3} \left[ 1 + B_{i,k-1}^{(n)} \theta_i - \frac{1}{2} \left( \beta_i + B_{i,k-1}^{(n)} \right)^2 \sigma_i^2 \right] z_{i,t}$$

and hence

$$A_k^{(n)} = A_{k-1}^{(n)} - \frac{1}{2} \beta_3^2 \sigma_3^2 + \sum_{i=1,3} B_{i,k-1}^{(n)} (1 - \theta_i) \mu_i$$

$$B_{i,k}^{(n)} = 1 + B_{i,k-1}^{(n)} \theta_i - \frac{1}{2} \left( \beta_i + B_{i,k-1}^{(n)} \right)^2 \sigma_i^2$$
5.A.4 Appendix: Log-Excess Returns and Risk Premium

Under our normality assumption, the equilibria condition is equivalent to

\[
0 = E_t \left[ m_{t-1}^{(j)} + r_{k,t-1}^{(j)} \right] + \frac{1}{2} var_t \left[ m_{t-1}^{(j)} + r_{k,t-1}^{(j)} \right] \\
= E_t \left( m_{t-1}^{(j)} \right) + E_t \left( r_{k,t-1}^{(j)} \right) + \frac{1}{2} var_t \left[ m_{t-1}^{(j)} \right] + \frac{1}{2} var_t \left[ r_{k,t-1}^{(j)} \right] + \frac{1}{2} cov_t \left[ m_{t-1}^{(j)}, r_{k,t-1}^{(j)} \right]
\]

\begin{equation}
\text{(5.39)}
\end{equation}

where \( m_{t-1}^{(j)} \equiv \log M_{t-1}^{(j)} \) and \( r_{k,t-1}^{(j)} \equiv \log Q_{k-1,t-1}^{(n)} - \log Q_{k,t}^{(n)} = q_{k-1,t-1}^{(j)} - q_{k,t}^{(j)} \). Equation (5.39) must hold for every maturity and we can therefore write the difference between a \( k \)-period bond and an one-period bond as

\[
0 = E_t \left( r_{k,t-1}^{(j)} - r_{1,t-1}^{(j)} \right) + \frac{1}{2} \left[ var_t \left( r_{k,t-1}^{(j)} \right) - var_t \left( r_{1,t-1}^{(j)} \right) \right] + \text{cov}_t \left[ m_{t-1}^{(j)}, r_{k,t-1}^{(j)} \right] - \text{cov}_t \left[ m_{t-1}^{(j)}, r_{1,t-1}^{(j)} \right]
\]

\begin{equation}
\text{(5.40)}
\end{equation}

It is important to realize that

\[
\text{cov}_t \left[ m_{t-1}^{(j)}, r_{1,t-1}^{(j)} \right] = var_t \left( r_{1,t}^{(j)} \right) = 0 \text{ as } E_t \left( r_{1,t}^{(j)} \right) = y_{1,t}
\]

and therefore non-stochastic conditioned on time \( t \). This means that the log excess return of a \( k \)-period bond over a 1-period bond is given by

\[
E_t \left( r_{k,t-1}^{(j)} \right) - y_{1,t} = -\frac{1}{2} var_t \left( r_{1,t}^{(j)} \right) - \text{cov}_t \left[ m_{t-1}^{(j)}, r_{k,t-1}^{(j)} \right]
\]

\begin{equation}
\text{(5.41)}
\end{equation}

Using the definitions of the affine structure model we can write \( r_{k,t-1}^{(j)} \) as

\[
\begin{align*}
-\sum_{i=1}^{1} \left( z_{t} - \frac{1}{2} \left( \beta_1 + B_{k,t-1}^{(j)} \right)^2 \sigma_t^2 z_{t,i} \right) \\
- \frac{1}{2} \beta_t^2 \sigma_t^2 + \sum_{i=1}^{1} \left( z_{t} - \frac{1}{2} \left( \beta_1 + B_{t,k-1}^{(n)} \right)^2 \sigma_t^2 z_{t,i} \right) \\
- \sum_{i=1}^{1} \left( B_{k,t-1}^{(n)} \right)^2 \sigma_t^2 z_{t,i}
\end{align*}
\]

