Asset prices, traders' behavior, and market design

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Asset Prices, Traders' Behavior and Market Design*

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Abstract

The dynamics in a financial market with heterogeneous agents is analyzed under different market architectures. We start with a tractable behavioral model under Walrasian market clearing and simulate it under more realistic trading protocols. The key *behavioral* feature of the model is the switching of agents between simple forecasting rules on the basis of fitness measure. Analyzing the dynamics under order-driven protocols we show that behavioral and structural assumptions of the model are closely intertwined. High responsiveness of agents to a fitness measure causes excess volatility, however the frictions of the order-driven markets may stabilize the dynamics. We also analyze and compare allocative efficiency and time series properties under different protocols.

*An earlier version of this paper was prepared while Mikhail Anufriev was visiting UNSW, whose hospitality he gratefully acknowledges. We wish to thank all the participants of the Symposium on Agent-Based Computational Methods in Economics and Finance in Aalborg, of the Workshop on "Complexity in Economics and Finance" in Leiden, and of the seminars in Amsterdam, Pisa, Sydney and Trieste for useful comments and stimulating questions. We especially thank Giulio Bottazzi, Buz Brock, Cars Hommes and Florian Wagener for providing very detailed comments during different stages of the work. We acknowledge financial support from the Complex Markets E.U. STREP project 516446 under FP6-2003-NEST-PATH-1. The usual disclaimers apply.

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JEL codes: G12, D44, D61, C62.

Keywords: Asset pricing model, Heterogeneous Beliefs, Learning, Trading protocols, Market Architecture

1 Introduction

Models of financial markets often assume the simplistic mechanism for a market clearing: the Walrasian scenario. This observation applies also to an innovative research area of heterogeneous agent models, in which the heterogeneity in expectations of traders is a key to explain the properties of markets. In reality, however, markets are functioning in a different way. Agents are allowed to transmit only finite amount of information in form of the orders to buy or sell. Furthermore, many markets employ continuous trade in the form of sequential orders. In this paper we study the impact of the market organization on dynamical properties of the asset pricing model populated by adaptive, boundedly rational agents with heterogeneous forecasting rules. We demonstrate that the adaptive abilities of the agents can be impaired by frictions inherent in the order-driven mechanisms. Surprisingly, the price dynamics can be stabilized via this channel. We also analyze how the market efficiency and statistical properties of prices are affected by this interplay of behavioral and institutional assumptions.

Statistical properties of real financial data have been thoroughly investigated in the past, see e.g. Fama (1970), Pagan (1996), Brock (1997) and Cont (2002). This line of research established a number of regularities in financial data, so-called "stylized facts", many of which are observed universally in all time periods and on different stock exchanges. Some of these regularities, e.g. absence of significant autocorrelations in price returns, are well in agreement with the prevailing theory called the Efficient Market Hypothesis which suggests that the markets are informationally efficient with asset prices immediately reflected a new information. At the same time, such regularities as large and persistent trading volume, significant positive autocorrelations in variance of returns (volatility clustering), heavier than normal tails of the return distribution are left unexplained within the classical paradigm. A seminal paper of Shiller (1981) detected that asset prices are more volatile than underlying fundamentals. The discovered excess volatility undermined a completeness of the Efficient Market Hypothesis.

Explaining these empirical properties by means of a simple theoretical model is an important but not a simple task and there are different directions to deviate from the classical paradigm with rational, representative agent (see e.g. Lucas, 1978) in a hope to accomplish this goal. One way is to acknowledge that rationality assumption is too demanding in a complex environment of financial markets. Models with heterogeneous agents using some bounded rational procedure as proposed in Sargent (1993) and Evans and Honkapohja (2001) can be more appropriate. A number of agent-based simulations of markets and more rigorous analytical "Heterogeneous Agent Models" (HAMs) have been developed, where agents with different expectations may coexist in one market.[1] If one group of agents, *fundamentalists*, believe that price typically reflects a fundamental value, while another group, *chartists*, extrapolate price trends, then the prices in a market can deviate from the fundamental value when chartists are in a majority. In Brock and Hommes (1998) this simple story is augmented by the evolutionary dynamics of relative fractions of fundamentalists and chartists. In such "Adaptive Belief System" agents not only update their forecasts as new data become available, but also switch from one forecasting technique to another depending on their past performances. Gaunersdorfer et al. (2008) show that even a simple version of such adaptive model can generate dynamics with some realistic properties. Since extrapolative expectations of chartists can be self-confirming, prices can deviate from the fundamental level even in the absence of considerable fundamental news. Thus, dynamics exhibit an excess volatility. Furthermore, for certain parameter values the underlying deterministic system possesses two attractors, the fundamental steady state and a cycle around it, with small volatility on the former and high volatility on the latter. When dynamic noise is added to the system, price trajectory can interchangeably visit basins of these two attractors generating volatility clustering. Gaunersdorfer and Hommes (2007) show that with sufficiently large level of noise this model indeed generates a dynamics qualitatively similar with real markets.

Alternatively, one can focus on the market design as a possible origin of stylized facts.

[1]Santa Fe artificial market introduced in Arthur et al. (1997) and the model of microscopic simulations in Levy et al. (2000) are two known examples of computational approach focused on bounded rationality in expectation formation. They are accompanied by parsimonious models in Day and Huang (1990), Lux (1995), Brock and Hommes (1998), Farmer and Joshi (2002), Diks and van der Weide (2005), Anufriev et al. (2006) and Anufriev and Dindo (2007). See LeBaron (2006) and Hommes (2006) for recent reviews focused, respectively, on computational and analytical models with heterogeneous agents.

Most of the classical models (with notable exceptions in Kyle, 1985 and Glosten and Milgrom, 1985) and all the HAMs quoted above use the Walrasian market clearing. It may be the case, however, that specific design features of the real markets bring some structure into the data. LiCalzi and Pellizzari (2003) show that an artificial market with realistic architecture, namely an order-driven market under electronic book protocol, is capable of generating satisfactory statistical properties of price series (e.g. leptokurtosis of the returns distribution) even with minimal behavioral assumptions. Furthermore, simulations in Bak et al. (1997) and Maslov (2000) suggest that desirable distributional properties can arise in the order-driven market even in the absence of any behavioral assumptions on the side of the agents.

These two approaches disentangle behavioral and structural assumptions and, therefore, may provide only partial explanation of statistical regularities of financial markets. As opposed to those studies, recent agent-based models in Chiarella and Iori (2002), LeBaron and Yamamoto (2006) and Chiarella et al. (2007) incorporate the agents' heterogeneity in the order-driven markets. But an interplay between behavioral and structural assumptions is far from trivial in these models, so that they suffer from the "curse of complexity", when it becomes virtually impossible to understand how the two sets of assumptions contribute to the models' results. Consequently, our approach in this paper will be to start with a parsimonious model, which is analytically tractable under the Walrasian market clearing, and then increase the complexity by adding more realistic, order-driven trading protocols. The latter versions of the model is investigated through computer simulations.

Our research strategy is largely inspired by the work of Bottazzi et al. (2005). Motivated by an empirical evidence from the world's stock exchanges that market micro-structure does influence statistical properties of returns, they compare dynamics under different trading protocols when two types of traders, chartists and noise traders, act in a market in fixed proportions. Bottazzi et al. (2005) conclude that market architecture plays larger role in shaping the time series properties than behavioral aspects of the model. The authors also analyze the allocative efficiency of the market and show that, as opposed to the time series properties, it depends mainly on the traders' behavior.

This paper is focused on the similar questions. We start, however, with a model built in the Adaptive Belief System of Brock and Hommes (1998). Thus, populational ecology consists of fundamentalists and trend-followers whose proportions are evolving on the basis of difference in past profits. A key behavioral parameter of the model is the intensity of choice, measuring the sensitivity of agents to this difference. If the market clears in the Walrasian way and the number of agents approach infinity, the model is approximated by the deterministic model similar to the one analyzed in Gaunersdorfer et al. (2008). With our choice of forecasting rules, there exist two regimes in the market, tranquil and volatile. When the intensity of choice is low, i.e. smaller than a certain critical value, there is no excess volatility and prices tranquilly stay on the fundamental level in the absence of the dividend payments. When the intensity of choice is high, i.e. larger than this critical value, the volatile regime occurs with persistent deviations of prices from the fundamental level and excess volatility is observed.

Our simulations reveal that similar two regimes are displayed also under two order-driven trading protocols, i.e. the batch auction and the order book. Interestingly, the critical value of the intensity of choice is higher in the order-driven markets, implying larger region of market tranquility. Given a noisy nature of the order-driven trade, it is surprising, but it can be well explained by the interplay of our behavioral assumptions and the market design. We also compare the properties of market dynamics over different market mechanisms, and show, in particular, that an order-driven trade brings volatility clustering to the model.

The paper is organized as follows. In the next Section we briefly describe different market mechanisms and introduce behavioral part of our model. In Section 3 we analyze the model for a simple case of Walrasian market with large number of agents and explain how two different market regimes arise. We then proceed by introducing the details of our implementation of different market mechanisms in Section 4. Results of simulations are discussed in Section 5. Section 6 provides some final remarks.

2 The Model

We consider a standard asset-pricing model with two assets. The numéraire of the economy is the elastically supplied riskless asset which yields constant gross return $R = 1 + r$ per period. The risky asset pays a random dividend y_t in the beginning of period t. Realizations of dividend are independently drawn from some distribution with positive support and mean $\bar{y}$. The fundamental price of the risky asset is defined as a discounted value of the expected dividends and equal to $p^f = \bar{y}/r$. The risky asset is traded in the market, and its actual price dynamics is influenced both by evolution in the demand/supply of traders and by the precise mechanism for price determination. The following three trading protocols will be compared.

