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THE MODAL LOGIC OF FORCING

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Abstract. A set theoretical assertion ψ is *forceable* or *possible*, written $\Diamond\psi$, if ψ holds in some forcing extension, and *necessary*, written $\Box\psi$, if ψ holds in all forcing extensions. In this forcing interpretation of modal logic, we establish that if ZFC is consistent, then the ZFC-provable principles of forcing are exactly those in the modal theory S4.2.

1. Introduction

What are the most general principles in set theory relating forceability and truth? We are interested in how the set theoretical method of forcing affects the first order theory of a model of set theory. As with Solovay's celebrated analysis of provability, both this question and its answer are naturally formulated with modal logic.[1] We aim to do for forceability what Solovay did for provability.

Forcing was introduced by Paul Cohen in 1962 in order to prove the independence of the Axiom of Choice AC and the Continuum Hypothesis CH from the other axioms of set theory. In an explosion of applications, set theorists subsequently used it to construct an enormous variety of models of set theory and prove many other independence results. With forcing one builds an extension of any model V of set theory, in an algebraic manner akin to a field extension, by adjoining a new ideal object G, a V-generic filter over a partial order $\mathbb{P}$ in the ground model V, while preserving ZFC. The resulting forcing extension $V[G]$ is closely related to the ground model V, but may exhibit different set theoretical truths in a way that can often be carefully controlled. The method has become a fundamental tool in set theory.

Because the ground model V has some access via names and the forcing relation to the objects and truths of the forcing extension $V[G]$, there are clear affinities between forcing and modal logic. (One might even imagine the vast collection of

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[1] [Sol76]; for a survey of the result and the subsequent development of the field of provability logic, see also [JdJ98].

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all models of set theory, related by forcing, as an enormous Kripke model.) Accordingly, we define that a statement of set theory φ is *forceable* or *possible* if φ holds in some forcing extension, and φ is *necessary* if it holds in all forcing extensions. The modal notations $\Diamond\varphi$ and $\Box\varphi$ express, respectively, that φ is possible or necessary. This forcing interpretation of modal logic was introduced by the first author in [Ham03] in connection with the Maximality Principle, a new forcing axiom, with related work in [Lei04] and [HW05]. An alternative but related connection between modal logic and forcing was explored by Fitting and Smullyan in [SF96], and Blass [Bla90] provides an interpretation of modal logic in set theory that is not directly related to forcing. In [Boo93, Chapter 13], Solovay analyzed the set theoretic modalities of "true in all transitive models of ZF" and "true in V_κ for every inaccessible cardinal κ," respectively. This work has been extended in unpublished work of Enayat and Togha.

The forcing modal operators, of course, are eliminable in the language of set theory, because their meaning can be expressed in the usual language of set theory by means of the forcing relation or Boolean values. For example, $\Diamond\varphi$ simply means that there is some partial order $\mathbb{P}$ and condition $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \varphi$, and $\Box\varphi$ means that for all partial orders $\mathbb{P}$ and $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}} \varphi$. In this way, one can interpret $\Diamond\varphi$ and $\Box\varphi$ in any model of set theory.[2] In ZFC we may freely use a mixed language of set theory with the modal operators $\Diamond$ and $\Box$, understood as abbreviations with the forcing interpretation. Our goal is to discover which modal principles forcing must obey.

For example, the scheme $\Box\varphi \to \varphi$ expresses the statement "if φ is true in all forcing extensions, then it is true." This is valid, because every model of set theory is a (trivial) forcing extension of itself. Similarly, $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$ is valid for forcing because a statement φ is not forceable if and only if all forcing extensions satisfy $\neg\varphi$. The principle $\Box\varphi \to \Box\Box\varphi$ is valid because if φ holds in all forcing extensions, then so does $\Box\varphi$, since any forcing extension of a forcing extension is a forcing extension. The reader may easily verify that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is valid. The principle $\Diamond\Box\varphi \to \Box\Diamond\varphi$ is valid for forcing, because if φ is necessary in $V^{\mathbb{P}}$ and $V^{\mathbb{Q}}$ is an arbitrary extension, then φ is true in the product extension $V^{\mathbb{P}\times\mathbb{Q}}$, as this extends $V^{\mathbb{P}}$; consequently, φ is forceable over every such extension $V^{\mathbb{Q}}$. These modal assertions axiomatize the modal theory known as $\mathsf{S4.2}$, and a bit of formalization will help us express what we have observed. A *modal assertion* is a formula of propositional modal logic, such as $(\Box q_0 \to q_0)$, expressed with propositional variables q_i, Boolean connectives $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$ and the modal operators $\Diamond, \Box$. We write $\varphi(q_0, \ldots, q_n)$ to denote a modal assertion whose propositional variables are among $q_0, \ldots, q_n$.

[2]By formalizing forcing in ZFC, rather than the metatheory, one can sensibly force over any model of ZFC, without needing it to be countable or transitive and regardless of the metatheoretical objects, such as generic filters, which may or may not exist in a larger universe. In this syntactic account, one considers what is forced by various conditions without ever building the forcing extension as a structure. A semantic account of forcing over an arbitrary $M \models \mathrm{ZFC}$ is provided by the quotient $M^{\mathbb{P}}/U$ of the Boolean valued universe $M^{\mathbb{P}}$ by any ultrafilter U, with no need for U to be M-generic (even $U \in M$ works fine!). The original model M maps elementarily into the ground model of $M^{\mathbb{P}}/U$, and so one obtains the forcing extension as an actual structure (although the new ground model may not be isomorphic to M and may not even be well founded). Vopěnka seems to have been the first to do forcing this way [Vop65].

Main Definition 1. A modal assertion $\varphi(q_0, \ldots, q_n)$ is a *valid principle of forcing* if for all sentences ψ_i in the language of set theory, $\varphi(\psi_0, \ldots, \psi_n)$ holds under the forcing interpretation of $\Diamond$ and $\Box$.

The validity of φ, therefore, is expressible as a scheme in set theory. We say that $\varphi(q_0, \ldots, q_n)$ is a ZFC-*provable principle of forcing* if ZFC proves all such substitution instances $\varphi(\psi_0, \ldots, \psi_n)$. This naturally generalizes to larger theories with the notion of a T-provable principle of forcing. For any model $W \models$ ZFC, the modal assertion $\varphi(q_0, \ldots, q_n)$ is a *valid principle of forcing in* W if all substitution instances $\varphi(\psi_0, \ldots, \psi_n)$ are true in W. Hamkins posed the question:

Main Question 2. What are the valid principles of forcing?

For the ZFC-provable principles of forcing, our Main Theorem 6 provides the answer. Meanwhile, a few paragraphs ago, we essentially observed the following.

Theorem 3. *Every assertion in the modal theory* $\mathsf{S4.2}$ *is a* ZFC-*provable principle of forcing.*

Proof. The modal theory $\mathsf{S4.2}$ is obtained from the axioms

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Dual	$\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$
S	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
.2	$\Diamond\Box\varphi \to \Box\Diamond\varphi$

by closing under modus ponens and necessitation (from φ, deduce $\Box\varphi$). We observed earlier that each axiom is a valid principle of forcing, provably so as a scheme in ZFC. The ZFC-provable principles of forcing are clearly closed under modus ponens, and they are closed under necessitation because if $\varphi(\psi_0, \ldots, \psi_n)$ holds in all models of set theory, then so does $\Box\varphi(\psi_0, \ldots, \psi_n)$. $\Box$

Let us quickly show that none of the modal theories most commonly considered beyond $\mathsf{S4.2}$ are ZFC-provable principles of forcing. This follows from our main theorem, but we find it informative to have explicit failing instances of the principles; they are what pointed to $\mathsf{S4.2}$ as the right choice.

5	$\Diamond\Box\varphi \to \varphi$
M	$\Box\Diamond\varphi \to \Diamond\Box\varphi$
W5	$\Diamond\Box\varphi \to (\varphi \to \Box\varphi)$
.3	$\Diamond\varphi \wedge \Diamond\psi \to (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\psi \wedge \Diamond\varphi))$
Dm	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to (\Diamond\Box\varphi \to \varphi)$
Grz	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$
Löb	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$
H	$\varphi \to \Box(\Diamond\varphi \to \varphi)$

Observation 4. *If* ZFC *is consistent, then none of the above assertions is a* ZFC-*provable principle of forcing.*

Proof. (5) Let φ be the assertion "V $\neq$ L," where L denotes Gödel's constructible universe. This is easily forced, and once true it remains true in all further extensions. Thus, $\Diamond\Box\varphi$ holds in L, but not φ itself. So 5 is not valid for forcing over L.

(M) This is the McKinsey axiom, also known as .1. The Continuum Hypothesis CH is forceable over any model of set theory (see [Jec03, ex. 15.15]), so $\Box\Diamond\,\mathrm{CH}$ holds. But $\neg\mathrm{CH}$ is also forceable ([Jec03, 14.32]), so $\Diamond\Box\,\mathrm{CH}$ fails. Thus, M is invalid for forcing in every model of ZFC.

(W5) Let φ be the assertion "$\omega_1^{\mathrm{L}} < \omega_1$ or CH," which is true in L, but not necessary there, since one can force $\neg\mathrm{CH}$ without collapsing ω_1^{L}. Since one could collapse ω_1^{L}, however, φ is forceably necessary in any model of set theory. Thus, $\Diamond\Box\varphi$ holds in L, but not $\varphi\to\Box\varphi$, and so W5 fails as a principle of forcing in L.

(.3) Let φ assert "ω_2^{L} is a cardinal, but ω_1^{L} is not" and ψ assert "ω_1^{L} is a cardinal, but ω_2^{L} is not." These are each forceable over L, since ω_1^{L} and ω_2^{L} can be collapsed independently. Yet, whenever one of them holds, the other becomes unforceable. So the antecedent of this instance of .3 is true in L, while the conclusion is false, so .3 is not valid in L.

(Dm) This axiom is also known as Dum. Let φ assert "$\omega_1^{\mathrm{L}} < \omega_1$ or $\neg\mathrm{CH}$." Since one may collapse ω_1^{L}, this is possibly necessary, but false, in L. So $\mathrm{L}\models\neg(\Diamond\Box\varphi\to\varphi)$, falsifying the conclusion of this instance of Dm. For the antecedent, suppose a forcing extension satisfies $\mathrm{L}^{\mathbb{P}}\models\omega_1^{\mathrm{L}}=\omega_1$. Since we may force $\neg\mathrm{CH}$ over $\mathrm{L}^{\mathbb{P}}$, and then recover CH, both without collapsing ω_1^{L}, we conclude $\mathrm{L}^{\mathbb{P}}\models\neg\Box(\varphi\to\Box\varphi)$. It follows vacuously that $\mathrm{L}^{\mathbb{P}}\models\Box(\varphi\to\Box\varphi)\to\varphi$. Alternatively, if $\mathrm{L}^{\mathbb{P}}\models\omega_1^{\mathrm{L}}<\omega_1$, then φ is true there, so $\mathrm{L}^{\mathbb{P}}\models\Box(\varphi\to\Box\varphi)\to\varphi$. Thus, every forcing extension satisfies $\Box(\varphi\to\Box\varphi)\to\varphi$, and so $\mathrm{L}\models\Box(\Box(\varphi\to\Box\varphi)\to\varphi)$, fulfilling the antecedent of Dm in L. So Dm is not valid for forcing over L.