\begin{equation}
\text{(5.42)}
\end{equation}

when \( j = r \)

when \( j = n \)

\[
\begin{align*}
- \sum_{i=1}^{1} \left( z_{t} - \frac{1}{2} \left( B_{k,t-1}^{(n)} \right)^2 \sigma_t^2 z_{t,i} \right) \\
- \sum_{i=1}^{1} \left( B_{k,t-1}^{(n)} \right)^2 \sigma_t^2 z_{t,i}
\end{align*}
\]

when \( j = n \)

where \( z_{t} \) is the log price of the \( k \)-period bond and \( z_{t,i} \) is the log price of the \( i \)-period bond.
The expressions in equation (5.12) tells us that this excess return is solely depending on \( B_{i,k-1}^{(j)} \). Therefore, the variances of \( r_{k,t-1}^{(j)} \) are given by

\[
\text{var}_t \left( r_{k,t-1}^{(j)} \right) = \begin{cases} \sum_{i=1}^{n} \left( B_{i,k-1}^{(j)} \right)^2 z_{i,t} \sigma_i^2 & \text{when } j = r \\ \sum_{i=1}^{n} \left( B_{i,k-1}^{(j)} \right)^2 z_{i,t} \sigma_i^2 & \text{when } j = n \end{cases}
\]

\[
\text{cov}_t \left( r_{k,t-1}^{(j)} \right) = \begin{cases} - \sum_{i=1}^{n} B_{i,k-1}^{(j)} z_{i,t} \sigma_i^2 & \text{when } j = r \\ - \sum_{i=1}^{n} B_{i,k-1}^{(j)} z_{i,t} \sigma_i^2 & \text{when } j = n \end{cases}
\]

which means that the log excess return can be written as

\[
E_t \left( r_{k,t-1}^{(j)} \right) - y_{1,t} = \begin{cases} - \frac{1}{2} \sum_{i=1}^{n} \left( \left( B_{i,k-1}^{(j)} \right)^2 z_{i,t} \sigma_i^2 + 2B_{i,k-1}^{(j)} \beta z_{i,t} \sigma_i^2 \right) & \text{when } j = r \\ - \frac{1}{2} \sum_{i=1}^{n} \left( \left( B_{i,k-1}^{(j)} \right)^2 z_{i,t} \sigma_i^2 + 2B_{i,k-1}^{(j)} \beta z_{i,t} \sigma_i^2 \right) & \text{when } j = n \end{cases}
\]

5.A.5 Appendix: The Real Interest Rate

For notational convenience, we assume that \( \beta_4 = 0 \). The result stays the same as the unexpected inflation factor is independent of the real interest rate and only added to the spread and henceforth does not change the interpretation of the argument. To see the error \( I \) that arises when we match two yields with different durations in our case, consider a nominal bond with duration \( k \) and a real bond with duration \( k^* \). If we falsely set the duration of a real bond at \( k \) (while in reality \( k^* > k \), the error equals

\[
I = \left[ y_{k,t}^{(n)} - y_{k,t}^{(r)} \right] - \left[ y_{k,t}^{(n)} - y_{k^*,t}^{(r)} \right] = y_{k^*,t}^{(r)} - y_{k,t}^{(r)} = \frac{1}{k^*} A_{k^*,t}^{(r)} - \frac{1}{k} A_{k,t}^{(r)} + \sum_{j=1}^{k^*} \left( \frac{1}{k^*} B_{j,k^*,t}^{(r)} - \frac{1}{k} B_{j,k,t}^{(r)} \right) z_{j,t} \quad (5.43)
\]
Using the difference equations (5.24) and (5.26), one can relate this error to the parameters of the model. If \( k^* = k + \Delta > k \), one can show that