Under *Walrasian market-clearing* (WA), agents submit complete demand and supply schedules, and the price of the risky asset p_t in period t is defined as an intersection of a sum of the individual demand curves with a sum of the individual supply curves. In a market organized as a *batch auction* (BA) agents simultaneously post the buy or sell orders [2]. Cumulative demand and supply curves are then derived, and the price of the risky asset p_t is an intersection of these curves. In the third market type the agents submit orders sequentially during the trading session, and the matching is accommodated by an electronic *order book* (OB) which stores unsatisfied orders. If submitted order finds a matching order of the opposite type in the book, it is satisfied (completely or partially). An unsatisfied part of the order is stored in the book. In such a market there is no unique price during period t, and notation p_t is used to denote the closing price in this market, i.e. the price of the last transaction.

These three mechanisms are interesting because they range from the settings preferred in theoretical literature to the protocols used in real markets. Moreover, they differ in a number of dimensions, such as information required from the traders to be submitted and timing of order submission. Indeed, the WA is a standard theoretical tool to model the market clearing process. However, it requires an infinite information from the agents, and therefore is not implementable in practice [3]. Architecture of the BA overcomes this problem, as agents have

[2] This mechanism is sometimes referred to as a *call auction* or a *sealed-bid auction* in the literature.

[3] The English clock auction with inter-period bids can provide a close approximation to the WA.

to submit only finite number of orders. It is used in a number of exchanges, typically to define a starting price of a trading session. Nowadays, however, most of the exchanges are using the OB mechanism as more efficient for continuous trading.

For comparison, the agents' behavior will be modeled in a similar way under these three institutional market settings. We populate the market by N myopic expected utility maximizers whose demand functions depend on expectations of next period price. The demands of agents are not homogeneous because there are two rules to form the expectations. *Fundamentalists* compute the fundamental value and expect that the price will move towards it. *Trend-followers* are less sophisticated, they simply extrapolate the past price changes. Relative fractions of fundamentalists and trend-followers affect, of course, the price determination in a given trading session. These fractions, in turn, are changing between trading sessions and depend on the relative past performances of the two groups using different rules. As a performance measure we take an average return earned by fundamentalists (trend-followers) during the last trading session.

In the remaining of this Section we explain how the demand of agents is defined and then introduce the evolutionary dynamics in the model.

2.1 Agents demand

Agents are risk-averse expected utility maximizers with common risk aversion coefficient a. Let $A_{i,t}$ and $B_{i,t}$ denote, respectively, the number of the risky and the riskless asset possessed by agent i at time t. In order to obtain the optimal portfolio composition, agent i maximizes at time t the conditional expectation of negative exponential utility of next period wealth $W_{i,t+1}$. The wealth is uncertain both because the market price of the risky asset may change and also because the random dividend is paid. Agent i solves the following problem

$$\max_{A_{i,t},B_{i,t}} \left\{E_{i,t}[-\exp(-a_i W_{i,t+1})]\right\} \tag{1}$$

subject to

$$
\begin{aligned}
W_{i,t+1} &= A_{i,t}\big(p_{t+1} + y_{t+1}\big) + B_{i,t}(1+r)\,, \\
W_{i,t} &= A_{i,t}\,p + B_{i,t}\,.
\end{aligned}
\tag{2}
$$

The notation $E_{i,t}$ in (1) stresses the fact that the expectation is conditional on the information available at the beginning of time t and that the expectation is agent-specific. From the constraints (2) the wealth evolution is derived

$$
W_{i,t+1} = W_{i,t}(1+r) + A_{i,t}\big(p_{t+1} + y_{t+1} - (1+r)p\big)\,.
$$

Assuming conditional normality of the wealth at time $t+1$, the above optimization problem is equivalent to the mean-variance optimization

$$
\max_{A_{i,t}} \Big\{ A_{i,t} E_{i,t}\big[\big(p_{t+1} + y_{t+1} - (1+r)p\big)\big] - \frac{a}{2} A_{i,t}^2\, V_{i,t}[p_{t+1} + y_{t+1}] \Big\}\,,
$$

where $V_{i,t}[p_{t+1} + y_{t+1}]$ stands for the conditional expectations of trader i about the variance of price cum dividend at time $t+1$. The first-order condition gives the standard demand function for the risky asset

$$
A_{i,t}(p) = \frac{E_{i,t}[p_{t+1} + y_{t+1}] - (1+r)\,p}{a\,V_{i,t}[p_{t+1} + y_{t+1}]}\,. \tag{3}
$$

As in Brock and Hommes (1998) we assume that traders have homogeneous and time-invariant expectations about conditional variance $V_{i,t}[p_{t+1} + y_{t+1}] = \sigma^2$ and share correct expectations about dividend $E_{i,t}[y_t] = \bar{y}$. It simplifies the model and allows us to concentrate on the heterogeneity in expectations of traders.

At any trading session every trader chooses one of two possible forecasting rules, reflecting two trading attitudes commonly observed in the real markets. Fundamental forecasting rule

$$
E_t^1[p_{t+1}] = p^f + v\,(p_{t-1} - p^f)\,, \qquad v \in [0,1]\,, \tag{4}
$$

predicts that any price deviation from the fundamental level will be corrected. In one limiting case, $v = 0$, immediate correction is expected, while in another limiting case, $v = 1$, agents rely on the market, expecting that current price gives the best prediction. Trend-following forecasting rule

$$
E_t^2[p_{t+1}] = p_{t-1} + g\,(p_{t-1} - p_{t-2})\,, \qquad g > 0\,, \tag{5}
$$

predicts that past trend in price will be kept, so it is extrapolated with coefficient g from past price level.

Notice that the former rule (as opposed to the latter) requires a knowledge of fundamental value. Consequently, we assume that to use the fundamental rule the agent has to pay cost $C > 0$ per period, while the second rule is available for free.

2.2 Evolutionary updating of expectations

At the end of every trading round agents update their forecasting strategy, i.e. choose which of the two rules, (4) or (5), will be used during the next session. The choice of the active forecasting rule is based upon the commonly available deterministic part reflecting the past performances of two rules. In addition, this measure is disturbed by the stochastic error component reflecting the measurement error or imperfect computations of agents. The choice process is modeled as follows.

In the end of trading round t, first, an individual *realized excess profit* is computed as a product of the holdings of the risky asset between trading rounds $t-1$ and t and its excess return

$$A_{i,t-1}\left(p_t + y_t - (1+r)\,p_{t-1}\right). \tag{6}$$

Notice that in the case of continuous trading the excess return is evaluated on the basis of closing prices. We stress also that under order-driven protocols, realized position of agent $A_{i,t-1}$ can differ from agents' demand $A_{i,t-1}(p_{t-1})$ due to possible rationing and/or difference between quoted and transacted prices. More details will be provided later, when we discuss the market protocols.

Having computed individual profits, the performances of fundamental and trend-following forecasting rules, U_t^1 and U_t^2, are defined as *average* profit earned by all the fundamentalists and all the trend-followers, respectively. From (6) it is clear that performance of the rule is an average position of the followers of this rule times the excess return. Thus, if the risky asset has earned positive (negative) return, then performance of the group with larger average possession of the asset is bigger (smaller).

Finally, agent i chooses the predictor for which the following maximum is realized

$$\max\left(U_t^1 - C + \xi_{i,t}; U_t^2 + \zeta_{i,t}\right), \tag{7}$$

where C is the cost of fundamental predictor, and $\xi_{i,t}$ and $\zeta_{i,t}$ are independent over time and among the agents random variables. The choice can be rewritten in terms of probabilities for the special case of a Gumbel distribution of error terms. In this case individual i chooses the predictors with *discrete choice* probabilities[4]

$$n_{t+1}^1 = \frac{\exp\left(\beta(U_t^1 - C)\right)}{\exp\left(\beta(U_t^1 - C)\right) + \exp\left(\beta U_t^2\right)}, \qquad n_{t+1}^2 = 1 - n_{t+1}^1, \tag{8}$$

with subscript indicating that these probabilities shape the population trading at period $t+1$. Parameter $\beta \geq 0$ is the *intensity of choice* measuring how sensitive agents are with respect to the difference in past performances of two strategies. If the intensity of choice is infinite, the traders always switch to the historically most successful strategy. On the opposite extreme, $\beta = 0$, agents are equally distributed between different types independent of the past performance. The intensity of choice β is inversely related to the variance of the noise terms $\xi_{i,t}$ and $\zeta_{i,t}$.

The timing of our model is as follows. At the end of period t the average profit earned by fundamentalists from their holdings between periods $t-1$ and t is computed and learned by all the traders. Analogously, the average profit of trend-followers is learned. On the basis of these two performances, at the beginning of period t agents independently choose their new types according to the probabilities defined in (8). To make this procedure feasible, in our simulations, we always assure that every forecasting type has at least one representative at any trading round. If an independent random draw did not produce any fundamentalist/trend-follower, we simply repeat the procedure until the population contain at least one fundamentalist/trend-follower.

[4]Our specification of the error terms are common in the literature on the random utility models, see Anderson et al. (1992). Implied probabilities are used to model a choice in a number of theoretical models with different range of application, see e.g. Brock (1993), Brock and Hommes (1997), Camerer and Ho (1999) and Weisbuch et al. (2000).

At the same time the demand functions for every type is computed on the basis of past prices by plugging the expectation rules (4) and (5) into the demand equation (3). Now, when the types are determined and demands are computed, the excess demand for every agent can be found and trading session $t+1$ starts. Then, under the BA and the OB, which are the order-driven protocols, every agent is allowed to submit only one order per period, which is a point from the excess demand curve.[5] Furthermore, in the OB market, where intra-session trade is sequential, the sequence in which agents enter the market is relevant for the outcome of trade. To control for this effect, we assume that in each period agents enter the market in a random order, independently distributed over the time. To the end of the trading session agents have fixed their profits for the holdings between periods t and $t+1$, and have their portfolio updated. The price p_{t+1} is defined according to the trading mechanism.

3 Walrasian Market Clearing and Large Market Limit

Let us first discuss the implications on the price dynamics of our behavioral assumptions of the heterogeneity in expectations and agents' adaptivity. For this purpose we consider the simplest way of clearing market, assuming that at every period it is in temporary Walrasian equilibrium with demand equal to supply.