(Grz) This is the Grzegorczyk axiom. Since Grz is stronger than Dm, it is invalid in L by the previous case. We will show, however, that Grz is invalid in every model of set theory. Let φ be the CH. Since $\neg\mathrm{CH}$ is forceable over any model of set theory, $\Box\varphi$ is always false, and so $\varphi\to\Box\varphi$ is equivalent to $\neg\varphi$. So this instance of Grz reduces to $\Box(\Box\neg\mathrm{CH}\to\mathrm{CH})\to\mathrm{CH}$. Since CH is always forceable, $\Box\neg\mathrm{CH}$ is false, making the first implication, and the hypothesis of this instance of Grz, vacuously true. So this axiom instance will fail whenever CH fails. Similar reasoning shows that if φ is $\neg\mathrm{CH}$, then the corresponding axiom instance fails if CH holds. So Grz fails as a principle of forcing in every model of set theory.

(Löb) This axiom expresses the content of Löb's theorem in provability logic, where one interprets $\Box\varphi$ as "φ is provable." Under the forcing interpretation, however, it is invalid, because in fact it contradicts $\mathsf{S4}$. If φ is any contradiction, such as $0=1$, then $\Box\varphi$ is false, and consequently $\Box(\Box\varphi\to\varphi)$ is always true. So in any model of set theory, the hypothesis of this instance of the Löb axiom is true, while the conclusion is false. So Löb is never a valid principle of forcing.

(H) Let φ be the CH. As we have mentioned, this is necessarily possible, so $\Diamond\varphi$ always holds. In this case, therefore, $\Box(\Diamond\varphi\to\varphi)$ is equivalent to $\Box\varphi$, which is false. So H fails in any model where CH is true. By using $\neg\mathrm{CH}$ in the other models, we conclude that H fails as a principle of forcing in every model of set theory. $\Box$

The corresponding modal theories, listed in Figure 1 with their implication diagram, are obtained by closing the axioms under modus ponens, substitution and necessitation. This list is not exhaustive, as there are continuum many modal theories above $\mathsf{S4.2}$ that are not listed. We refer the reader to [CZ97] and [BdRV01] for excellent developments of modal logic, including the analysis of these and many other theories.

$$
\begin{aligned}
\mathsf{S5} &= \mathsf{S4} + 5 \\
\mathsf{S4W5} &= \mathsf{S4} + \mathrm{W5} \\
\mathsf{S4.3} &= \mathsf{S4} + .3 \\
\mathsf{S4.2.1} &= \mathsf{S4} + .2 + \mathrm{M} \\
\mathsf{S4.2} &= \mathsf{S4} + .2 \\
\mathsf{S4.1} &= \mathsf{S4} + \mathrm{M} \\
\mathsf{S4} &= \mathsf{K4} + \mathrm{S} \\
\mathsf{Dm.2} &= \mathsf{S4.2} + \mathrm{Dm} \\
\mathsf{Dm} &= \mathsf{S4} + \mathrm{Dm} \\
\mathsf{Grz} &= \mathsf{K} + \mathrm{Grz} = \mathsf{S4} + \mathrm{Grz} \\
\mathsf{GL} &= \mathsf{K4} + \text{Löb} \\
\mathsf{K4H} &= \mathsf{K4} + \mathrm{H} \\
\mathsf{K4} &= \mathsf{K} + 4 \\
\mathsf{K} &= \mathsf{K} + \mathrm{Dual}
\end{aligned}
$$

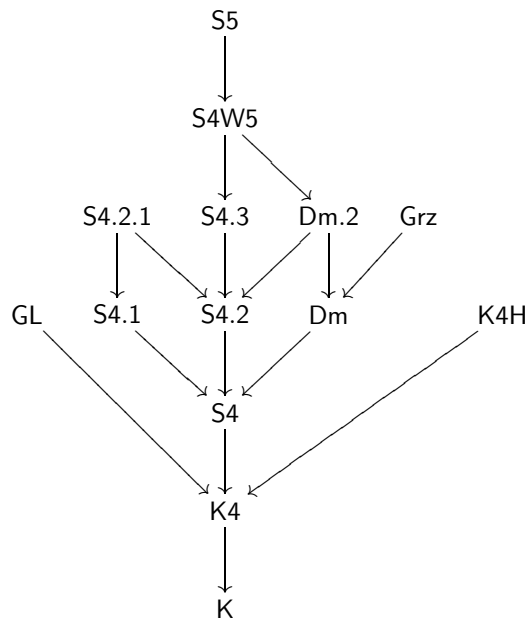

FIGURE 1. Some common modal theories

Corollary 5. *If* ZFC *is consistent, then none of the modal theories* S5*,* S4W5*,* S4.3*,* S4.2.1*,* S4.1*,* Dm.2*,* Dm*,* K4H*,* GL *or* Grz *are* ZFC*-provable principles of forcing, and all are invalid in* L*. The modal theories* S4.1*,* S4.2.1*,* K4H*,* GL *and* Grz *are invalid as principles of forcing in every model of set theory.*

So if the ZFC-provable principles of forcing constitute any previously known modal theory, then the best remaining candidate is S4.2.

This article is intended primarily for two audiences: set theorists interested in the fundamental principles of forcing and modal logicians interested in the application of their subject to set theory. While we felt it necessary in the arguments to assume a basic familiarity with forcing, we do provide references to specific elementary forcing results in the standard graduate set theory textbooks where this might be helpful. We were able to provide in our arguments a complete account of the necessary concepts from modal logic. Sections 2 and 3 cover the proof of our Main Theorem 6, which answers our Main Question 2 above. Section 2 contains a complete proof of the main theorem written primarily with the set theoretical reader in mind, and Section 3 emphasizes certain aspects of the proof for the modal logicians. After this, we apply our technique fruitfully in Sections 4–6 to various other instances of Main Question 2, by restricting the focus to a given model of set theory, by investigating the role of parameters in the valid principles of forcing, and by restricting attention to a natural class of forcing notions, such as those with the countable chain condition.

2. The main theorem

Our main theorem provides an answer to Question 2.

Main Theorem 6. If ZFC is consistent, then the ZFC-provable principles of forcing are exactly those in the modal theory $\mathsf{S4.2}$.

The rest of this section is devoted to the proof of this theorem. We begin with the key concepts of buttons and switches, which generalize the important features of the substitution statements arising in the proof of Observation 4. Specifically, a *switch* is a statement φ of set theory such that both φ and $\neg\varphi$ are necessarily possible, so that by forcing φ can be switched on or off at will. For example, the CH is a switch, because you can ensure either CH or $\neg$CH by forcing over any model of set theory. In contrast, a *button* is a statement that is (necessarily) possibly necessary. The button is *pushed* when it is necessary, and otherwise *unpushed*. The idea is that you can always push a button by making it necessary, but having done so, you cannot unpush it again. The assertion "ω_1^{L} is countable" is a button because it can be forced over any model of set theory and once it becomes true it remains true in all further extensions. This button is unpushed in L. The reader is invited to check that a statement is possibly necessary if and only if it is necessarily possibly necessary; at bottom, this amounts to the $\mathsf{S4.2}$ validity of $\Diamond\Box\varphi \leftrightarrow \Box\Diamond\Box\varphi$. Thus, a button remains a button in every forcing extension. Although it may seem at first that buttons and switches are very special sorts of statements, we invite the reader to check that in fact every statement in set theory is either a button, the negation of a button, or a switch (and these types are disjoint).

A collection of buttons b_n and switches s_m is *independent* in a model if first, all the buttons are unpushed in the model and second, necessarily (that is, in any forcing extension), any of the buttons can be pushed and any of the switches can be switched without affecting the value of any of the other buttons or switches. In other words, the collection of buttons and switches is independent in W if the buttons are unpushed in W, but in any forcing extension $W^{\mathbb{P}}$, whatever the pattern of buttons and switches is in $W^{\mathbb{P}}$, any button can be turned on by forcing to some $W^{\mathbb{P}*\dot{\mathbb{Q}}}$ without affecting the value of any of the other buttons or switches, and any switch can be turned either on or off by forcing to some $W^{\mathbb{P}*\dot{\mathbb{R}}}$ without affecting the value of any of the other buttons or switches. It follows, of course, that any finite pattern of buttons and switches being on or off is possible by forcing over W. For an account of buttons, switches and independence in an arbitrary Kripke model, see Definition 8 and the accompanying text.

We note that the counterexample substitution instances showing the forcing invalidity of the modal assertions in Observation 4 were each Boolean combinations of independent buttons and switches.

Lemma 6.1. *If* $\mathrm{V} = \mathrm{L}$*, then there is an independent collection of infinitely many buttons and infinitely many switches.*

Proof. For positive natural numbers n and m, let the button b_n be the assertion "ω_n^{L} is not a cardinal," and let the switch s_m be the assertion "the GCH holds at $\aleph_{\omega+m}$." If $\mathrm{V} = \mathrm{L}$, then it is clear that none of the buttons is true, but in any model of set theory, the button b_n can be made true by collapsing ω_n^{L}, without affecting the truth of any other button or the properties of the GCH above $\aleph_\omega$ (see [Jec03, 15.21]). Once the button b_n becomes true, it is clearly necessary,

because the cardinal ω_n^{L} will remain collapsed in any further forcing extension. The switches s_m are clearly switches, because with forcing one can arrange the values of the continuum function at $\aleph_{\omega+m}$ at will by forcing that adds no new bounded sequences below $\aleph_\omega$ (see [Jec03, 15.18 and related]). Thus, the switches can be set to any desired pattern without affecting any of the buttons. $\square$

A *lattice* is a partial order such that any two nodes a and b have a greatest lower bound or *meet*, denoted $a \wedge b$ and a least upper bound or *join*, denoted $a \vee b$. It follows that every nonempty finite set A has a meet $\bigwedge A$ and join $\bigvee A$, and that a finite lattice has a least and a greatest element.

Lemma 6.2. *If F is a finite lattice and W is a model of set theory with a sufficiently large independent family of buttons b_i, then to each node $w \in F$ we may assign an assertion p_w, a Boolean combination of the buttons, such that W satisfies:*

(1) *In any forcing extension, exactly one of the p_w is true, and $W \models p_{w_0}$, where w_0 is the minimal node of F.*
(2) *In any forcing extension satisfying p_w, the statement p_v is forceable if and only if $w \leq v$ in F.*

Proof. Let us associate a button b_u with each node $u \in F$. For any $A \subseteq F$, let $b_A = (\bigwedge_{u\in A} \Box\, b_u) \wedge (\bigwedge_{u\notin A} \neg \Box\, b_u)$ be the sentence asserting that exactly the buttons in A are pushed and no others. Let $p_w = \bigvee\{\, b_A \mid w = \bigvee A \,\}$ be the sentence asserting that the pattern of buttons that have been pushed corresponds to a set A with least upper bound w in F. Since every forcing extension must have some pattern A of buttons b_u pressed and every such A has a least upper bound in F, it is clear that p_w will be true in the model if and only if w is the least upper bound of A. Thus, in any forcing extension exactly one of the p_w is true, and $W \models p_{w_0}$, as all buttons are unpushed in W.