\[
\frac{1}{k^*} A_k^{(r)} - \frac{1}{k} A_k^{(r)} = \left( \frac{1}{k^*} - \frac{1}{k} \right) A_k^{(r)} + \frac{1}{k^*} \sum_{j=1,2} \sum_{h=1}^{\Delta} B_{j,k}^{(r)} (1 - \theta_j) \mu_j \quad (5.44)
\]

\[
\frac{1}{k^*} B_{j,k}^{(r)} - \frac{1}{k} B_{j,k}^{(r)} = \left( \frac{1}{k^*} \theta_j^2 - \frac{1}{k} \right) B_{j,k}^{(r)} \quad + \frac{1}{k^*} \sum_{h=1}^{\Delta} \left( \theta_j^h - 1 \right) \left( 1 - \frac{1}{2} \left( \beta_j + B_{j,k}^{(r)} \right)^2 \sigma_1^2 \right) \quad (5.45)
\]

and this is clearly a function of the real rate parameters. The mismatch in duration can be ignored however if \( \sigma_1 \to 0 \) for a sufficiently large \( k \). For example, let us consider the limit case where \( \sigma_1^2 = 0 \) and \( k \to \infty \), which implies \( z_{1,t-1} = z_{1,t} \) and hence \( \theta_1 = 1 \). We have

\[
B_{1,k}^{(r)} = \sum_{h=0}^{h=k-2} \theta_1^h \frac{1}{1 - \theta_1} \quad (5.46)
\]

as \( |\theta| < 1 \) and the error as depicted in equation (5.45) goes to zero. In addition, the second term in equation (5.44) can be written as \( \frac{1}{k^*} \sum_{i=1}^{\Delta} B_{1,k}^{(r)} (1 - \theta_1) \mu_i \frac{\Delta}{k^*} \to 0 \) while

\[
A_k^{(r)} = A_0^{(r)} + \sum_{i=1}^{\Delta} \sum_{h=1}^{k} B_{i,k}^{(r)} (1 - \theta_i) \mu_i
\]

\[
= A_0^{(r)} + (1 - \theta_2) \mu_2 \sum_{h=1}^{k} B_{2,k}^{(r)} \quad (5.47)
\]

The total error term is therefore given by

\[
y_{k^*,t}^{(r)} - y_{k,t}^{(r)} = \frac{1}{k^*} A_k^{(r)} - \frac{1}{k} A_k^{(r)} + \left( \frac{1}{k^*} B_{2,k}^{(r)} - \frac{1}{k} B_{2,k}^{(r)} \right) z_{2,i}
\]

\[
= (1 - \theta_2) \mu_2 \sum_{h=1}^{k} B_{2,k}^{(r)} + \left( \frac{1}{k^*} B_{2,k}^{(r)} - \frac{1}{k} B_{2,k}^{(r)} \right) z_{2,i} \quad (5.48)
\]

and therefore independent from the real state parameters \( (\mu_1, \beta_1, \theta_1, \sigma_1) \). The error as depicted in equation (5.48) depends solely on the parameters of the second state (liquidity) and not on third state (inflation) because a real bond was falsely matched to a nominal
bond ($k^*$ was fixed at $k$). If the maturity of a nominal bond was falsely matched to the maturity of a real bond (i.e. $k$ was fixed at $k^*$), the error would be described by the third state rather than the second state.

5.A.6 Appendix: A State Space Framework

Once we linearize the model, we can use the Kalman filter for the Gaussian linear model as depicted in equations (5.34) and (5.35). The Kalman filter recursion is a set of equations that allows an estimator to be updated once a new observation becomes available. The Kalman filter gives an optimal prediction of the (unobserved) state variables using the previously estimated values. The estimates for these state variables are then updates using the information from the observed yields. The by-product of this procedure is the prediction error and can be used to evaluate the likelihood-function. For notational convenience, let us restate the model in here as

$$y_t = \hat{y}_{t-1} + B_t (z_t - \zeta_t) + \varepsilon_t$$

$$z_{t-1} = (I - \Theta) \mu + \Theta z_t + R_t \xi_{t-1}$$

(5.49)