Thus out first mechanism, *Walrasian protocol* (WA), assumes that at time t every agent submits the excess demand function $\Delta A_{i,t}(p)$, which is the difference between demand $A_{i,t}$ defined in (3) and the current position of the investor in the risky asset, $A_{i,t-1}$. The price of the risky asset is determined from the market clearing condition $\sum_i \Delta A_{i,t}(p) = 0$. Since the demand function in (3) is strictly decreasing, there exists a unique equilibrium price, which we denote as p_t.

In this paper we concentrate on a special case of zero outside supply of the shares of the risky asset.[6] The pricing equation becomes $\sum_i A_{i,t}(p_t) = 0$, which we now rewrite in *deviations*

[5] It implies that the WA can be viewed as a limit version of the BA when the number of orders per agent is infinite.

[6] Model in Brock and Hommes (1998) is solved under the same assumption. As they show, this assumption

from fundamental price, $x_t = p_t - p^f$. Furthermore, we normalize the risk aversion coefficient so that $a\sigma^2 = 1$. Using (3), (4) and (5), the price deviation from p^f at time t is given by

$$x_t = \frac{1}{R\,N} \sum_{i=1}^{N} \Big(v\, x_{t-1}\, \mathcal{I}^1_{i,t} + \big(x_{t-1} + g(x_{t-1} - x_{t-2})\big)\, \mathcal{I}^2_{i,t} \Big)\,, \tag{9}$$

where for $h \in \{1,2\}$ index function $\mathcal{I}^h_{i,t}$ is equal to 1 if agent i forms at time t expectation $E^h_t[p_{t+1}]$, and it is equal to 0, otherwise.

Pricing equation (9) shows that the market price is a discounted sum of individual expectations. For instance, if price was on the fundamental level during the last two periods, both fundamentalists and trend-followers expect no deviation, so that the realized price will be indeed equal to p^f. If, instead, the asset was equally overestimated during the last two periods ($x_{t-1} = x_{t-2} > 0$), trend-followers will expect no change in price, while fundamentalists with $v < 1$ will expect a price correction. As a result, price will move in the direction of the fundamental level. Its exact value will depend on the relative number of fundamentalists.[7]

Let us turn now to the question of how the relative number of fundamentalists and trend-followers is determined. The setting with the WA clearing is also the simplest in this respect. Indeed, the agents' demands are always satisfied in such a market. Therefore, at any given period the positions of all the agents with a given forecast are the same, as well as the realized excess profits (6). The performance measures of fundamentalists and trend-followers are then given by

$$U^1_t = \big(v\, x_{t-2} - R\, x_{t-1}\big)\, \big(x_t - R\, x_{t-1} + \delta_t\big) \tag{10}$$

and

$$U^2_t = \big(x_{t-2} + g\, (x_{t-2} - x_{t-3}) - R\, x_{t-1}\big)\, \big(x_t - R\, x_{t-1} + \delta_t\big)\,, \tag{11}$$

respectively. Random term $\delta_t = y_t - \bar{y}$ represents the shock due to the dividend realization.

can be made without lose of generality, since positive supply case is equivalent to a re-definition of the dividends. Hommes et al. (2005) consider the model with positive supply.

[7]Notice that the zero total supply of shares implies that the populations of fundamentalists and trend-followers always take opposite positions. In the first example, both groups have zero amount of shares. In the second example, fundamentalists are short and trend-followers are long in the risky asset.

3.1 Large Market Limit

An important feature of our setting is that the dynamics under the WA can be studied by means of the dynamical system theory in a special case, when the number of agents becomes large. Indeed, for $N \to \infty$ the Law of Large Numbers guarantees a convergence of the actual fractions of fundamentalists and trend-followers, which can be used in (9) to compute the price, to the probabilities defined in (8). The model is described then by one equation of the fourth order (or, equivalently by the 4-dimensional system) consisting of the market clearing equation coupled with an update of the fractions of fundamentalists (which we will denote simply as n_t omitting the superscript)

$$\begin{aligned} x_{t+1} &= \frac{1}{R}\Big(vx_t\, n_{t+1} + \big(x_t + g(x_t - x_{t-1})\big)\,(1 - n_{t+1})\Big) + \varepsilon_{t+1} \\ n_{t+1} &= \exp\Big(\beta\Big[\big(v\, x_{t-2} - R\, x_{t-1}\big)\,\big(x_t - R\, x_{t-1} + \delta_t\big) - C\Big]\Big)\Big/ Z_{t+1}\,, \end{aligned} \tag{12}$$

where normalization factor $Z_{t+1} = \exp\big(\beta(U_t^1 - C)\big) + \exp\big(\beta U_t^2\big)$ with performance measure of fundamentalists (10) and performance measure of trend-followers (11). Dynamics in (12) is stochastic and there are two sources of noise. First source is the dividend realizations δ_t entering the performance measure. Second term, ε_{t+1}, was added to the pricing equation to represent the dynamic noise. When both of these terms are zeros, the corresponding system is deterministic and the following result takes place.

Proposition 3.1. *Consider system* (12) *with $\delta_t = 0$, i.e. when $y_t = \bar{y}$ and with $\varepsilon_t = 0$. This system has a unique steady-state with $x^* = 0$ and $n^* = e^{-\beta C}/(1 + e^{-\beta C})$.*

(i) For $g \le R$, this steady-state is locally stable.

(ii) For $g > R$, this steady-state is locally stable for $\beta < \beta^* = \ln\big((g - R)/R\big)/C$. When $\beta = \beta^*$ the steady-state exhibits the Neimark-Sacker bifurcation, and for $\beta > \beta^*$ it is locally unstable.

Proof. From the first equation of (12), one gets that in an equilibrium, it is $R\,x^* = vx^*\,n^* + x^*\,(1 - n^*)$. Since $v < 1$ and the fraction of fundamentalists n^* should belong to the interval $[0, 1]$, this equation has a unique solution $x^* = 0$. Substituting zero deviation into the performance measure, we derive $n^* = e^{-\beta C}/(1 + e^{-\beta C})$.

To derive the stability conditions, the Jacobian matrix of the system should be computed. Substituting the second equation of (12) into the first and introducing the lagged variables, the Jacobian matrix in the fundamental equilibrium reads:

$$\boldsymbol{J} = \left\| \begin{matrix} \big((v-1-g)n^*+1+g\big)/R & g(n^*-1)/R & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} \right\| .$$

This matrix has two zero eigenvalues, while two others are derived from 2×2 matrix with trace $T = \big((v-1-g)n^*+1+g\big)/R$ and determinant $D = g(1-n^*)/R$. Standard conditions for eigenvalues of this matrix to be insider the unit circle are $D < 1$, $T < D+1$ and $T > -D-1$. The last two conditions are always satisfied, while the first is simplified to $n^* > 1 - R/g$. When $g \leq R$, this is also satisfied and the fundamental steady-state is locally stable. If $g > R$, the bifurcation value is a solution of $e^{-\beta C}/(1+e^{-\beta C}) = 1 - R/g$, which gives result of Proposition 3.1*(ii)*. □

The only steady-state of system (12) is "fundamental", with price staying on the level p^f, implying deviation $x^* = 0$. In this situation both forecasting rules give correct predictions, but fundamentalists have to pay positive cost C. Consequently, they have smaller relative share than the trend-followers: $n^* < 0.5$. With growing β, the equilibrium fraction of trend-followers increases. When the trend-followers extrapolate weakly ($0 < g < 1+r$), the fundamental steady state is stable. When they extrapolate strongly ($g > 1+r$), the fundamental steady-state is stable for small β and unstable for high β. When β crosses its critical value, the stable quasi-cyclic attractor is created through the Neimark-Sacker bifurcation. Notice that the bifurcation value of β does not depend on the value of parameter v in the fundamentalists' forecasting rule.

Qualitatively, Proposition 3.1 implies that depending on the intensity of choice market dynamics can be in one of two regimes: *tranquil*, with price staying on the fundamental level, and *volatile*, with systematic large deviations from it. In the first regime there is no excess

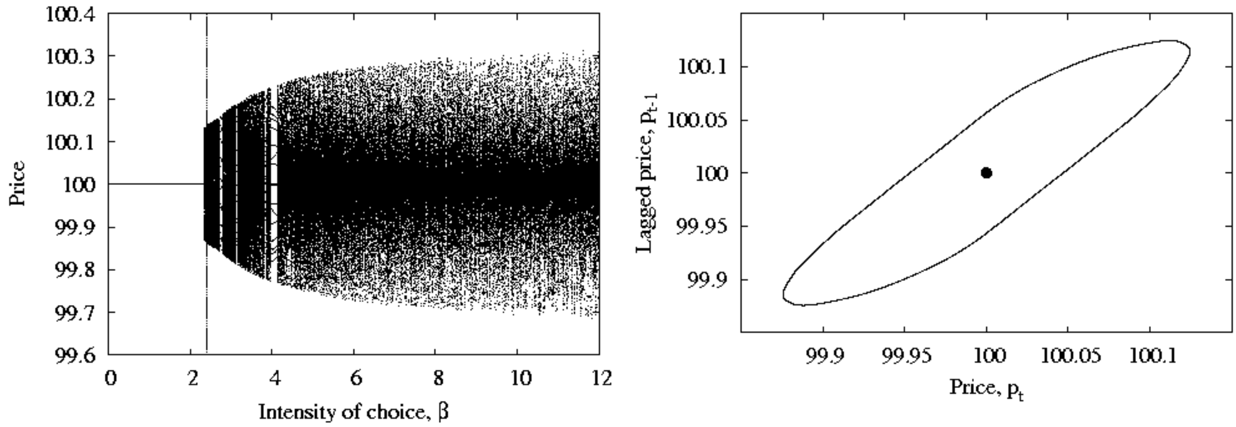

Figure 1: Bifurcation diagrams for the Walrasian model in the limit $N \to \infty$. For each $\beta \in (0, 12)$, 300 points after 10000 transitory periods are shown. **Left panel:** Neimark-Sacker bifurcation of fundamental steady-state. Parameters are: $r = 0.1$, $\bar{y} = 10$, $v = 0.1$, $g = 1.2$ and $C = 1$. **Right panel:** Phase portrait of the system in coordinates (p_t, p_{t-1}). Quasi-periodic cycle coexists with fundamental steady-state. The same parameters as in the left panel with $\beta = 2.3$.

volatility in price and the trading volume is zero. In the second regime periods of overvaluation of the asset are followed by the periods of its undervaluation, and price exhibits bubbles and crashes with positive trading volume and excess volatility. Coexistence of two regimes and its dependence from simple behavioral parameter, makes the case of strong extrapolation especially interesting.[8] Benchmark parameters $r = 0.1$ and $g = 1.2$ of our simulation are chosen to guarantee the coexistence of two regimes, since we are interested in the dependence of "bifurcation scenario" on the market architecture. Other parameters are set to $\bar{y} = 10$, implying fundamental price $p^f = 100$, and $C = 1$. The precise values of both of them are not important for qualitative behavior. Finally, we choose $v = 0.1$, implying that fundamentalists expect very small deviation from fundamental value. Again, the precise value of v is not important. However, as our simulations show, parameter v must be small enough in order the price dynamics to be bounded.