For the second claim, suppose that $W[G]$ is a forcing extension where p_w holds. Let $A = \{\, u \in F \mid W[G] \models b_u \,\}$ be the set of buttons that are pushed in $W[G]$. Since p_w is true, it must be that w is the join of A in F. If $w \leq v$ in F, then by pushing the button b_v and no others, we arrive at a forcing extension $W[G][H]$ with buttons pushed in $A \cup \{v\}$. Since this has join v, this means that p_v is true in $W[G][H]$, and hence p_v is forceable in $W[G]$, as desired. Conversely, suppose that p_v is forceable in $W[G]$. Thus, there is some further extension $W[G][H]$ satisfying p_v. This extension exhibits some pattern of buttons $\{\, b_u \mid u \in B \,\}$, where the join of B is v. Since $W[G][H]$ is a forcing extension of $W[G]$, the buttons in A remain pushed in $W[G][H]$, and so $A \subseteq B$. Thus, v must be at least as large as the join of A, which is w, so $w \leq v$ in F. $\square$

A *pre-lattice* is obtained from a lattice by replacing each node with a cluster of one or more equivalent nodes, all related by $\leq$. Equivalently, it is a partial pre-order $\leq$ (a reflexive and transitive relation) on a set F, such that the quotient of F by the equivalence relation $u \equiv v \leftrightarrow u \leq v \leq u$ is a lattice under the induced quotient relation $\leq$.

Lemma 6.3. *Suppose that F is a finite pre-lattice and $\{\, b_i, s_j \,\}_{i,j}$ is a sufficiently large finite independent family of buttons and switches in a model of set theory W. Let w_0 be any node in the minimal cluster of F. Then to each $w \in F$ we may*

assign an assertion p_w, a Boolean combination of the buttons and switches, such that $W \models p_{w_0}$ and:

(1) *In any forcing extension, exactly one of the p_w is true.*
(2) *In any forcing extension satisfying p_w, the statement p_v is forceable if and only if $w \leq v$ in F.*

Proof. The idea is to use the buttons as in Lemma 6.2 to determine which cluster is intended in the quotient lattice, and then use the switches to determine which node is intended within this cluster. Let $[u]$ denote the equivalence class of u in the quotient lattice $F/{\equiv}$, and let $p_{[u]}$ be the label assigned to $[u]$ in Lemma 6.2. Thus, $p_{[u]}$ is the disjunction of various complete patterns of buttons having supremum $[u]$. Suppose that the largest cluster of F has k nodes, and $k \leq 2^n$. For each subset $A \subseteq \{0, \ldots, n-1\} = n$, let $s_A = (\bigwedge_{i \in A} s_i) \wedge (\bigwedge_{i \notin A} \neg s_i)$ assert that the pattern of switches is specified by A. Since every pattern of switches is possible by forcing over W, every s_A is necessarily possible, and in any forcing extension of W, exactly one s_A holds. For each cluster $[u]$, assign to every $w \in [u]$ a nonempty set $\vec{A}_w = \{A^w_0, \ldots, A^w_{j_w}\}$ of subsets of n in such a way that the various $\vec{A}_w$ for $w \in [u]$ partition the subsets $A \subseteq n$. We may assign the pattern A of switches that happen to hold in W to any desired node w_0 in the minimal cluster of F. Let $s_w = \bigvee_{A \in \vec{A}_w} s_A$ assert that the switches occur in a pattern appearing in $\vec{A}_w$. Finally, define $p_w = p_{[w]} \wedge s_w$.

We now prove that this works. In any forcing extension of W, we know by Lemma 6.2 that exactly one $p_{[u]}$ is true. Furthermore, any forcing extension exhibits some pattern A of switches being true, and this A is assigned to exactly one $w \in [u]$, so exactly one s_w is true for $w \in [u]$. Thus, in any forcing extension, exactly one $p_w = p_{[u]} \wedge s_w$ is true. We arranged that p_{w_0} is true in W by the assignment of the pattern of switches holding in W to the world w_0.

If $W^{\mathbb{P}}$ is a forcing extension satisfying p_w, then both $p_{[w]}$ and s_w hold in $W^{\mathbb{P}}$. If $w \leq v$ in F, then we already know that $p_{[v]}$ is forceable in $W^{\mathbb{P}}$, and s_v is forceable from any extension of W without affecting the buttons, so $p_v = p_{[v]} \wedge s_v$ is forceable over $W^{\mathbb{P}}$, as desired. Conversely, if p_v is forceable over $W^{\mathbb{P}}$, this implies that $p_{[v]}$ is forceable over $W^{\mathbb{P}}$, and so $[w] \leq [v]$ in the quotient lattice. It follows that $w \leq v$ in F. $\square$

We now state some definitions from modal logic. A *propositional world*, also called a *state*, is a map of the propositional variables to the set {true, false}. This is simply a row in a truth table. A *Kripke model* M is a set U of propositional worlds, together with a relation R on U called the *accessibility* relation. The Kripke semantics define when a modal assertion φ is true at a world w in a Kripke model M, written $(M, w) \models \varphi$. Namely, for atomic assertions, $(M, w) \models q$ if q is true in w; for Boolean connectives, one uses the usual inductive treatment; for necessity, $(M, w) \models \Box\varphi$ if whenever $w \mathrel{R} v$, then $(M, v) \models \varphi$; and for possibility, $(M, w) \models \Diamond\varphi$ if there is v with $w \mathrel{R} v$ and $(M, v) \models \varphi$. The underlying *frame* of the model M is the structure $\langle U, R\rangle$, ignoring the internal structure of the elements of U. The reader may easily check that every Kripke model whose frame is a partial pre-order satisfies $\mathsf{S4}$, and every Kripke model on a directed partial pre-order satisfies $\mathsf{S4.2}$. A deeper fact is Lemma 6.5, that the finite pre-lattice frames are complete for $\mathsf{S4.2}$. If Λ is a modal theory and every Kripke model with frame F satisfies Λ at every world, then F is a Λ-*frame*. A modal theory Λ has the *finite frame* property if

whenever $\Lambda \nvdash \varphi$, then there is a finite Λ-frame F and a Kripke model having frame F in which φ fails. A class $\mathbb{F}$ of frames is *complete* for a modal theory Λ if every $F \in \mathbb{F}$ is a Λ-frame and any φ true in all Kripke models having frames in $\mathbb{F}$ is in Λ. In this case, we also say that Λ is *defined* by $\mathbb{F}$. Two Kripke models M and M' are *bisimilar* if there is a correspondence of their worlds $a \sim a'$ for $a \in M$, $a' \in M'$, not necessarily functional or one-to-one, such that corresponding worlds have the same truth assignments to the propositional variables, and whenever a accesses b in M and $a \sim a'$, then there is some b' in M' such that $b \sim b'$ and a' accesses b' in M', and conversely, whenever $a \sim a'$ and a' accesses b' in M', then there is some $b \sim b'$ such that a accesses b. It follows by induction that corresponding worlds in M and M' have exactly the same modal truths.

The next lemma is the heart of our argument, where we prove that the behavior of any Kripke model on a finite pre-lattice can be exactly simulated by forcing.

Lemma 6.4. *If M is a Kripke model whose frame is a finite pre-lattice with a world w_0 and W is a model of set theory with a sufficiently large independent family of buttons and switches, then there is an assignment of the propositional variables q_i to set theoretical assertions ψ_i, such that for any modal assertion φ we have*

$$(M, w_0) \models \varphi(q_0, \dots, q_n) \quad \textit{if and only if} \quad W \models \varphi(\psi_0, \dots, \psi_n).$$

Proof. Each ψ_i will be a certain Boolean combination of the buttons and switches. We have assumed that the frame F of M is a finite pre-lattice. We may assume without loss of generality that w_0 is an initial world of M, by ignoring the worlds not accessible from w_0. Let p_w be the assertions assigned according to Lemma 6.3. Since w_0 is an initial world of F, we may ensure that $W \models p_{w_0}$. Let $\psi_i = \bigvee\{\, p_w \mid (M, w) \models q_i \,\}$. We will prove the lemma by establishing the following stronger claim:

$$(M, w) \models \varphi(q_0, \dots, q_n) \quad \text{if and only if} \quad W \models \Box\left(p_w \to \varphi(\psi_0, \dots, \psi_n)\right).$$

This is true for atomic φ, since q_i is true at w if and only if p_w is one of the disjuncts of ψ_i, in which case $p_w \to \psi_i$ in any forcing extension of W, and conversely if $p_w \to \psi_i$ is true in a forcing extension where p_w is true, then ψ_i must be true there, in which case q_i is true at w in M. If the statement is true for φ_0 and φ_1, then it is also true for $\varphi_0 \wedge \varphi_1$. For negation, suppose that $(M, w) \models \neg\varphi(q_0, \dots, q_n)$. By induction, $W \not\models \Box(p_w \to \varphi(\psi_0, \dots, \psi_n))$, so there is a forcing extension $W^{\mathbb{P}}$ satisfying p_w and $\neg\varphi(\psi_0, \dots, \psi_n)$. Since the truth values of ψ_i necessarily depend only on the values of the various p_u, it follows that all forcing extensions with p_w will satisfy $\neg\varphi(\psi_0, \dots, \psi_n)$. So we have proved $W \models \Box(p_w \to \neg\varphi(\psi_0, \dots, \psi_n))$, as desired, and reversing the steps establishes the converse. Finally, $(M, w) \models \Diamond\, \varphi(q_0, \dots, q_n)$ if and only if $\exists u \geq w\, (M, u) \models \varphi(q_0, \dots, q_n)$, which occurs if and only if $W \models \Box(p_u \to \varphi(\psi_0, \dots, \psi_n))$. Since $W \models \Box(p_w \to \Diamond\, p_u)$, this implies $W \models \Box(p_w \to \Diamond\, \varphi(\psi_0, \dots, \psi_n))$, as desired. Conversely, if $W \models \Box(p_w \to \Diamond\, \varphi(\psi_0, \dots, \psi_n))$, then $\varphi(\psi_0, \dots, \psi_n)$ is forceable over any extension of W with p_w. Since all such extensions have p_u for some $u \geq w$ and the ψ_i depend only on the values of p_v, it must be that $W \models \Box(p_u \to \varphi(\psi_0, \dots, \psi_n)$ for some $u \geq w$. By induction, this is equivalent to $(M, u) \models \varphi(q_0, \dots, q_n)$ and consequently to $(M, w) \models \Diamond\, \varphi(q_0, \dots, q_n)$, as desired. $\Box$

The next step of our proof relies on a fact about $\mathsf{S4.2}$. A *tree* is a partial order $\leq$ on a set F such that the predecessors of any node are linearly ordered. A *pre-tree*

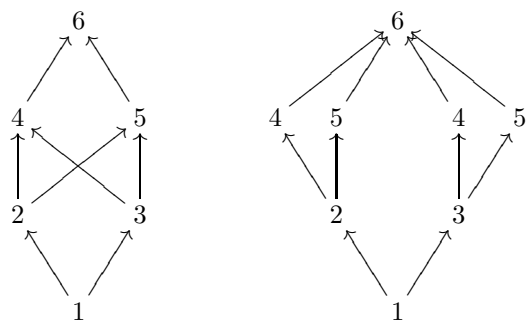

FIGURE 2. A partial order (left) and its partial unravelling, a baled tree (right)

is a partial pre-order $\leq$ on a set F such that the quotient $F/{\equiv}$ is a tree; each node of this tree is effectively replaced in F with a cluster of equivalent nodes. A *baled tree* is a partial order $\leq$ on a set F having a largest node $b \in F$, such that $F \setminus \{b\}$ is a tree (imagine baling or tying the top branches of a tree together, as in Figure 2). A *baled pre-tree* is the result of replacing each node in a baled tree with a cluster of equivalent nodes; equivalently, it is a partial pre-order whose quotient by $\equiv$ is a baled tree. Note that every baled tree is a lattice, and every baled pre-tree is a pre-lattice. A partial pre-order is *directed* if any two nodes have a common upper bound.