(5.50)

where $\varepsilon_t \sim N(0, H), \xi_{t+1} \sim N(0, Q_t), R_t = \sqrt{z_t} \Gamma_t, z_t \sim N(\mu, P_t)$ and $\zeta_t = E(z_t|Y_{t-1})$. In here we set the initial values of the state to its expectation. Because we are working with normal distributed residuals, the subsets of variables given the information set $Y_{t-1} = (y_1, \ldots, y_{t-1})$ is also normal distributed and we have to find the conditional distribution of $z_{t-1}$ given $Y_t$ which is determined by $\zeta_{t-1}$ and $P_{t-1} = var(z_{t-1}|Y_t) = var(\zeta_{t-1})$. Since

$$z_{t-1} = (I - \Theta) \mu + \Theta z_t + z_t^{1/2} \xi_{t-1}$$

We have

$$\zeta_{t-1} = E(z_{t-1}|Y_t)$$

$$= (I - \Theta) \mu + \Theta E(z_t|Y_t)$$

(5.51)

$$P_{t-1} = var(z_{t-1}|Y_t)$$

$$= \Theta var(z_t|Y_t) \Theta^t + R_t Q_t R_t^t$$

(5.52)
The one-step forecast of \( y_t \) given \( Y_{t-1} \) is given by \( v_t \) and equals

\[
v_t = y_t - E(y_t | Y_{t-1}) = y_t - \hat{y}_{t|t-1}
\]  

(5.53)

The first key recursion for the Kalman filter is given by

\[
E(z_t | Y_t) = E(z_t | Y_{t-1}, v_t) = E(z_t | Y_{t-1}) + \text{cov}(z_t, v_t) [\text{var}(v_t)]^{-1} v_t = \zeta_t + K_t F_t^{-1} v_t
\]

(5.54)

where

\[
K_t = \text{cov}(z_t, v_t) = E \left[ E \left( z_t (y_t - \hat{y}_{t|t-1})' | Y_{t-1} \right) \right] = E \left[ E \left( z_t (z_t - \zeta_t)' B_t' | Y_{t-1} \right) + E (z_t \varepsilon_t | Y_{t-1}) \right] = E \left[ E \left( z_t (z_t - \zeta_t)' B_t' | Y_{t-1} \right) \right] = \zeta_t + K_t F_t^{-1} v_t
\]

(5.55)

\[
F_t = \text{var}(v_t) = B_t P_t B_t' + H_t
\]

(5.56)

Substituting equation (5.54) into (5.51) gives us

\[
\zeta_{t+1} = (I - \Theta) \mu + \Theta E(z_t | Y_t) = (I - \Theta) \mu + \Theta \zeta_t + \Theta K_t F_t^{-1} v_t
\]

(5.57)

Moreover, the second key recursion in the Kalman filter tells us that

\[
\text{var}(z_t | Y_t) = \text{var}(z_t | Y_{t-1}, v_t) = \text{var}(z_t | Y_{t-1}) + \text{cov}(z_t, v_t) [\text{var}(v_t)]^{-1} \text{cov}(z_t, v_t)' = P_t - K_t F_t^{-1} K_t'
\]

(5.58)

and substituting this equation back into (5.52) yields

\[
P_{t-1} = \Theta P_t \Theta' - \Theta K_t F_t^{-1} K_t' \Theta' + R_t Q_t R_t'
\]

(5.59)

In here, equations (5.53),(5.55),(5.56),(5.57) and (5.59) constitutes the Kalman filter which we estimate in combination with the quasi-maximum likelihood procedure.
5.B Graphs Chapter 5

Figure 5.1: The state dynamics of the model. Recall that real interest is a common factor driving the price kernel of both nominal and real bonds. A liquidity premium drives the price kernel of real bonds while an inflation premium drives the price kernel of nominal bonds.
Figure 5.2: Graphics reports the residuals of the nominal bonds in basis points.

Figure 5.3: Graphics reports the residuals of the real bonds in basis points.