[8]Similar regimes were identified in simulations of Santa Fe artificial market model of Arthur et al. (1997), and analytical treatment in Brock and Hommes (1998).

The results of Proposition 3.1 are illustrated for these benchmark parameters in Fig. 1. The left panel shows a bifurcation diagram, where for each $\beta \in (0, 12)$ we simulate the long-run behavior of price. In accordance with our result for any $\beta < \beta^* \approx 2.398$ the price converges to the fundamental price p^*. When the intensity of choice increases to β^* the fundamental equilibrium loses stability, and a stable quasi-periodic cycle around p^* is created. In the volatile regime, the price dynamics is fluctuating around p^*. The bifurcation diagram shows that the amplitude of these fluctuations slightly increases with the intensity of choice.

An interesting feature of the model is not captured by the bifurcation diagram and Proposition 3.1. The right panel of Fig. 1 illustrates that a stable quasi-periodic cycle exists in the model even for $\beta = 2.3$ which is less than β^*. Numerical analysis of the system shows that the fundamental steady state is globally stable for $\beta < \beta^{**} = 2.23$. Then, when the intensity of choice belongs to the interval $[\beta^{**}, \beta^*]$, locally stable fundamental steady-state coexists with a quasi-periodic attractor.[9]

Typical patterns of price and return dynamics for the volatile regime of the model ($\beta = 5$ in this example) are shown in Fig. 2. The deterministic simulation in the left panel shows that in such regime, characterized by persistent deviations from the fundamental level, dynamics repeatedly go through qualitatively similar phases. At the beginning of each phase price fluctuates only slightly around fundamental value. With time, however, fluctuations are getting wilder, but at a certain point the price stabilizes and exhibits only small oscillations. The right panel shows the dynamics of the same model when both noise terms, random dividend and dynamic noise, are added.

To get insight into economic explanation of the endogenous fluctuations, in Fig. 3 we show a snapshot of the previous deterministic simulation for 23 periods. Dynamics of the prices (top panel), of the fraction of fundamentalists (middle, in the log scale) and of the excess return $x_t - Rx_{t-1}$ (bottom) are shown. When price is close to the fundamental level both fundamentalists and trend followers have similar return, but the former group pays a

[9] Such coexistence of attractors seems to be a consequence of a so-called Chenciner bifurcation, thoroughly discussed in Gaunersdorfer et al. (2008) in a similar model.

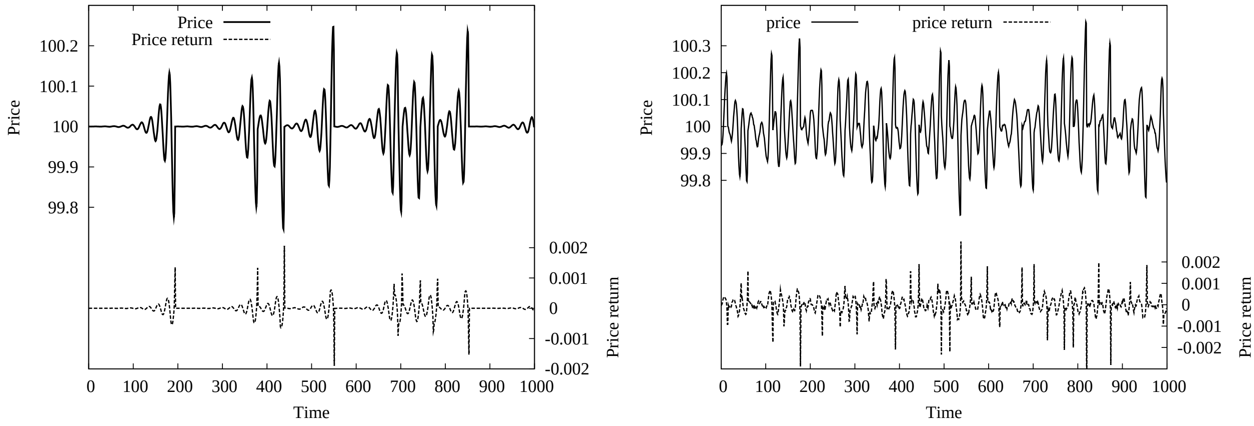

Figure 2: Price (left axis) and return (right axis) dynamics in the limit of Walrasian model. The same parameters as in the left panel of Fig. 1 and $\beta = 5$. **Left panel:** Deterministic dynamics. **Right panel:** Dynamics with random dividend and small dynamic noise. Both noise terms are i.i.d. normal with standard deviation equal to 0.005.

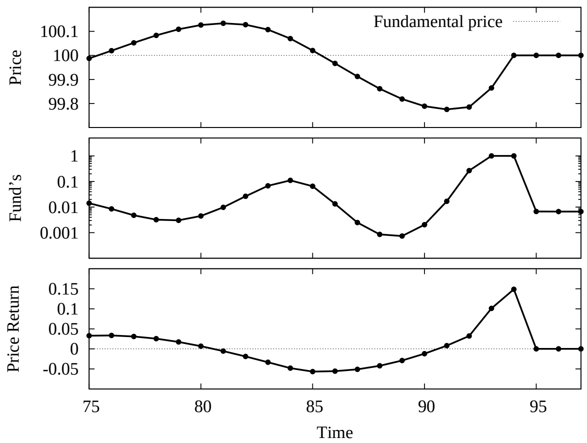

Figure 3: Dynamics in the limit of Walrasian model for the benchmarking values of parameters and $\beta = 5$.

positive cost. The trend-followers then dominate the market and an upward price trend is developed. Trend-followers hold the asset and fundamentalists sell short waiting for the price correction. However, the short-term return of the asset is positive (due to the capital gain) and the wealth of trend-followers increases. Between periods 76 and 79 their share grows. This process ends because, on the one hand, the extrapolative expectations are not strong enough to sustain a trend, and, on the other hand, the asset is overvalued and the dividend yield is low. As the fraction of fundamentalists grows, the price trend slows down, the excess return becomes negative, and at period 81 trend is reverting. Self-fulfilling downward trend is now developed and the fraction of the trend-followers increases again. However, as before, the price extrapolation is not enough, and at period 91 the excess return becomes positive with prices still below the fundamentals. It brings a high return to the fundamentalists who holds the asset. This return overcomes positive costs, and the fraction of fundamentalists grows almost to 1. It brings price to the fundamental level and story repeats itself.

To summarize, the special case of the Walrasian market with a large number of agents shows that the model has a unique, fundamental steady-state. If the trend-followers extrapolate strongly enough, this steady-state loses its stability when the intensity of choice increases. Consequently, two different regimes are observed, tranquil and volatile. The second regime, when the intensity of choice is high, is consistent with excess volatility. Will these two regimes present in a market with alternative trading mechanism? How the critical value of intensity of choice depend on the market trading rules? And how the properties of price dynamics are affected by the mechanisms? These are the questions which we analyze below.

4 Market Mechanisms

A market mechanism is a well-defined procedure which transforms input from agents to the output as price and quantity traded. There are numerous market mechanisms in a literature and reality, among which we take three stylized procedures. The model was simulated separately for every mechanism and the results were compared.

4.1 Walrasian Auction

The implementation of Walrasian auction (WA) was described in the beginning of the previous Section. Dynamics is given by the pricing equation (9), while the forecasting type of any trader is determined by probabilities (8). Recall that the demand of agents are always satisfied under the WA implying that the performance measures are given by (10) and (11). The only difference between the Large Market Limit analyzed in Section 3.1 and the WA simulations reported below is the number of agents, which is infinite in the former case and finite in the latter.

Simple but informative illustration of the WA is given in the left panel of Fig. 4. In this example there are five fundamentalists and five trend-followers arriving to the market at time t with initial endowments of zero shares of the risky asset (i.e. demand and excess demand coincide for these agents). Let us assume that $p_{t-2} < p_{t-1} = p^f$. Since fundamentalists forecast price p^f for the next period, they have net demand for $p_t < p^f$ and net supply, otherwise. The thin curves show individual demand (supply) schedules of fundamentalists. Trend-followers expect that prices will be raising, so that they have net demand for $p_t < p^*$ and net supply for $p_t > p^*$ with some $p^* > p^f$. The dashed lines show individual demand (supply) schedules for trend-followers. All individual demand and supply have the same slope, R, in absolute value. To obtain the aggregate curves one has to sum up individual schedules for every given price, i.e. horizontally. When price is inside the interval $[p^f, p^*]$ all the demand is generated by the trend-followers and all the supply is generated by the fundamentalists. Summation of five corresponding curves gives the aggregate demand and aggregate supply both shown as thick curves. When price is below p^f, all ten agents want to buy. Thus, the aggregate supply is zero, while the aggregate demand curve has a kink at price p^f. Analogously, the aggregate demand is zero above p^* and the aggregate supply curve has a kink at price p^*.