Lemma 6.5. *If a modal assertion φ is not in* $\mathsf{S4.2}$*, then it fails in some Kripke model M whose frame is a finite baled pre-tree, and hence a finite pre-lattice.*

Proof. It is easy to see that every Kripke model whose frame is a directed partial pre-order satisfies $\mathsf{S4.2}$, and conversely that any frame that is not a directed partial pre-order has a Kripke model violating $\mathsf{S4.2}$ (see [CZ97, Theorem 3.38]). Thus, $\mathsf{S4.2}$ is defined by the class of directed partial pre-orders. By [CZ97, Theorem 5.33], it is known that $\mathsf{S4.2}$ has the finite frame property, and so if $\mathsf{S4.2} \nvdash \varphi$, then φ fails in a Kripke model M_0 whose frame F_0 is a finite $\mathsf{S4.2}$ frame, which is to say, a finite directed partial pre-order. We will construct a Kripke model M that is bisimilar with M_0, and which consequently has the same modal theory, but whose frame is a finite baled pre-tree, and consequently a finite pre-lattice. Our construction is a minor modification of the standard technique of *tree unravelling* as described in the proof of [CZ97, Theorem 2.19]. We know that φ fails at some world w_0 in M_0, and we may assume that w_0 is in the smallest cluster of F_0. By directedness, F_0 has a largest cluster $[b]$. The quotient $F_0/{\equiv}$ is a finite directed partial order. For each $[u] \in F_0/{\equiv}$, let us say that t is a path from $[w_0]$ to $[u]$ in $F_0/{\equiv}$ if it is a maximal linearly ordered subset of the interval $[[w_0],[u]]$ in $F_0/{\equiv}$. Such paths form a tree when ordered by end-extension. Let F be the *partial unravelling* of F_0, except for the largest cluster. That is, F consists of the maximal cluster $[b]$ of F_0, together with the set of all pairs $\langle u, t\rangle$, where t is a path from $[w_0]$ to $[u]$ in $F_0/{\equiv}$ and $[u] \neq [b]$. The order on F is by end-extension of the paths t and the F_0 order on u, with $[b]$ still maximal. The worlds within any copy of a cluster are still equivalent and consequently still form a cluster, and so F is a baled pre-tree. Let M be the

resulting Kripke model on F, obtained by also copying the propositional values from every world $u \in F_0$ in M_0 to the copies $\langle u, t\rangle$ of it in F. It is easy to see that M is bisimilar with M_0, according to the correspondence that we have defined, because every world accesses in M copies of the worlds its corresponding world accesses in M_0 and only such copies. It follows that every world in M_0 satisfies exactly the same modal truths in M_0 that its copies satisfy in M. Consequently, φ fails at the copy of w_0 in M. Thus, φ fails in a Kripke model whose frame is a finite baled pre-tree, and all such frames are pre-lattices. $\square$

Proof of Theorem 6. Finally, we prove the theorem. By Theorem 3, the set of ZFC-provable principles of forcing includes $\mathsf{S4.2}$. If φ is not in $\mathsf{S4.2}$, then by Lemma 6.5, there is a Kripke model M on a finite pre-lattice in which φ fails at an initial world. It is well known that if ZFC is consistent, then so is ZFC+V = L, and so by Lemma 6.1, there is a model of set theory L having an infinite independent family of buttons and switches. By Lemma 6.4, there is an assignment of the propositional variables of φ to sentences ψ_i such that $\mathrm{L} \models \neg\varphi(\psi_0, \ldots, \psi_n)$. Therefore, φ is not a valid principle of forcing in L, and hence not a ZFC-provable principle of forcing. $\square$

3. The Jankov-Fine formula

While the previous section was written with a set theoretical reader in mind, let us now emphasize certain points for the modal logicians. The main theorem can be restated in a way (as follows) that aligns it with many other completeness theorems in modal logic.

Theorem 7. *If* ZFC *is consistent, then*

$$\mathsf{S4.2} \vdash \varphi(q_0, \ldots, q_n) \quad \textit{if and only if} \quad \forall \psi_0, \ldots, \psi_n \ \mathrm{ZFC} \vdash \varphi(\psi_0, \ldots, \psi_n),$$

where the ψ_i range over the sentences in the language of set theory and $\Box$ and $\Diamond$ are understood in ZFC *with the forcing interpretation.*

Modal logicians will recognize that Lemmas 6.2 and 6.3 assert exactly that the assertions p_w fulfill the relevant Jankov-Fine formula, which we now define. For any graph $F = \langle U, E\rangle$, assign a propositional variable p_w to each vertex $w \in U$ of the graph and let $\delta(F)$ be the following formula, the *Jankov-Fine formula*. It asserts that, necessarily, exactly one p_w is true, and if p_w is true, then $\Diamond p_v$ if and only if $w\ E\ v$.

$$\begin{aligned} \delta(F) = \quad & \Box \textstyle\bigvee_{w \in U} p_w && \wedge \\ & \Box \textstyle\bigwedge_{w \neq v} (p_w \to \neg p_v) && \wedge \\ & \Box \textstyle\bigwedge_{w\ E\ v} (p_w \to \Diamond p_v) && \wedge \\ & \Box \textstyle\bigwedge_{\neg w\ E\ v} (p_w \to \neg \Diamond p_v) \end{aligned}$$

Lemma 6.4 has nothing essentially to do with set theory, but rather only with Kripke models and the Jankov-Fine formula:

Lemma 7.1. *Suppose that M is a Kripke model whose frame F is a finite partial pre-order and w_0 is a world of M. If N is any other Kripke model satisfying* $\mathsf{S4}$ *(at some world u_0) and the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$, then there is an assignment of the propositional variables q_i of M to assertions ψ_i in N such that for any modal assertion φ,*

$$(M, w_0) \models \varphi(q_0, \ldots, q_n) \quad \textit{if and only if} \quad (N, u_0) \models \varphi(\psi_0, \ldots, \psi_n).$$

Proof. We argue just as in Lemma 6.4. Let p_w be the assertions in N satisfying the Jankov-Fine formula. Let $\psi_i = \bigvee\{\, p_w \mid (N,w) \models q_i \,\}$. We establish the following stronger claim by induction on φ:

$$(M,w) \models \varphi(q_0,\dots,q_n) \quad \Leftrightarrow \quad (N,u_0) \models \Box(p_w \to \varphi(\psi_0,\dots,\psi_n)).$$

The atomic case holds by the definition of ψ_i. Conjunction follows because $\Box$ distributes over $\wedge$. Negation follows via the properties of the Jankov-Fine formula, because every world in N (accessible from u_0) satisfies exactly one p_w, and any two such worlds agree on every ψ_i. Possibility follows using the Jankov-Fine formula again, since p_u is possible from a world with p_w if and only if $w \leq u$ in F. Finally, the stronger claim implies the lemma, because p_{w_0} is true at u_0 in N. $\Box$

A modal theory Λ is *closed under substitution* if $\varphi(\psi_0,\dots,\psi_n)$ is in Λ whenever $\varphi(q_0,\dots,q_n)$ is, for any modal assertions ψ_i.

Lemma 7.2. *Suppose that F is a finite partial pre-order. If a modal theory $\Lambda \supseteq \mathsf{S4}$ is closed under substitution and is consistent, for any $w_0 \in F$, with the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$, then F is a Λ-frame.*

Proof. If F is not a Λ-frame, then there is a Kripke model M with frame F and some $\varphi \in \Lambda$ such that $(M,w_0) \models \neg\varphi(q_0,\dots,q_n)$ for some $w_0 \in F$. If Λ is consistent with $\delta(F) \wedge p_{w_0}$, then there is a Kripke model satisfying $(N,u_0) \models \Lambda \wedge \delta(F) \wedge p_{w_0}$. By Lemma 7.1, there is an assignment $q_i \mapsto \psi_i$, where ψ_i is a Boolean combination of the p_w in N, such that $(N,u_0) \models \neg\varphi(\psi_0,\dots,\psi_n)$. This contradicts a substitution instance of $(N,u_0) \models \Lambda$, since $\varphi \in \Lambda$. $\Box$

Similarly, the concepts of button and switch are not set theoretic; they make sense in any Kripke model. Specifically:

Definition 8. A *button* is a statement that is necessarily possibly necessary, and a *switch* is a statement such that it and its negation are necessarily possible. A family of buttons $\{b_i\}_{i\in I}$ and switches $\{s_j\}_{j\in J}$ is *independent* in M at world u if none of the buttons is necessary at u and necessarily, any button can be turned on and any switch can be turned either on or off without affecting the other buttons and switches.

This can be expressed precisely in modal logic as follows. For any $A \subseteq I$ and $B \subseteq J$, let $\Theta_{A,B} = (\bigwedge_{i\in A} \Box b_i) \wedge (\bigwedge_{i\notin A} \neg\Box b_i) \wedge (\bigwedge_{j\in B} s_j) \wedge (\bigwedge_{j\notin B} \neg s_j)$ assert that the pattern of the buttons and switches is specified by A and B. The family $\{b_i\}_{i\in I} \cup \{s_j\}_{j\in J}$ of buttons and switches is independent if

$$\Big(\bigwedge_{i\in I} \neg\Box b_i\Big) \wedge \bigwedge_{\substack{A\subseteq I\\ B\subseteq J}} \Box\Big(\Theta_{A,B} \to \bigwedge_{\substack{A\subseteq A'\\ B'\subseteq J}} \Diamond\,\Theta_{A',B'}\Big).$$

Thus, the buttons are off initially, and necessarily, from any possible pattern of buttons and switches, any larger pattern of buttons and any pattern of switches is possible. The main technique in our proofs of Lemmas 6.2 and 6.3 appears to be very reminiscent of Smoryński's [Smo70] proof of de Jongh's theorem [dJ70] on Heyting's Arithmetic.