The aggregate demand and the aggregate supply curves intersect in the point labeled "WA", whose ordinate is the equilibrium price and abscissa is the equilibrium quantity under Walrasian market-clearing. All the agents trade their respective quantity on the equilibrium price. Notice that in this example, the equilibrium price is on the half way between p^f and p^*, which are "no-trade" prices for the two forecasting types. Other distribution of agents among types would lead to different outcome. It is geometrically clear from Fig. 4 that when agents with one forecasting type are outnumbered by the agents of another type, then equilibrium price is closer to the no-trade price of this dominating type, while the equilibrium volume is

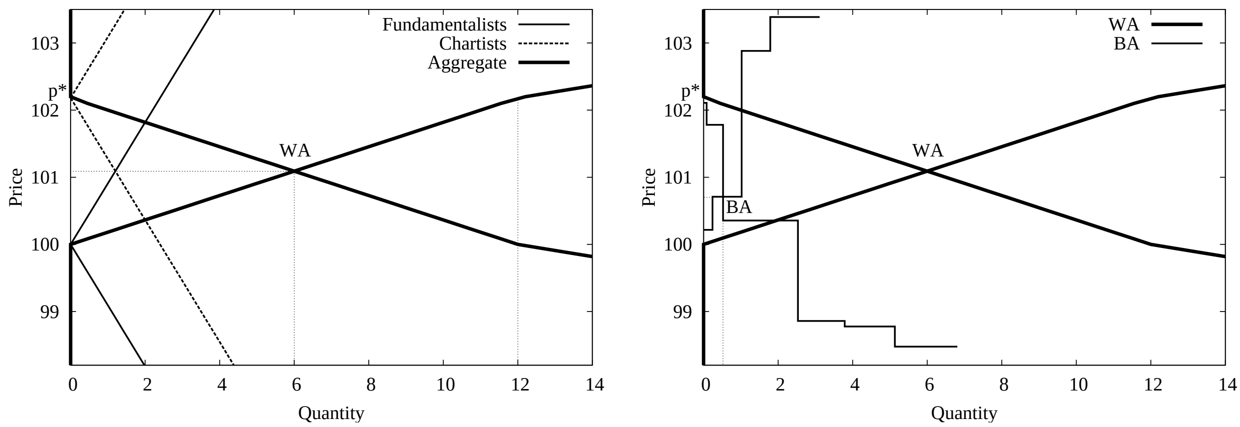

Figure 4: Comparison of different market-clearing mechanisms. **Left panel:** Walrasian price and quantity are found as intersections of the aggregate demand and supply schedules (thick) built starting from the individual curves. **Right panel:** Batch auction compared with Walrasian auction.

smaller than under equal type distribution.

4.2 Batch Auction

Under the batch auction (BA) agent i submits at time t one *limit order*, which is a price/quantity combination $(p_{i,t}, q_{i,t})$. When the ordered quantity is positive (negative), the order is of a buy (sell) type. The price in the limit order defines the largest (smallest) price accepted to the submitter for execution of the buy (sell) order.

To make a comparison between the BA and the WA meaningful, we will require that agents submit those price/quantity combinations, which belong to their demand or supply schedules. We use simple strategic considerations to determine the price in the order generation process. Namely, we assume that the price of the limit order $p_{i,t}$ of agent i is determined as a random draw from a normal distribution with mean p_{t-1}, the price of the previous trading session, and standard deviation σ_o. The realizations are independent over time and agents. Under this price selection rule an agent reasonably believes that there is a high chance that her order will be executed at a price which is close to the last closing price p_{t-1}. The larger is the deviation from this price, the higher may be potential gains from the trade, but the lower is the likelihood of such an order execution. Therefore, only in rare occasions an agent will experiment with an order priced considerably far from the previous closing price p_{t-1} [10].

Given price $p_{i,t}$, the desired quantity is computed as

$$q_{i,t}(p_{i,t}) = A_{i,t}(p_{i,t}) - A_{i,t-1}\,, \tag{13}$$

where the first term in the right hand-side is a point from the demand curve (3), while $A_{i,t-1}$ is a current holding of the risky asset.

All N orders are submitted simultaneously. Then, all the limit buy orders are sorted such that their price sequence is decreasing. It gives us a step-level market demand curve. Analogously, the limit sell orders are sorted so that their price is increasing to define a step-level market supply curve. The price p_t is determined as an intersection of constructed demand and supply curves (or the midpoint of the interval between the lowest and the highest clearing price, if there are multiple intersections). In those cases when demand and supply curves do not intersect, price p_t is set to the price of previous period, p_{t-1}. The corresponding quantities are traded on this price between those agents who submitted bids (asks) no lower (no higher) than p_t. Traders who submitted orders exactly at p_t may be rationed accordingly, while all the other traders do not trade at all and keep their previous portfolios.

In the right panel of Fig. 4 we use the previous example to construct the schedules for the BA. For every agent one price was generated randomly around price $p_{t-1} = p^f$ and then the corresponding individual demanded or supplied quantity was found. Sum of the quantities gave two step curves whose intersection, labeled "BA", determines equilibrium price and quantity. Of course, the precise schedules depend on the particular random draw, but certain general tendency can be seen from this example. Obviously, the quantity traded under BA is always smaller than the quantity traded under WA, while the equilibrium price under the BA can be both higher or smaller than under the WA.

In computation of time-t performance measures, notice that a number of traders did not trade during time $t-1$, so that their positions were left unchanged. Thus, their perfor-

[10] In the context of the OB, Farmer and Mike (2008) find that the shape of the actual distribution of prices of submitted orders can be well approximated by Student t distribution around the best price

mances are evaluated as $A_{i,t-2}\left(p_t + y_t - (1+r)p_{t-1}\right)$, which is an earned excess profit on the old holdings. Instead, those traders who did trade (and were not rationed) changed their positions on the basis of their submitted quantities. Their performance is then equal to $A_{i,t-1}(p_{i,t-1})\left(p_{t-1} + y_{t-1} - (1+r)p_{t-2}\right)$, and it is different from the performance under WA because the transactions happened not on the submitted price of bid, $p_{i,t-1}$. The BA mechanism, by its nature, introduces distortions in the individual agents performances.

4.3 Order Book

In the order-book market, there are many transactions during one trading session at time t. Each agent can place only one buy or one sell order during the session. To make a reasonable comparison with the BA, the order generation process is identical to the one described in Section 4.2, while the sequence in which agents place their orders is determined randomly and independent for different sessions.

During the session the market operates according to the following mechanism. There is an electronic book containing unsatisfied agents' buy and sell orders placed during current trading session. When a new buy or sell order arrives to the market, it is checked against the counter-side of the book. The order is partially or completely executed if it finds a *match*, i.e. a counter-side order at requested or better price, starting from the best available price. An unsatisfied order or its part is placed in the book. At the end of the session all unsatisfied orders are removed from the book. The mechanisms for determining type of the order, its price and quantity are equivalent to those described for the batch auction. Price p_t is the closing price of the session, i.e. the price of the last transaction.

In Fig. 5 we show a possible order book realization for the same limit orders as were generated for the BA illustration in Fig. 4. The integer labels show the sequence in which the buy/sell orders arrive, while different types of horizontal lines show which part of the order is satisfied. Order 2 partially matches order 1 and after the corresponding transaction one sell order (part of order 2) remains in the book. Then buy order 3 arrives, but its price is worse than the best price in the book, thus it goes to the buy side of the book. Analogously, order

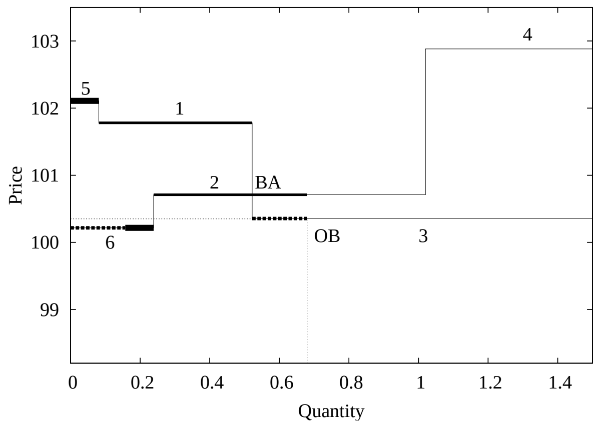

Figure 5: Order-book market compared with BA.

4 is added to the sell part of the book, and order 5 is added to the buy side of the book. Moreover, this order has the best price to buy at this moment. When order 6 arrives, first it partially matches with order 5, and then its remaining part matches with order 3. Point OB shows the outcome of the trade with the OB mechanism. The price p_t is the price of order 3, while the volume for the session is the total traded number of shares. Notice that the flexibility of this trading mechanism allows larger traded volume than under the BA. At the same time, the OB price p_t is typically different from the BA price and may significantly depend on the order in which transactions happen.

Agents' performance measure is computed as in (6), but similarly to the BA mechanism the agents' position $A_{i,t-2}$ may be different from the agents' demand $A_{i,t-2}(p_{t-2})$. Contrary to the BA case, however, the order in which agents arrive to the market is now important for the outcome of trade, that is whether an agent is rationed or not.

5 Simulations and Results

We simulate the system with finite number of agents under different market mechanisms, namely the WA, the BA and the OB mechanism [11]. In all cases we keep the dividend constant

[11] The software for the simulation is written in C++ and is a modification of the YAFiMM package created for Bottazzi et al. (2005). The YAFiMM package is publicly available at `http://www.sssup.it/~bottazzi`

Parameter	Symbol	Value (Range)
Intensity of choice	β	$[0, 12]$
Interest rate	r	0.1
Mean dividend	$\bar{y}$	10
Normalized risk-aversion	$a\sigma^2$	1.0
Trend-followers' extrapolation	g	1.2
Fundamentalists' reversion	v	0.1
Fundamentalists' costs	C	1.0
Stand. deviation of limit order price	σ_o	3.0
Number of agents	N	1000
Transient period	Tr	2000

Table 1: Parameter values used in simulations.

and do not add any dynamic noise. While under these two assumptions in the LML with Walrasian market clearing the system is deterministic, in the simulations with a finite number of agents the amount of randomness will increase from one mechanism to another. With the WA the system becomes stochastic since the realized fractions of fundamentalist and trend-followers are no longer equal to their analytic probabilities. In the BA the amount of stochasticity is higher due to the fact that agents choose a random points on their individual demand schedules. In the OB mechanism the level of stochasticity is further increased by random sequencing of order submissions.

The parameters that we use for the simulations are summarized in Table 1. The model parameters are the same as we used in the analysis of the LML, when non-trivial and non-divergent dynamics was generated. The behavioral parameter we mainly focus on is the intensity of choice β. In all our simulations we set the number of agents $N = 1000$. This number is high enough to obtain a dynamics close to the deterministic LML with WA. The transient period is set to 2000 to avoid any transitory effects.

Before starting a detailed analysis of the stochastic system let us have a look on time series

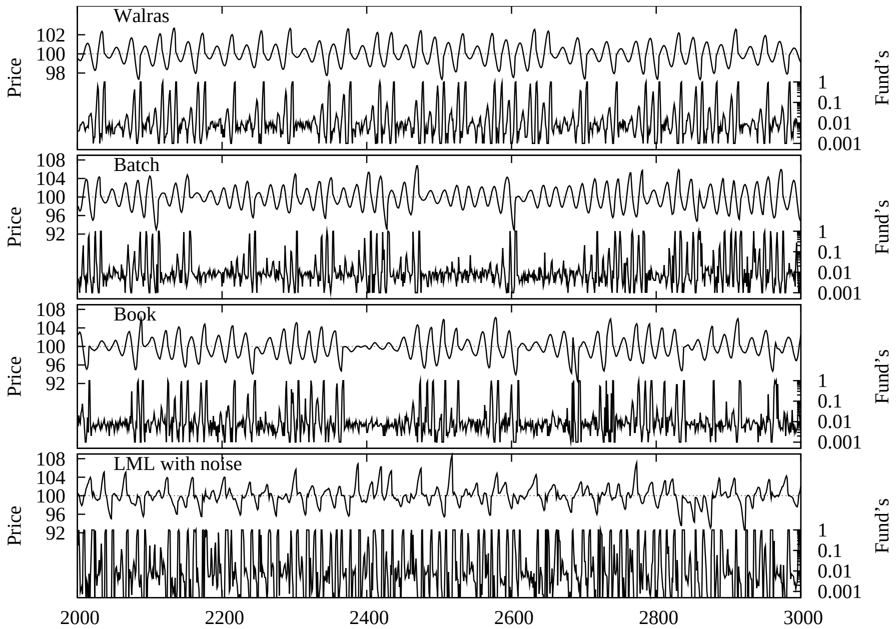

Figure 6: Dynamics of prices (left axis) and share of fundamentalists (right axis in the log-scale) for $\beta = 5$ in WA, BA, OB and LML with model approximation error. In the latter case the error is independently drawn from normal distribution with zero mean and standard deviation $\sigma_\varepsilon = 0.3$.

simulated from the model under different market mechanisms. In Fig. 6 we show the dynamics of the price (upper part, left axis) and the share of fundamentalists (lower part, right axis) for $\beta = 5$. Recall from Figs. 2 and 3 that for this level of the intensity of choice, the system in the LML does not converge to the fundamental steady-state, but oscillates around it. The top panel shows that under the WA the system behaves similarly to the deterministic LML buffeted with small dynamic noise (cf. the right panel of Fig. 2). Under the BA (the second panel from the top) and the OB (the third panel from the top) the dynamics is similar but certain differences can be observed even by the naked eye.

First, the price deviations from the fundamental level have larger amplitude than under the WA. Second, looking at the dynamics of the fraction of the fundamentalists in all three markets, one can distinguish the two alternating phases in the dynamics, but these phases are much more visible under the order-driven protocols than under the Walrasian market. These phases are (1) stable ecology, when the fractions of fundamentalists/trend-followers in the market exhibit only moderate changes and (2) turbulent ecology, when every period a large fraction of agents switch from one type to another. Third, under the OB this populational dynamics translates into price dynamics and the observed phases of stable and turbulent ecology correspond to the phases of small and large price oscillations, respectively. These phases can be linked to the phenomenon of volatility clustering observed in the real financial data. Under the BA the changes between small and large price fluctuations are not so abrupt.

Finally, we want to verify whether similar price behavior can be obtained by adding a dynamic noise to the LML.[12] In the bottom panel we add the Gaussian noise to the LML dynamics given by the system (12). The standard deviation 0.3 is chosen to match the amplitude of the price fluctuations under the order-driven mechanisms. Contrary to the last two mechanisms, we no longer observe clear-cut phases in the time series. Therefore, the randomness inherent to the order-driven market mechanisms distorts the price evolution differently from simple dynamic noise.

5.1 Change of Market Regimes

Next, we investigate the interplay of agents behavior and market mechanism on price dynamics in details.

Fig. 7 depicts the dependence of the stable distribution of prices on the intensity of choice parameter β. The parameter β ranges from 0 to 12 with a linear step of 0.05 which gives 240 points in total. The distribution for each level of beta is represented by a gray-shade coded histogram. Darker shades correspond to the areas of higher density. The histogram is computed using price levels from 10000 periods after 2000 transient periods. A bifurcation in the stochastic system corresponds to the qualitative changes in the stable distribution which we attempt to identify graphically. In the case of the WA around the point $\beta = 2.23$, we observe a dramatic increase in the variance of the distribution and an emergence of bimodal distribution. Around this point of bifurcation the system transits from the tranquil regime to

[12] For instance, Gaunersdorfer and Hommes (2007) try to reproduce the "stylized facts" by adding the dynamic noise to a similar deterministic model.

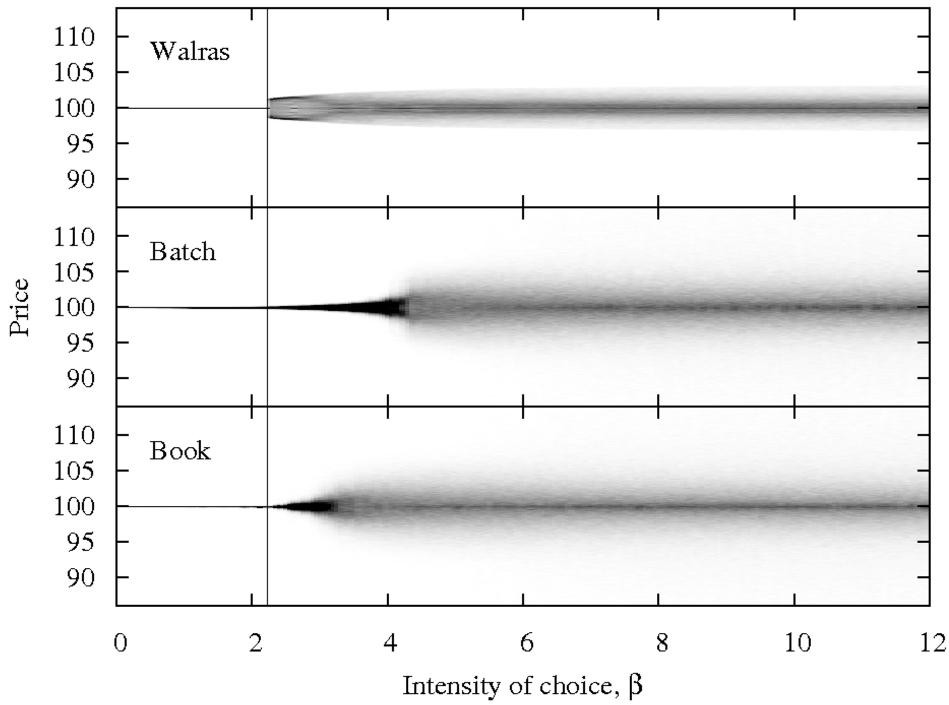

Figure 7: Distribution of prices for different market mechanisms as a function of the intensity of choice. A critical value of β (or range of the values) for which the change of market regime happens is smaller for WA and larger for BA.

the volatile regime discussed in details in Section 3.1. The bifurcation is delayed for the BA and occurs around $\beta = 4$. For the OB mechanism the bifurcation occurs for the value of β higher then for the WA and lower then for the BA, around the point $\beta = 3$. For additional evidence about the point of bifurcation under three different mechanisms see Fig. 8 (left panel) which shows standard deviation of the price.

The delay in bifurcation observed for order-driven mechanisms can be explained by the interplay between agents behavior and the characteristics of the mechanism. As we pointed out in Section 2.2, the intensity of choice parameter β is inversely proportional to the level of noise in the average performance of the forecasting rule (see Eq. 7). Thus, the higher level of noise in the performance measure would correspond to the lower level of β. Recall also that the performance measure is the average of individual performances taken over all the agents using the same forecasting rule, and that the individual performances are computed on the basis of the agents' holdings of the risky asset and the excess return of this asset as in 6. When describing trading mechanisms in Section 4 we emphasized that the agents' (excess) demands are fully satisfied only under the WA. Under the BA and the OB mechanisms the agents' demands are translated into orders, some of which may be rationed. Risky asset holdings of the agents whose orders were rationed may be inconsistent with the chosen predictor resulting in an inconsistent performance measure. This in turn will result in a higher level of noise in the average performance measure of the strategy lowering the "effective" value of the intensity of choice parameter β. The larger is the amount of rationed orders, the higher is the level of noise. Note that the amount of rationed orders is larger under the BA than under the OB (see Fig. 5). It suggests that the actual "bifurcation" level of β is higher under the BA mechanism than under the OB mechanism, which is exactly what we observe in Fig. 7. Additional evidence of the amount of rationed orders under the order-driven mechanisms is provided in Fig. 8 (right panel) where we report an average traded volume of the risky asset.