Lemma 7.3. *If F is a finite pre-lattice, $w_0 \in F$ and $\Lambda \supseteq \mathsf{S4}$ is a modal theory consistent with a sufficiently large independent family of buttons and switches, then Λ is consistent with the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$.*

Proof. Suppose that $(M, u_0) \models \Lambda$ has a sufficiently large independent family of buttons and switches. The proofs of Lemmas 6.2 and 6.3 work in M. Specifically, in those arguments we assigned to each node w in F an assertion p_w, a Boolean combination of buttons and switches, so that at any world u accessible from u_0 in M, exactly one of the p_w is true, and if p_w is true, then $\Diamond p_v$ holds if and only if $u \leq v$ in F. Thus, $(M, u_0) \models \delta(F)$. By assigning the pattern of switches that happens to hold in M at u_0 to the node w_0, we also arranged $(M, u_0) \models p_{w_0}$. So Λ is consistent with the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$. □

The lemmas combine to prove Theorems 6 and 7 as follows. Let Λ be the ZFC-provable principles of forcing. It is easy to see, as in the proof of Theorem 3, that Λ is closed under substitution, modus ponens and necessitation. By Theorem 3, we know $\mathsf{S4.2} \subseteq \Lambda$. By Lemma 6.1, if ZFC is consistent, then there are models of set theory having infinite independent families of buttons and switches. It follows that Λ is consistent with arbitrarily large finite independent families of buttons and switches. By Lemma 7.3, therefore, Λ is consistent with the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$ for any finite pre-lattice F. By Lemma 7.2, therefore, all such F are Λ-frames. By Lemma 6.5, any statement not in $\mathsf{S4.2}$ fails in a Kripke model having such a frame and consequently is not in Λ. So $\Lambda \subseteq \mathsf{S4.2}$ and, consequently, $\Lambda = \mathsf{S4.2}$.

Let us now push these techniques a bit harder, in order to arrive at a new class of frames complete for $\mathsf{S4.2}$ and some useful characterizations of $\mathsf{S4}$, $\mathsf{S4.2}$ and $\mathsf{S5}$. A partial pre-order $(B, \leq)$ is a *pre-Boolean algebra* if the quotient partial order $B/{\equiv}$ is a Boolean algebra.

Lemma 9. *For any natural numbers n and m, there is a Kripke model N whose frame is a finite pre-Boolean algebra, such that at any initial world in N, there is an independent family of n buttons and m switches.*

Proof. Using power sets, let $F = \mathrm{P}(n) \times \mathrm{P}(m)$, so that the nodes of F consist of pairs (A, B), where $A \subseteq n = \{0, \ldots, n-1\}$ and $B \subseteq m = \{0, \ldots, m-1\}$. The order is determined by the first coordinate only, so that $(A, B) \leq (A', B')$ if and only if $A \subseteq A'$. This is clearly a partial pre-order. The corresponding equivalence relation is $(A, B) \equiv (A', B')$ if and only if $A = A'$, and so the quotient is isomorphic to the power set $\mathrm{P}(n)$, which is a finite Boolean algebra. So F is a finite pre-Boolean algebra. Let N be the Kripke model on F in which b_i is true at (A, B) when $i \in A$ and s_j is true at (A, B) when $j \in B$. Clearly, every b_i is a button in N and every s_j is a switch in N, and they are independent at any initial world of N, because whatever the pattern (A, B) of buttons and switches in any world of N, any larger pattern of buttons $A' \supseteq A$ and any pattern of switches $B' \subseteq m$ is possible. □

The Kripke model N produced in Lemma 9 has the smallest frame supporting an independent family of n buttons and m switches, because for independence one needs worlds realizing every pattern (A, B) of buttons and switches.

Lemma 10. *A class $\mathbb{F}$ of frames is complete for $\mathsf{S4.2}$ if and only if every $F \in \mathbb{F}$ is an $\mathsf{S4.2}$ frame and there are Kripke models, with frames in $\mathbb{F}$, for arbitrarily large finite independent families of buttons and switches.*

Proof. The forward implication is immediate, because $\mathsf{S4.2}$ is consistent by Lemma 9 with the existence of large independent families of buttons and switches. Conversely, suppose the latter property. If φ_0 is not in $\mathsf{S4.2}$, then by Lemma 6.5 there is some Kripke model M whose frame F is a finite pre-lattice such that

$(M, w_0) \models \neg\varphi_0(q_0, \ldots, q_n)$. Let (N, u_0) be a Kripke model with frame in $\mathbb{F}$ having an independent family of n buttons and m switches, where n is the number of clusters in F and the size of any cluster is at most 2^m. By the proof of Lemma 7.3, there are assertions p_w for $w \in F$ such that $(N, u_0) \models \delta(F) \wedge p_{w_0}$. By Lemma 7.1, there is an assignment of the propositional variables q_i of M to assertions ψ_i in N such that $(M, w_0) \models \varphi(q_0, \ldots, q_n)$ if and only if $(N, u_0) \models \varphi(\psi_0, \ldots, \psi_n)$. By the assumption on φ_0, this means that $(N, u_0) \models \neg\varphi_0(\psi_0, \ldots, \psi_n)$. Thus, a substitution instance of φ_0 fails at a world in N, a Kripke model whose frame is in $\mathbb{F}$. So this class of frames is complete for $\mathsf{S4.2}$. □

Theorem 11. *The following sets of frames are complete for* $\mathsf{S4.2}$.

(1) *Finite directed partial pre-orders.*
(2) *Finite pre-lattices.*
(3) *Finite baled pre-trees.*
(4) *Finite pre-Boolean algebras.*

Proof. All of these frames are directed partial pre-orders, and so they are all $\mathsf{S4.2}$ frames. Lemma 6.5 shows that any statement not in $\mathsf{S4.2}$ fails in a Kripke model whose frame is a finite baled pre-tree, and hence a finite pre-lattice and a finite directed pre-order, so these classes are complete for $\mathsf{S4.2}$. The new part of this theorem is (4). By Lemma 9 the class of finite pre-Boolean algebras has Kripke models for arbitrarily large finite independent families of buttons and switches. So by Lemma 10, this class is also complete for $\mathsf{S4.2}$. □

Let us summarize what we have proved about $\mathsf{S4.2}$:

Theorem 12. *Suppose that the modal theory* Λ *contains* $\mathsf{S4}$ *and is closed under substitution. For any class* $\mathbb{F}$ *of finite frames complete for* $\mathsf{S4.2}$*, the following are equivalent:*

(1) Λ *is consistent with arbitrarily large finite independent families of buttons and switches.*
(2) Λ *is consistent with the Jankov-Fine formula* $\delta(F) \wedge p_w$ *for any frame* $F \in \mathbb{F}$ *and world* $w \in F$.
(3) *Every frame in* $\mathbb{F}$ *is a* Λ*-frame.*
(4) $\Lambda \subseteq \mathsf{S4.2}$.

Proof. Lemma 7.2 shows that (2) implies (3), as the elements of $\mathbb{F}$ must be finite directed partial pre-orders. For (3) implies (4), observe that if $\varphi \notin \mathsf{S4.2}$, then it must fail in a Kripke model whose frame is in $\mathbb{F}$, contrary to (3). For (4) implies (2), note that $\mathsf{S4.2}$ and $\delta(F) \wedge p_{w_0}$ are true together in the Kripke model having frame F, with p_w true exactly in the world w. For (4) implies (1), if $\Lambda \subseteq \mathsf{S4.2}$, then Λ is true in the Kripke models constructed in Lemma 9, which have large independent families of buttons and switches. Finally, Lemma 7.3 shows that (1) implies (2) in the special case where $\mathbb{F}$ is the class of all finite pre-lattices, and hence (1) implies (4) for any $\mathbb{F}$, since they do not depend on $\mathbb{F}$, completing the proof. □

Our later analysis will benefit from similar characterizations of $\mathsf{S4}$ and $\mathsf{S5}$. For $\mathsf{S5}$, we use the fact that the class of finite complete reflexive graphs is complete for $\mathsf{S5}$. This result will be applied in Theorem 17.

Theorem 13. *Suppose that a modal theory Λ contains* $\mathsf{S4}$ *and is closed under substitution. Then the following are equivalent:*

(1) *Λ is consistent with arbitrarily large finite independent families of switches.*
(2) *Λ is consistent with the Jankov-Fine formula $\delta(F) \wedge p_w$ for any finite complete graph F and world $w \in F$.*
(3) *Every finite complete reflexive graph is a Λ-frame.*
(4) $\Lambda \subseteq \mathsf{S5}$.

Proof. For (1) implies (2), the point is that when F has only one cluster, the argument of Lemma 6.3 does not require any buttons. Suppose that N is a Kripke model having an independent family of switches $\{s_j\}_{j \in J}$. For any $A \subseteq J$, define s_A as in Lemma 6.3 to assert that the pattern of switches is A. Partition the collection of $A \subseteq J$ among the worlds $u \in F$ by assigning a nonempty set $\vec{A}_u$ of sets to each world u in F. Let $s_u = \bigwedge_{A \in \vec{A}_u} s_A$ as in Lemma 6.3. By assigning whatever pattern of switches holds at u_0 to the world w, we can arrange that $(N, u_0) \models s_w$. Since every world must have some unique pattern of switches, it follows that (M, u_0) satisfies that necessarily, exactly one s_u is true. Furthermore, since the switches are independent, we also know that $(M, u_0) \models \Box\, s_u$ for any $u \in F$. Since all worlds in F are accessible from each other, this implies $(N, u_0) \models \delta(F) \wedge p_w$, as desired.

Lemma 7.2 shows that (2) implies (3). Also, (3) implies (4) because any statement not in $\mathsf{S5}$ is known to fail in a Kripke model whose frame is a finite complete reflexive graph (see [CZ97, Proposition 3.32, Corollary 5.29]). Finally, if $\Lambda \subseteq \mathsf{S5}$, then Λ holds in any Kripke model whose frame is a complete graph. It is easy to arrange independent families of switches in such Kripke models, just by ensuring that every possible pattern of switches is exhibited in some world. $\Box$

In the $\mathsf{S4}$ context, one can generalize Theorems 12 and 13 to the following, which we expect will be relevant for Question 36. Recall that a modal logic Θ has the finite frame property if whenever $\Theta \not\vdash \varphi$, then there is a finite Θ-frame F and a Kripke model M with frame F in which φ fails. Such theories as $\mathsf{S4}$, $\mathsf{S4.2}$, $\mathsf{S4.3}$ and $\mathsf{S5}$ are known by [CZ97, 5.29, 5.32, 5.33] to have the finite frame property.

Theorem 14. *Suppose that Θ is a modal logic containing* $\mathsf{S4}$ *and having the finite frame property, such as* $\mathsf{S4}$, $\mathsf{S4.2}$, $\mathsf{S4.3}$ *or* $\mathsf{S5}$. *If Λ contains* $\mathsf{S4}$ *and is closed under substitution, then for any class $\mathbb{F}$ of frames complete for Θ, the following are equivalent:*

(1) *Λ is consistent with the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$ for any frame $F \in \mathbb{F}$ and world $w_0 \in F$.*
(2) *Every frame in $\mathbb{F}$ is a Λ-frame.*
(3) $\Lambda \subseteq \Theta$.