Thus, in the Adaptive Belief Scheme, where heterogeneous agents choose their active forecasting rules on the basis of past performances, the order rationing inefficiencies introduced by the order-driven mechanisms, lead to market stabilization in a sense of wider interval of the intensity of choice parameter β for which the market is in the tranquil regime. It is, however, important to keep in mind that in the volatile regime, when the intensity of choice is high enough, the market fluctuations are larger in amplitude under the order-driven markets (see Figs. 6 and 7).

5.2 Informational Efficiency

The Efficient Market Hypothesis postulates that in the efficient market the price should reflect all available information about the asset value. In our setting with random i.i.d. dividend, the fundamental value of the asset is simply discounted sum of all future expected dividends, i.e. fundamental price p^f. The informational efficiency is often measured by comparing the volatility of the observed price with the volatility of the fundamental dividend process (see Shiller, 1981). To abstract from an effect of time-varying dividend in our model we keep

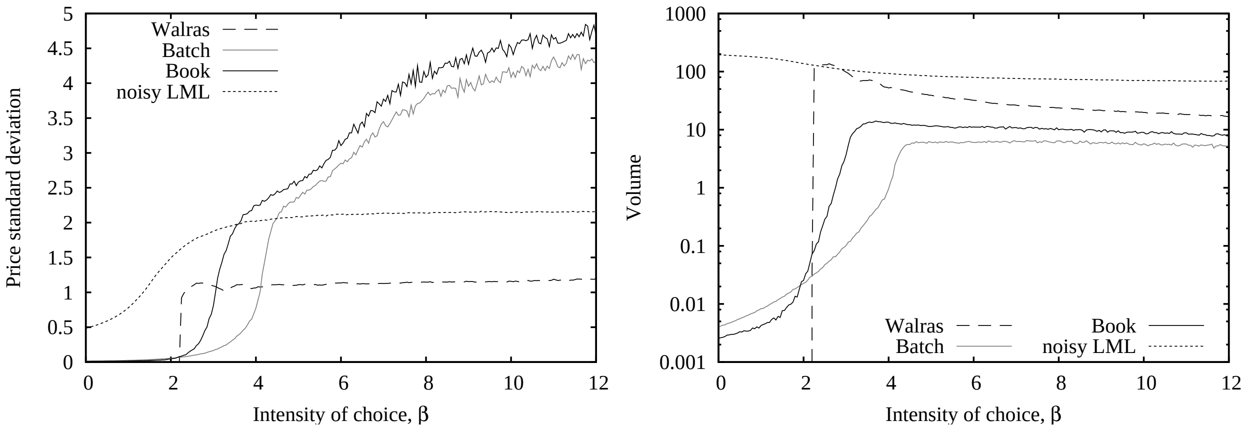

Figure 8: Two measures of market information in-efficiency for different market mechanisms as a function of the intensity of choice β. **Left panel:** Standard deviation of price. **Right panel:** Traded volume.

the dividend process constant. Under this assumption the Efficient Market Hypothesis would predict constant price over time and zero trading volume. Therefore, price volatility and trading volume can be used as measures of information efficiency.

In Fig. 8 we compare the standard deviation of the price (left panel) and the average traded volume (right panel) under different values of β for three market mechanisms and the LML with the Gaussian noise ($\sigma_\varepsilon = 0.3$). As before, the intensity of choice parameter β varies from 0 to 12 with a linear step of 0.05 which gives 240 points of evaluation in total. As usually, we ignore the first 2000 transitory steps and compute the standard deviation of the price and the average traded volume over the next 1000 periods. To eliminate the dependence of our results on a particular realization of random seed, we repeat this process for 100 random seeds and report an average of the statistics of interest. Averaging over different random seeds accounts for possible dependence of our results on initial conditions. We also compute 95% confidence bounds for the reported averages. Given the length of the series and the large number of random seeds the confidence bounds are very tights. For clarity, we do not plot them on the figures, but they can be easily inferred from statistics variations for neighboring values of β.

The standard deviation of the price (Fig. 8, left panel) depends on both the intensity of choice parameter β and the market mechanism. For $\beta < 2$ all of the mechanisms without added dynamic noise have standard deviation close to zero. It rises rapidly at the point of bifurcation and quickly converges to the level around 1.0 for WA, while for the BA and OR is continues to grow with β and shows some signs of stabilization to the level of 4.0 and 4.5 respectively when $\beta > 11$. The standard deviation for the OB is always higher then for the BA because of earlier bifurcation and extra layer of stochasticity (order sequencing) specific for the OB mechanism. The dynamic noise added to the LML is magnified from the initial level of $\sigma_\varepsilon = 0.3$ to the level increasing from 0.5 to 2.0 and stabilizing at the level of 2.0 when $\beta > 4$. The observed volatility pattern for the LML with the dynamic noise is different from the pattern produced by the order-driven mechanisms, which confirms that the time-series produced under the latter could not be produced by adding dynamic noise to the analytic LML. Based on volatility measure we conclude that information efficiency depends on both behavioral and institutional assumptions. For $\beta > 5$ the WA provided the most informationally efficient outcome, followed by the BA while the OB give the least informationally efficient outcome. However, when $2 < \beta < 5$ the order-driven mechanism are superior to the WA.

The average traded volume (Fig. 8, right panel) also depends on the value of β. For $\beta < 2$, when the price is very close to the fundamental, the average traded volume is 0 for all mechanisms besides LML with the dynamic noise. The dynamic noise added to the LML creates very high level of volume which slowly levels off with β increasing, which as before is in sharp contrast with the patterns produced by the order-driven mechanisms. For $\beta > 2.3$ the traded volume is always higher for the WA, which is followed by the OB and the BA. The reason for lower volume is the order rationing which is higher for the BA than for the OB (see Fig. 5). Interestingly, the average traded volume decreases in β, for $\beta > 5$. For large values of β the fraction one type of agents is much larger than the fraction of the other type, which leads to the lower volume as explained in details in Section 4.1.

In this Section we showed that in the volatile regime, i.e. when the intensity of choice is large enough, the order-driven markets are less informationally efficient from the point of view of price volatility, but more efficient in the sense that they lead to smaller trading volume. Of course, these measures of information efficiency should be interpreted with some care. For

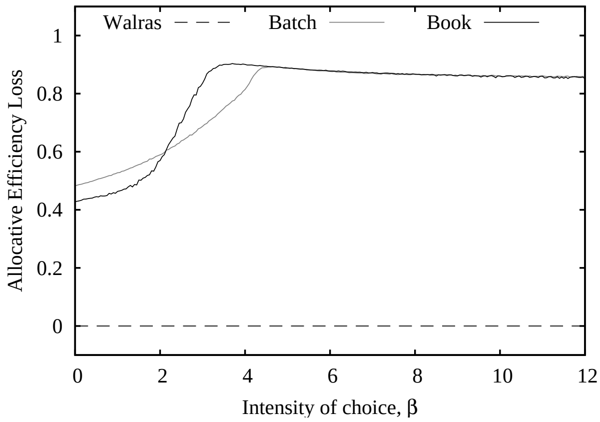

Figure 9: Allocative efficiency loss for different market mechanisms as a function of the intensity of choice β.

example, smaller traded volume in a market with rich endogenous dynamics can be a sign of lesser efficiency. Finally, notice that in their model of heterogeneous agents with fixed fractions Bottazzi et al. (2005) find similar results.

5.3 Allocative Efficiency

The main purpose of any trading mechanism is an efficient allocation of resources, that is an allocation which fully satisfies agents' demands at a realized price. By construction the WA always achieves an efficient allocation. For more realistic trading mechanism such as the BA and the OB, an efficient allocation is not necessarily guaranteed. Following Bottazzi et al. (2005) we define a measure of allocative efficiency loss, $L_{i,t}$:

$$L_{i,t} = 1 - \frac{1}{1 + |A_{i,t}(p_t) - A_{i,t}|\, p_t}\,, \tag{14}$$

where $A_{i,t}(p_t)$ is a desired amount of risky asset, i.e. a point on the demand schedule of agent i at closing price p_t of the period, and $A_{i,t}$ is the realized holding of the risky asset of agent i at the end of period t. By construction the measure is always between 0 and 1. Obviously under the WA the (excess) demands of the agents are fully satisfied and $L_{i,t} = 0$.

Fig. 9 shows the measure of allocative efficiency loss averaged across 1000 agents as a function of the intensity of choice β for the BA and the OB mechanisms. As before we take an average over 1000 time periods after the 2000 transitory periods which is averaged again over 100 random seeds. Consistently with our previous result we observe that the allocative efficiency loss depends on β. Before the bifurcation point the inefficiency is lower since we are close the the fundamental price and agents' demands are relatively small. After the bifurcation the price amplitude increases which translates into larger demands and the allocative efficiency loss increases. For $\beta > 5$ the allocative efficiency loss stabilizes at the level close to 0.85 for both order-driven mechanism and then slowly levels off. This small increase in efficiency is again explained by the higher concentration of price distribution in the neighborhood of fundamental price for larger values of the intensity of choice.

The allocative efficiency of the two order-driven mechanisms is exactly the same for a given $\beta > 5$. While the effect of the order rationing is more pronounced under the BA, the OB produces higher price deviations. Apparently both effects are of the same magnitude in terms of an influence on the allocative efficiency loss. Similarly to Bottazzi et al. (2005), in our model the precise implementation of the clearing system on the order-driven market does not affect the allocative efficiency.

5.4 Time Series Properties

We compare the times series of the price returns generated under different market mechanisms through the prism of "stylized facts" established in the literature that was shortly discussed in the Introduction. The returns are defined as $r_t = (p_t - p_{t-1})/p_{t-1}$, i.e. as relative price changes. All the statistics were computed over 1000 periods after 2000 transient and averaged over 100 random seeds.