Proof. If the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$ holds in a Kripke model $(N, u_0) \models \Lambda$, then by Lemma 7.1, any Kripke model M with frame F has an assignment $q_i \mapsto \psi_i$ such that $(M, w_0) \models \varphi(q_0, \ldots, q_n)$ if and only if $(N, u_0) \models \varphi(\psi_0, \ldots, \psi_n)$. In particular, if φ fails in such an M, then the corresponding substitution instance of it will fail in $N \models \Lambda$. So every such F is a Λ-frame, and we have proved (1) implies (2). For (2) implies (3), observe that if φ is not in Θ, then it fails in a Kripke model M having a finite frame $F \in \mathbb{F}$. Since F is a Λ-frame, we know that $M \models \Lambda$ and consequently φ is not in Λ. So $\Lambda \subseteq \Theta$. For (3) implies (1), observe that the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$ is easily satisfied at a Kripke model M having

frame F, where p_w is true exactly at w. Since F is a Θ-frame, we know that $M \models \Theta$ and consequently $M \models \Lambda$ by (3), so (1) holds, and the proof is complete. $\square$

We do not have a button-and-switch characterization of $\mathsf{S4}$ in Theorem 14, because the frame of an $\mathsf{S4}$ model need not be directed, and it is not true that every possibly necessary statement is necessarily possibly necessary. Under $\mathsf{S4}$, one can have assertions φ such that, simultaneously, φ is possibly necessary, $\neg\varphi$ is possibly necessary, and φ is possibly a switch. Thus, for the $\mathsf{S4}$ context, we emphasize the official definition of button as a statement that is *necessarily* possibly necessary. Such examples show that unlike $\mathsf{S4.2}$, under $\mathsf{S4}$ it is no longer true that every statement is a button, the negation of a button or a switch.

4. Forcing over a fixed model of set theory

While our main theorem establishes that the ZFC-provable principles of forcing are exactly those in $\mathsf{S4.2}$, it is not true that every model of set theory observes only these validities. For any $W \models \mathrm{ZFC}$, recall that a modal assertion φ is a *valid principle of forcing in* W if for all sentences ψ_i in the language of set theory we have $W \models \varphi(\psi_0, \ldots, \psi_n)$. For meta-mathematical reasons connected with Tarski's theorem on the non-definability of truth, there is initially little reason to expect that the collection of such φ should be definable in W; rather, the assertion that φ is valid in W is expressed as a scheme, asserting all substitution instances $\varphi(\psi_0, \ldots, \psi_n)$ in W. So this is formally a second-order notion. Because a statement is provable in ZFC exactly when it holds in all models of ZFC, our main theorem establishes that the modal assertions φ that are valid in all models of set theory are exactly those in $\mathsf{S4.2}$. Our proof, however, established the following stronger result.

Theorem 15. *If W is a model of set theory with arbitrarily large finite independent families of buttons and switches, then the valid principles of forcing in W are exactly* $\mathsf{S4.2}$.

This result is stronger because it shows that the minimal set of forcing validities is realized in a single model of set theory (such as any model of $\mathrm{V} = \mathrm{L}$), rather than arising as the intersection of the validities of several models. Nevertheless, there are models of set theory whose valid principles of forcing go beyond $\mathsf{S4.2}$. For example, the Maximality Principle MP of [Ham03] asserts all instances of the scheme $\Diamond\Box\psi \to \psi$, so that any set theoretic statement ψ that holds in some forcing extension and all further extensions is already true. In other words, MP asserts that $\mathsf{S5}$ is valid for forcing. Because it is established in [Ham03] that if ZFC is consistent, then so is $\mathrm{ZFC} + \mathrm{MP}$, we conclude:

Theorem 16. *If* ZFC *is consistent, then it is consistent with* ZFC *that all* $\mathsf{S5}$ *assertions are valid principles of forcing.*

The forcing validities of a model, however, never go beyond $\mathsf{S5}$.

Theorem 17. *The valid principles of forcing in any model of set theory are included within* $\mathsf{S5}$.

Proof. Let Λ be the set of forcing validities in a model W of set theory. By Theorem 3, this includes all of $\mathsf{S4.2}$. Also, Λ is easily seen to be closed under substitution. Observe next that any model of set theory has an infinite independent family of switches, such as s_n = "the GCH holds at $\aleph_n$." These and their negations are

forceable in any finite pattern over any model of set theory by well known forcing arguments. It follows that Λ is consistent with the modal assertions that there are large independent families of switches. By Theorem 13, consequently, $\Lambda \subseteq \mathsf{S5}$. $\square$

Corollary 18. *If* ZFC *is consistent, then there is a model of set theory whose valid principles of forcing are exactly* $\mathsf{S5}$.

Our results establish that both $\mathsf{S4.2}$ and $\mathsf{S5}$ are realized as the exact set of forcing validities of a model of set theory (realized, respectively, in models of $\mathrm{V} = \mathrm{L}$ or of MP).

Question 19. Which modal theories arise as the valid principles of forcing in a model of ZFC?

For example, can there be a model of set theory whose valid principles of forcing are exactly $\mathsf{Dm.2}$ or exactly $\mathsf{S4.3}$? We have seen that if a model of set theory has sufficiently many independent buttons and switches, then the valid principles of forcing will be only $\mathsf{S4.2}$. At the other extreme, if there are no buttons, then MP holds and so the valid principles of forcing in the model will be $\mathsf{S5}$. Is it possible to have a model with a finite bound on the size of an independent family of buttons?

Question 20. Is there a model of ZFC with one unpushed button but not two independent buttons?

What are the valid principles of forcing in such models? It is clear that if there are only finitely many buttons in a model (meaning that the independent families of buttons have some bounded finite size), then we could use Lemmas 6.3 and 6.4 to simulate any Kripke model on a pre-lattice frame with a correspondingly bounded number of clusters. The various classes of such frames generate modal theories strictly between $\mathsf{S4.2}$ and $\mathsf{S5}$. Are these realizable as the forcing validities of models of set theory? Are these the only modal theories that arise? The next theorem is a start on these questions. Denote the collection of modal assertions φ that are valid in W by Force^W. Two buttons b_0 and b_1 are *semi-independent* if both are unpushed and one can push b_0 without pushing b_1.

Theorem 21. *Suppose that W is a model of* ZFC *set theory.*

(1) $\mathsf{S4.2} \subseteq \mathrm{Force}^W \subseteq \mathsf{S5}$.
(2) *If W has no unpushed buttons, then* $\mathrm{Force}^W = \mathsf{S5}$.
(3) *If W has at least one unpushed button, then* $\mathrm{Force}^W \subsetneq \mathsf{S5}$.
(4) *If W has two semi-independent buttons, then* W5 *is not valid in W, and so* $\mathsf{S4W5} \not\subseteq \mathrm{Force}^W$.
(5) *If W has two independent buttons, then* .3 *is not valid in W, and so* $\mathsf{S4.3} \not\subseteq \mathrm{Force}^W$.
(6) *If W has an independent family of one button and one switch, then* Dm *is not valid in W, and so* $\mathsf{Dm} \not\subseteq \mathrm{Force}^W$.

Proof. Statement (1) is the content of Theorems 3 and 17. Over $\mathsf{S4}$, the additional $\mathsf{S5}$ axiom is equivalent to $\Diamond\Box\varphi \to \Box\varphi$, which exactly asserts that every button is pushed. So (2) and (3) hold. For (4), suppose that b_0, b_1 are semi-independent buttons in W, and let $\varphi = (\neg\Box b_0 \wedge \neg\Box b_1) \vee \Box(b_0 \wedge b_1)$, which asserts that either neither or both buttons are pushed. Thus, φ is both true and possibly necessary in W, since the buttons are initially unpushed and we could push both, but φ is

not necessary, since we could push just b_0; this violates W5. For (5), suppose that W has two independent buttons b_1 and b_2. We argue as in Observation 4. Let $\varphi = \Box b_1 \wedge \neg \Box b_2$ and $\psi = \Box b_2 \wedge \neg \Box b_1$. Since both buttons are unpushed in W and either may be pushed, we conclude that $W \models \Diamond \varphi \wedge \Diamond \psi$. But in any forcing extension of W, if φ is true, then ψ is impossible and vice versa. So this instance of the conclusion of .3 fails in W. Thus, .3 is not valid in W, and so Force^W does not include $\mathsf{S4.3}$. For (6), suppose that W has an independent family of one button b and one switch s. We follow the argument of Observation 4, case Dm. We may assume that both $\Box b$ and s are false in W. Let $\varphi = \Box b \vee s$. This is possibly necessary in W, since one could push the button b, but not true in W, so $\Diamond \Box \varphi \to \varphi$ is false in W. In any forcing extension of W, if $\Box(\varphi \to \Box \varphi)$, then it must be that the button b has been pushed there, since otherwise one could make φ true and then false again by flipping the switch s. So we have argued that $(\Box(\varphi \to \Box \varphi)) \to \varphi$ holds in every forcing extension of W. Thus, the hypothesis of this instance of Dm holds in W, while the conclusion fails, so Dm is not valid in W. This establishes (6). $\Box$

Semi-independent buttons are part of the following more general arrangement. A list of assertions $\varphi_1, \ldots, \varphi_n$ is a *volume control* if each is a button, pushing any of them necessarily also pushes all the previous, and any of them can be pushed without pushing the next. More precisely:

$$\Box \bigwedge_{i<n} (\Box \varphi_{i+1} \to \Box \varphi_i) \wedge (\Diamond \Box \varphi_{i+1}) \wedge \big(\neg \Box \varphi_{i+1} \to \Diamond(\neg \varphi_{i+1} \wedge \Box \varphi_i)\big).$$

The idea is that one can turn up the volume to level j by forcing $\Box \varphi_j$, but there is no turning it down again. The volume control has volume zero if $\neg \Box \varphi_1$ (and so a volume control of length n has $n+1$ many volume settings). These volume controls exhibit the linear dependence of buttons, rather than independence. If buttons b_0 and b_1 are semi-independent, then $\Box b_0$, $\Box(b_0 \wedge b_1)$ is a volume control of length 2; conversely, every volume control of length 2 at volume zero consists of two semi-independent buttons. Similar ideas with more buttons produce arbitrarily long volume controls. A Kripke model with a linear frame of $n+1$ clusters admits volume controls of length n but no independent buttons. A family of volume controls, buttons and switches is *independent* in a model if all the volume controls have zero volume in that model, all the buttons are unpushed in that model, and in any forcing extension, one can operate any of the volume controls, buttons and switches without affecting any of the others.

Theorem 22. *If W is a model of set theory exhibiting arbitrarily long volume controls independent from arbitrarily large families of independent switches, then* $\mathrm{Force}^W \subseteq \mathsf{S4.3}$.