The returns averages over time are close to zero for all three mechanisms and for all considered values of β. Similarly, the skewness of return, which measures the asymmetry of the distribution is close to zero for all mechanisms and all β. Both these statistics are in agreement with real data. In discussing other statistics, notice that in the tranquil regime under the WA, the price dynamics converge to the fundamental level, so that the higher order statistics are not defined.

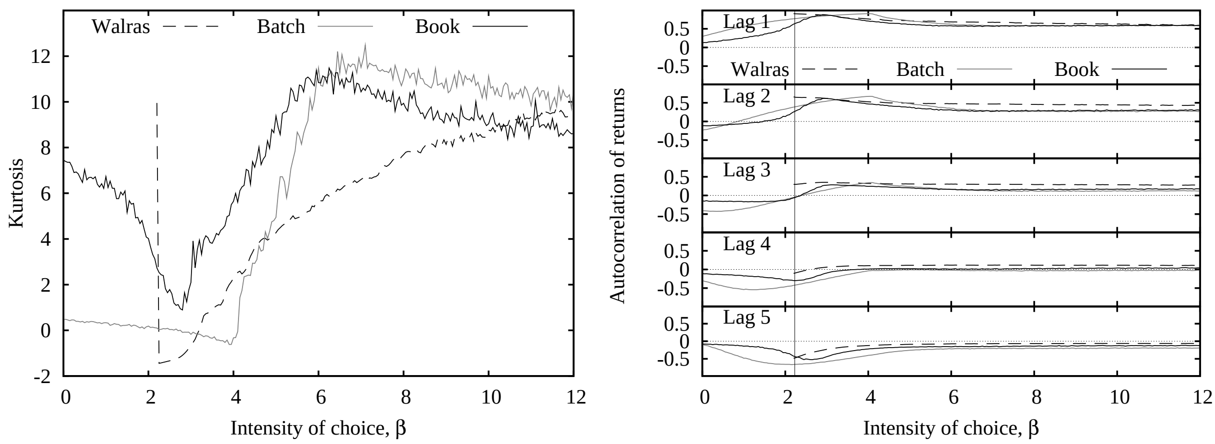

Figure 10: Time-series properties as a function of the intensity of choice β. **Left panel:** Excess kurtosis of the return distribution. **Right panel:** Autocorrelation of returns.

Comparison of the empirical return distributions with the normal distribution reveals that in the real markets returns exhibit a higher concentration around the mean and also the fatter tails. These properties of the distribution can be measured by *excess kurtosis* with respect to the kurtosis of the normal distribution, which is equal to 3. We find that the (excess) kurtosis of returns (see the left panel of Fig. 10) depends both on the value of the intensity of choice and on the market clearing mechanism. With increase of β, when it reaches the critical value of the market regime change, the kurtosis drops sharply, then it grows monotonically and then levels off converging to the relatively stable level. Close to the point of the regime change, the kurtosis is the highest under the BA and the lowest under the WA. For higher values of the intensity of choice, under all three mechanisms the kurtosis converges to the value similar to the one observed for the S&P 500 Index, which is 8.5 according to Gaunersdorfer and Hommes (2007). The dependence of kurtosis on behavioral parameters is in sharp contrast to the conclusions of Bottazzi et al. (2005) who find that skewness and kurtosis values depend only on market mechanism.

Linear unpredictability of the stock returns is another well-established regularity. It is usually verified by computing the autocorrelations of returns, which die out fast for the real data being insignificant already on the first lag. In the right panel of Fig. 10, we show the autocorrelations of returns for the first five lags as a function of the intensity of choice

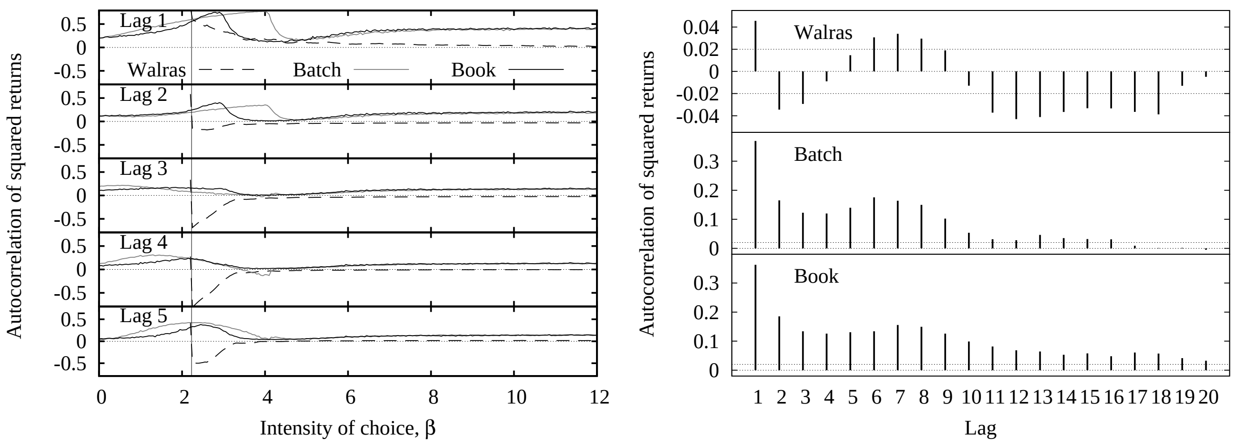

Figure 11: Autocorrelations of squared returns. **Left panel:** Autocorrelations for lags $1-5$ (top to bottom) as a function of the intensity of choice β and **Right panel:** Autocorrelation function for lags $1-20$ for fixed $\beta = 8.0$.

for all three mechanisms. In all cases we observe relatively large autocorrelations for the lags $1-3$ which is in contrast to the stylized facts. Relatively large autocorrelations of the returns produced by the model are the consequence of our behavioral assumptions, and, in particular, of the dominating trend-following behavior. Even if the modeling of a more realistic market architecture cannot "kill" the autocorrelations completely, it contributes to a certain improvement in the statistics. Indeed, conditional that the market is in the volatile regime, the autocorrelations are the largest under the WA.[13] Bottazzi et al. (2005) do not find conclusive evidence about the sign and magnitude of the autocorrelations of returns produced by their model, but indicate that they also significant on the first few lags.

Finally, we turn to the volatility clustering universally observed in the data. Volatility clustering, which suggests that despite the linear unpredictability of returns, they are not independent, can be identified as a presence of significant autocorrelations in the squared returns for a number of lags. In the left panel of Fig. 11 we show the autocorrelations of squared returns for lags $1-5$ (top to bottom) as a function of β. One immediately observes not only an

[13]The returns autocorrelations can be lowered to the zero level by adding a sufficiently large amount of dynamic noise, see e.g. Gaunersdorfer and Hommes (2007). Since it comes in a cost of understanding the dynamics of the model, we do not follow this tempting path.

expected dependence on the intensity of choice parameter β, but also a significant dependence on the market architecture. Namely, under both order-driven protocols the autocorrelations of the squared returns are always positive and relatively large. They decay slowly, which is consistent with the volatility clustering. In turn for the WA the autocorrelations of squared returns are generally close to zero or even negative.

To verify whether the squared returns generated by our model exhibit long memory, in the right panel of Fig. 11 we plot the autocorrelation function for the fixed value of $\beta = 8$ for three different mechanisms. The thin lines indicate 0.95 confidence limits. For the BA and the OB the squared returns show positive slow decaying correlation, while under the WA the auto-correlations are small and their pattern is atypical for financial series. We conclude that in our model the realistic patterns of volatility clustering can be attributed to the realistic order-driven mechanisms. Similarly Bottazzi et al. (2005) find volatility clustering and long memory of squared returns under the BA and the OB mechanisms.

6 Conclusion

Simulations presented in this paper contribute to the analysis of the interplay between behavioral ecologies of markets with heterogeneous traders and institutional market settings. We motivated our work by a presence of many regularities observed in financial markets and different approaches which economists exploit for explanation of them. However, since the dynamics of financial market is an outcome of a complicated interrelation between behavioral patterns and underlying market mechanism, we offer a route in between, starting with simple, analytically tractable model based on flexible behavioral assumptions and simulating it for a more realistic market setting.

Our gradual approach of introducing different market mechanisms in the market with heterogeneous agents was inspired by the work of Bottazzi et al. (2005). As opposed to our set-up, in their model agents did not change their strategies over time. As a result, Bottazzi et al. (2005) suggest that the time series properties are largely driven by market architecture.

We, however, clearly see that certain behavioral features are also important. In our model, no matter which type of market clearing is used, two different regimes with completely different dynamical properties occur depending on the value of the intensity of choice. On the other hand, trading protocol strongly affects the critical value of the intensity of choice, playing the role of the border line between two regimes. Furthermore, provided that the market is in volatile regime, the trading protocol also dictates the time series property.

We have also investigated an allocative efficiency of the market. The seminal paper of Gode and Sunder (1993) suggests that the continuous double auction leads to an allocatively efficient outcome even when agents trade at random. LiCalzi and Pellizzari (2007) explore this line of research and compare performances of four market protocols in terms of different criteria such as the time needed to converge to the equilibrium, traded volume and price volatility generated during this convergence. Agents valuations, or so called environment, is fixed in both paper. We consider a dynamic model with ever-changing environment, and find in this setting that there is simply not enough time to converge to an allocative efficiency outcome under the order- driven mechanisms. We find that the allocative efficiency loss is comparable for both mechanisms.

For better understanding of causal effects we keep our current model as parsimonious as possible. There is a number of extentions to the model that we consider in the future research. First, we plan to investigate the role of market orders in the order-driven protocols. Then we wish to increase the level of strategic behaviour in selecting the price level for the limit orders and also in choosing between the market order and the limit order. In terms of the market setup, we would like to consider the situation when the dividends are paid not every period, but only after a certain number of periods. It would also be interesting to consider an exogenous news arrival process. On the behavioural side, we would to extend the number of trading rules, and allow our trend-followers to learn the coeffcient of extrapolation and fundametalists to learn the coefficient of reversion.

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