Proof. We know by [CZ97, 3.31, 3.32, 3.40, 5.33] that the finite linear pre-orders are a complete class of $\mathsf{S4.3}$ frames. Suppose that F is such a finite linear pre-order and w_0 is a node in the minimal cluster of F. Let $v_1, \ldots, v_n$ be a volume control in W, where n is the number of clusters of F, and suppose that this volume control forms an independent family with the switches $s_0, \ldots, s_m$, where the size of any cluster is at most 2^m. As in Lemmas 6.2 and 6.3, we will assign to each node $w \in F$ an assertion p_w, so that W satisfies the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$. Specifically, within each cluster $[u]$, assign to each node $w \in [u]$ a nonempty set $\vec{A}_w$

of subsets $A \subseteq n$ in such a way that these partition all subsets $A \subseteq n$, and define $s_w = \bigvee_{A \in \vec{A}_w} s_A$, as in Lemma 6.3. In the least cluster, we assign the pattern A of switches holding in W to the node w_0. Now, for any node $w \in F$, if w is in the ith cluster, then we define $p_w = v_i \wedge s_w$. That is, the volume control indicates the intended cluster and the switch indicates the intended node within that cluster. Since every forcing extension of W exhibits some volume setting and some pattern of switches, it is clear that it will satisfy exactly one of the p_w, and in any extension of W where p_w holds, then $\Diamond p_v$ holds if and only if $w \leq v$. Thus, the Jankov-Fine formula $\delta(F) \wedge p_{w_0}$ is satisfied. It now follows by Theorem 14 that F is a Force^W-frame. So every $\mathsf{S4.3}$-frame is a Force^W-frame, and so $\text{Force}^W \subseteq \mathsf{S4.3}$. $\square$

George Leibman has observed that a volume control of length ω^2, with sufficiently uniform assertions, can be used to construct an independent family of one volume control of length ω and infinitely many switches, as follows. One divides the ω blocks into increasingly large finite blocks, with the position within these sub-blocks indicating the configuration of the switches, and the current ω block indicating the overall new volume of length ω. Thus, it is sufficient in Theorem 22 to have merely a (sufficiently uniform) volume control of length ω^2, or actually volume controls of length $\omega \cdot n$ for arbitrarily large n, without worrying about independent switches.

We close this section with a curious question. A modal theory is *normal* if it is closed under modus ponens, substitution and necessitation. All the named modal theories that we have considered, such as $\mathsf{S4}$, $\mathsf{S4.2}$, $\mathsf{S4.3}$, $\mathsf{Dm.2}$, $\mathsf{S5}$ and so on, are normal. Meanwhile, the set of valid principles of forcing in any model of ZFC is easily seen to be closed under modus ponens and substitution. But is it closed under necessitation?

Question 23. If φ is a valid principle of forcing, does it remain valid in all forcing extensions?

In other words, is $\Box\varphi$ also valid? Equivalently, if $W \models \text{ZFC}$, then is Force^W normal? Of course, if Force^W is necessarily $\mathsf{S4.2}$ or $\mathsf{S5}$, then the answer is yes, so this question is related to Questions 19 and 26.

5. The modal logic of forcing with parameters

We know from [Ham03] that parameters play a subtle role in the strength of the Maximality Principles. While MP is equiconsistent with ZFC, allowing real parameters in the scheme results in the principle $\text{MP}(\mathbb{R})$, which has some large cardinal strength; allowing uncountable parameters leads to inconsistency. Allowing real parameters from all forcing extensions leads to a principle $\Box\text{MP}(\mathbb{R})$ with a large cardinal strength of at least infinitely many Woodin cardinals (but less than $\text{AD}_{\mathbb{R}} + \Theta$ is regular). So let us analyze the role played by parameters in the valid principles of forcing. Specifically, define that $\varphi(q_0, \ldots, q_n)$ is a *valid principle of forcing in a model W with parameters in X* if for any set theoretical formulas $\psi_i(\vec{x})$ we have

$$W \models \forall \vec{x} \in X \ \varphi(\psi_0(\vec{x}), \ldots, \psi_n(\vec{x})).$$

We denote the collection of such φ by $\text{Force}^W(X)$. The next theorem provides another answer to Question 2.

Theorem 24. *In any model of set theory, the modal assertions φ that are valid principles of forcing with all parameters are exactly those in* $\mathsf{S4.2}$. *Succinctly,* $\mathrm{Force}^W(W) = \mathsf{S4.2}$.

Proof. Certainly any $\mathsf{S4.2}$ assertion is a valid principle of forcing with any parameters, by the argument of Theorem 3, which did not depend on whether there were parameters or not. For the converse direction, we argue as in the Main Theorem, using the following fact, that with sufficient parameters one can always construct independent families of buttons and switches.

Lemma 24.1. *If W is any model of* ZFC, *then with parameters ω_n^W for $n \in \omega$, there is an infinite independent family of buttons and switches.*

Proof. Let b_n be the assertion "ω_{n+1}^W is not a cardinal" and for positive m, let s_m be "the GCH holds at $\aleph_{\omega+m}$," referring to $\aleph_{\omega+m}$ *de dicto*, rather than with a parameter. In any forcing extension of W, any of the buttons can be forced without affecting the truth of the other buttons, by collapsing ω_{n+1}^W to its predecessor (see [Jec03, 15.21]). After this, the switches s_m and their negations can be forced in any desired pattern without adding bounded sets below $\aleph_\omega$ and, consequently, without affecting the buttons b_n (see [Jec03, 15.18 and related]). □

The result now follows from Theorem 15 and the observation that the existence of parameters can simply be carried through that argument. Specifically, if φ is not in $\mathsf{S4.2}$, then it fails at some world w_0 in a Kripke model M whose frame is a finite pre-lattice. As in the Main Theorem, we use the buttons and switches to define p_w for each world w in M verifying the Jankov-Fine formula as in Lemma 6.3. If we define ψ_i as in Lemma 6.4, then we observe as before that $(M, w_0) \models \varphi(q_0, \ldots, q_n)$ if and only if $W \models \varphi(\psi_0, \ldots, \psi_n)$. Since φ fails at w_0 in M and the ψ_i are Boolean combinations of the buttons and switches, this produces a failing substitution instance of φ in W using the same parameters. □

Theorem 25. *For any $W \models$ ZFC and any sets $X \subseteq Y$ of parameters in W, then*

$$\mathsf{S4.2} \subseteq \mathrm{Force}^W(Y) \subseteq \mathrm{Force}^W(X) \subseteq \mathrm{Force}^W \subseteq \mathsf{S5}.$$

Proof. Certainly any $\mathsf{S4.2}$ assertion is valid, even with parameters, so $\mathsf{S4.2} \subseteq \mathrm{Force}^W(Y)$. If $X \subseteq Y$, then it is at least as hard for a modal assertion φ to be valid for all substitution instances using parameters in Y as for parameters in X, so $\mathrm{Force}^W(Y) \subseteq \mathrm{Force}^W(X)$. Finally, $\mathrm{Force}^W = \mathrm{Force}^W(\emptyset)$, which is contained in $\mathsf{S5}$ by Theorem 17. □

Question 26. Can the set of forcing validities $\mathrm{Force}^W(X)$ be strictly between $\mathsf{S4.2}$ and $\mathsf{S5}$? When is it equal to $\mathsf{S4.2}$ or to $\mathsf{S5}$?

For example, in the proof of Theorem 24, we only used parameters ω_n^W, so we may conclude:

Corollary 27. *For any model W of set theory,* $\mathrm{Force}^W(\aleph_\omega^W) = \mathsf{S4.2}$. *More specifically,* $\mathrm{Force}^W(\{\, \omega_n^W \mid n \in \omega \,\}) = \mathsf{S4.2}$.

Corollary 28. *If a model W of set theory is absolutely definable in all forcing extensions (by the same formula, without parameters), then* $\mathrm{Force}^W = \mathsf{S4.2}$.

Proof. If W is absolutely definable, then the ω_n^W are also absolutely definable, thereby avoiding in Theorem 24 the need for them to appear explicitly as parameters. $\square$

An essentially identical argument shows, more generally, that if W is absolutely definable from parameters in X, then $\mathrm{Force}^W(X) = \mathsf{S4.2}$. In particular, no such W is a model of the Maximality Principle with parameters in X. We now push the parameters a bit lower. Let H_{ω_2} denote the collection of sets having hereditary size less than ω_2.

Theorem 29. *For any model W of set theory,* $\mathrm{Force}^W(H_{\omega_2}) = \mathsf{S4.2}$.

Proof. We will build an independent family of buttons and switches. In W, let $\omega_1 = \bigsqcup_n S_n$ be a partition of ω_1 into infinitely many disjoint stationary subsets S_n. Let b_n assert "S_n is not stationary". Each assertion b_n is false in W, since the S_n are stationary there, but in any forcing extension, by shooting a club through the complement of any S_n, we can force b_n to be necessary, while preserving all stationary subsets of the complement of S_n (see [Jec03, 23.8, ex. 23.6]). Thus, in any forcing extension of W, we can push button b_n without affecting any of the other buttons. (Note that this forcing collapses ω_1 when S_n is the sole remaining stationary set on the list.) So the buttons b_n are independent in W. For switches, let s_k assert that the GCH holds at $\aleph_{k+2}$. By forcing over W or any extension we can arrange the switches in any finite pattern, without adding subsets to ω_1^W and consequently without affecting the buttons. So W has an infinite independent family of buttons and switches using parameters in H_{ω_2}, and so by Theorem 15 we conclude $\mathrm{Force}^W(H_{\omega_2}^W) = \mathsf{S4.2}$. $\square$

This stationary set idea provides an alternative source of independent buttons for Lemma 6.1, because if $\mathrm{V} = \mathrm{L}$, then one can use the L-least partition of ω_1 into ω many stationary sets; one advantage here is that this provides arbitrarily large finite independent families of buttons that can be pushed without collapsing cardinals. Indeed, the button to collapse ω_1^{L} is equivalent to the infinite conjunction of these independent buttons.

Returning to Question 26, we observe that if W is a model of $\mathrm{V} = \mathrm{L}$ or if $\mathrm{Force}^W = \mathsf{S4.2}$, then clearly all the classes line up on the left side with $\mathrm{Force}^W(X) = \mathsf{S4.2}$. If $W \models \mathrm{MP}$, then $\mathrm{Force}^W = \mathsf{S5}$ is on the right. If $W \models \mathrm{MP}(\mathbb{R})$, then $\mathrm{Force}^W(\mathbb{R}) = \mathsf{S5}$. If X has any uncountable parameter x, then the assertion $\psi =$ "x is countable" is possibly necessary but not true in W; consequently, $\Diamond\Box\psi \to \psi$ fails in W, and so $\mathrm{Force}^W(X) \neq \mathsf{S5}$. More generally, if X has an element from which ω_1^W is absolutely definable, then $\mathrm{Force}^W(X) \neq \mathsf{S5}$. In fact, since CH is always a switch and independent from "ω_1^W is countable," we can conclude in this case that Dm is not valid, so $\mathrm{Dm} \not\subseteq \mathrm{Force}^W(X)$. If the set of parameters supports long volume controls with independent switches, then $\mathrm{Force}^W(X) \subseteq \mathsf{S4.3}$. If it supports many independent buttons and switches, then $\mathrm{Force}^W(X) = \mathsf{S4.2}$. The situation $\mathsf{S4.2} \subsetneq \mathrm{Force}^W(X) \subsetneq \mathsf{S5}$ would occur if one could construct at least one button in W using parameters in X, but not arbitrarily large finite independent families of buttons and switches.

The hypothesis $\mathrm{Force}^W(\mathbb{R}) = \mathsf{S5}$ is equivalent to $W \models \mathrm{MP}(\mathbb{R})$, which has large cardinal consistency strength. Specifically, [Ham03] shows it to be equiconsistent over ZFC with the existence of a stationary proper class of inaccessible cardinals. It

is natural to inquire about the strength of weaker hypotheses concerning $\mathrm{Force}^W(\mathbb{R})$. For example, we prove next that $\mathrm{Dm} \subseteq \mathrm{Force}^W(\mathbb{R})$ already has large cardinal strength. Define that ω_1 is *inaccessible to reals* if ω_1 is an inaccessible cardinal in $\mathrm{L}[x]$ for every real x (equivalently, if $\omega_1^{\mathrm{L}[x]} < \omega_1$ for all $x \in \mathbb{R}$). The assertion that there is a stationary proper class of inaccessible cardinals is expressed as a scheme in ZFC, asserting that every class of ordinals (definable from parameters) that is closed and unbounded in the class of all ordinals contains an inaccessible cardinal.

Theorem 30. *If* Dm *is valid for forcing with real parameters, then* ω_1 *is inaccessible to reals and every* $\mathrm{L}[x]$ *for* $x \in \mathbb{R}$ *has a stationary proper class of inaccessible cardinals. Indeed,* $\mathrm{L}_{\omega_1} \prec \mathrm{L}$ *and even* $\mathrm{L}_{\omega_1}[x] \prec \mathrm{L}[x]$.

Proof. We show $\mathrm{L}_{\omega_1} \prec \mathrm{L}$ by verifying the Tarski-Vaught criterion. This claim should be understood metatheoretically and proved as a scheme. Suppose that $\mathrm{L} \models \exists v\, \psi(u,v)$ where $u \in \mathrm{L}_{\omega_1}$. If there is no such v inside L_{ω_1}, then the assertion $\varphi(u) =$"there is $v \in \mathrm{L}_{\omega_1}$ such that $\mathrm{L} \models \psi(u,v)$" is false in W. But it is certainly forceably necessary, because we could make $\varphi(u)$ true by collapsing cardinals until the least witness v in L is hereditarily countable. In other words, $\varphi(u)$ is a button in W. The parameter u, being in L_{ω_1}, is hereditarily countable and can therefore be coded with a real. Meanwhile, the assertion CH is a switch, and this switch is independent of $\varphi(u)$, because the CH and its negation can be forced over any model without collapsing ω_1 and therefore without affecting the truth of $\varphi(u)$. Since we have an independent button and switch in W using a real parameter, it follows by Theorem 21, case (6), that Dm is not valid in W with real parameters, contrary to our assumption that Dm is valid for forcing over W with real parameters.

By relativizing to any real x, we similarly conclude that $\mathrm{L}_{\omega_1}[x] \prec \mathrm{L}[x]$, again proved as a scheme. It follows by elementary set theory that ω_1 is (strongly) inaccessible in $\mathrm{L}[x]$. Also, if $C \subseteq \mathrm{ORD}$ is a definable proper class club in $\mathrm{L}[x]$, defined from parameters in $\mathrm{L}_{\omega_1}[x]$, it follows that $C \cap \omega_1$ is unbounded in ω_1, and consequently $\omega_1 \in C$. Since $\mathrm{L}_{\omega_1}[x] \prec \mathrm{L}[x]$, we conclude that every proper class club definable from parameters in $\mathrm{L}_{\omega_1}[x]$ contains an inaccessible cardinal. Thus, the inaccessible cardinals form a stationary proper class in $\mathrm{L}_{\omega_1}[x]$. By elementarity again, we conclude that the inaccessible cardinals form a stationary proper class in $\mathrm{L}[x]$, as desired. □

Corollary 31. *The following are equiconsistent over* ZFC*:*

(1) $\mathrm{ZFC} + \mathrm{MP}(\mathbb{R})$.
(2) $\mathrm{ZFC} +$ S5 *with real parameters is valid for forcing.*
(3) $\mathrm{ZFC} +$ S4W5 *with real parameters is valid for forcing.*
(4) $\mathrm{ZFC} +$ Dm.2 *with real parameters is valid for forcing.*
(5) $\mathrm{ZFC} +$ Dm *with real parameters is valid for forcing.*
(6) $\mathrm{ZFC} +$ *the inaccessible cardinals form a stationary proper class.*

Proof. The Maximality Principle $\mathrm{MP}(\mathbb{R})$ asserts exactly that Axiom 5 is valid for forcing with real parameters, so (1) $\leftrightarrow$ (2). Clearly (2) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (5), since S5 $\supseteq$ S4W5 $\supseteq$ Dm.2 $\supseteq$ Dm. In fact, (4) $\leftrightarrow$ (5), since the axiom .2 is always valid. Statement (5) implies, by Theorem 30, that (6) is true in L. Finally, if there is a model where (6) holds, then results in [Ham03] produce a model satisfying $\mathrm{ZFC} + \mathrm{MP}(\mathbb{R})$. □

The assertions in Corollary 31 should be understood as schemes asserting the relevant substitution instances $\forall \vec{x} \in \mathbb{R}\ \varphi(\psi_0(\vec{x}), \ldots, \psi_n(\vec{x}))$.

Question 32. Which modal theories arise as the valid principles of forcing with real parameters?

We know that $\mathsf{S4.2} \subseteq \mathrm{Force}^W(\mathbb{R}) \subseteq \mathsf{S5}$, and both of these endpoints can occur. But is there any model of set theory W giving rise to any intermediate modal theory?

6. Restricting to a class of forcing notions

One can naturally restrict the scope of forcing extensions to those obtained by the members of a particular class Γ, so that $\Box_\Gamma \varphi$ means that φ holds in all forcing extensions by forcing in Γ and $\Diamond_\Gamma \varphi$ means that φ is forceable by some forcing in Γ. Natural classes Γ would include ccc forcing, proper forcing, and so on. The analogue of the Main Question becomes:

Question 33. For a given class of forcing Γ, what are the valid principles of Γ forcing?

There are many open questions here, which we leave for future projects. Let us close the paper by observing that even with ccc forcing, the situation changes dramatically.

Theorem 34. *If* ZFC *is consistent, then the* ZFC*-provable principles of* ccc *forcing do not include* $\mathsf{S4.2}$.

Proof. It is easy to deduce in $\mathsf{S4.2}$ the following Directedness axiom:

$$(\Diamond\Box\varphi \wedge \Diamond\Box\psi) \to \Diamond\Box(\varphi \wedge \psi).$$

We will show that the ccc interpretation of this Directedness axiom is not valid in L. This proof relies on the fact, a part of mathematical folklore, that there are two fundamentally different ways to destroy a Suslin tree by ccc forcing. If T is a Suslin tree, then of course forcing with T itself adds a branch through T, making it non-Aronszajn. Alternatively, if T is Suslin, then there is ccc forcing making T into a special Aronszajn tree, a union of countably many antichains (see [Jec03, 16.19 and related]). No further forcing can add a branch through this tree without collapsing ω_1, since the branch would have to contain at most one node from each antichain. Thus, these two alternatives cannot be amalgamated by ccc forcing. If φ is the assertion "the L-least Suslin tree is not Aronszajn" and ψ is the assertion "the L-least Suslin tree is special," then each of these statements is possibly necessary by ccc forcing over L, but their conjunction is false in all ccc extensions of L. Thus, $(\Diamond_{\mathrm{ccc}}\Box_{\mathrm{ccc}}\varphi) \wedge (\Diamond_{\mathrm{ccc}}\Box_{\mathrm{ccc}}\psi)$ holds in L, but not $\Diamond_{\mathrm{ccc}}\Box_{\mathrm{ccc}}(\varphi \wedge \psi)$, violating Directedness. $\Box$

Corollary 35. *The same conclusion holds for any class* Γ *of forcing notions containing all* ccc *forcing, whose members in any* ccc *extension preserve* ω_1. *This includes the classes of proper forcing, semi-proper forcing, cardinal-preserving forcing, and so on.*

Proof. In the proof of Theorem 34, the assertions φ and ψ are each ccc-forceable over L and once true, they are necessary by all forcing. Consequently, they are each Γ-forceably necessary in L. The conjunction $\varphi \wedge \psi$, however, is not forceable

by ω_1-preserving forcing over L, and hence not Γ-forceable over L. So the Γ-forcing validities in L do not include $\mathsf{S4.2}$. $\square$

Nevertheless, one can easily verify that $\mathsf{S4}$ remains valid for ccc forcing (and also for the other classes).

Question 36. What is the modal logic of ccc forcing? Does it go beyond $\mathsf{S4}$?

A complete set of $\mathsf{S4}$ frames consists of finite pre-trees (partial pre-orders whose quotients are trees), and with them one might try to carry out a similar analysis as in our Main Theorem, by finding set theoretical assertions to fulfill the Jankov-Fine assertions. The point is that branching in these trees gives rise to behavior totally unlike either buttons or switches. Branching corresponds in set theory to the possibility of ccc forcing extensions that cannot be amalgamated by further ccc forcing, as in the folklore fact above, where one chooses either to specialize a Suslin tree of L or to make it not Aronszajn. What is needed, therefore, is an elaborate generalization of this folklore idea, in which one can successively make choices with ccc forcing that cannot later be amalgamated by ccc forcing, in such a way that every ccc forcing extension is included.

While Theorem 34 shows that the ZFC-provable ccc modal validities do not include all of $\mathsf{S4.2}$, it is not currently clear conversely whether they are included within $\mathsf{S4.2}$. To prove that they are, it would suffice to find a model of ZFC with arbitrarily large independent families of ccc buttons and switches.

Under Martin's Axiom MA, of course, the product of ccc posets is again ccc, and this implies that the Directedness Axiom is valid for ccc forcing. But we seem unable to prove that this axiom is necessary under MA, since MA itself is easily destroyed by ccc forcing: adding even a single Cohen real creates Souslin trees. In the ccc analogue of Question 23, it is not clear whether the collection of the modal validities of ccc forcing is closed under necessitation. Thus, we seem unable to conclude that $\mathsf{S4.2}$ is valid for ccc forcing over any model of MA.

Question 37. What are the ZFC + MA-provably valid principles of ccc forcing?

Lastly, we mention that for ccc forcing, unlike the general situation with Corollary 31, there is no large cardinal strength to the hypothesis that $\mathsf{S5}$ is valid for ccc forcing with real parameters. Specifically, [Lei04] proves that if ZFC is consistent, then there is a model of ZFC in which every $\mathsf{S5}$ assertion is valid for ccc forcing with real parameters. If one wants $\mathsf{S5}$ to be valid for ccc forcing with parameters in H_{2^ω}, however, then [Ham03] shows that it is equiconsistent, as in Corollary 31, with a stationary proper class of inaccessible cardinals.

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