Bifurcation of random maps
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Link to publication

Citation for published version (APA):

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Download date: 06 Feb 2019
1 Introduction

"Probability theory has always generated its problems by its contact with other areas. There are very few problems that are generated by its own internal structure. This is partly because, one stripped of everything else, a probability space is essentially the unit interval with the Lebesgue measure."


1.1 Background

In this section we provide general background to the theory of random dynamical systems and their bifurcations, the object of this thesis. Our discussion is broader then strictly needed for this thesis, in the sense that we mention e.g. iterated function systems and stochastic differential equations which are not the central topic of this thesis. The issues discussed at length in the following chapters in this thesis are summarized in Section 1.2.

1.1.1 Dynamical systems: heuristics

Dynamical systems have been studied for many years, in a variety of different ways, and with many different motivations and applications. The essence of a dynamical system is to tell how some state changes with time under the action of mathematical laws. These are important things to study as they give information about the working of the real world, physical instruments. For example, swinging pendulums, hydrology, changing populations and genetics, climate forecast, financial markets and neural networks in the brain can all be treated using dynamical systems models. Some dynamical systems can be described as unforced systems - that is, they propagate in isolation from external influences. However, very few systems in the real world are genuinely isolated, and so it is often appropriate to describe dynamical systems as perturbed systems. In this case we have the dynamics of a system (perturbation) driving, or influencing, the dynamics of another. Regularly forced systems have been extensively studied for a long time. A simple example is a pendulum which experiences at regular intervals an external 'push' of the same magnitude each time. By contrast, forcing with irregularly (random) frequencies has received comparatively less attention and is in general much less understood.

It is thus widely accepted that understanding the effect of random perturbations on
nonlinear dynamical systems is an important challenge. Random perturbations can be small and irrelevant, or they can be so large as to overwhelm the underlying dynamics. The middle ground is the most interesting, the perturbations can be small, yet contribute non-trivially to the overall dynamics, so that one must consider the interaction of the deterministic dynamics with the stochastic perturbation. This is the approach we take in this thesis.

To respond to this need stochastic differential equations (SDE) were introduced by Itô at the beginning of the 1940s. He gives an explicit construction of diffusion processes which were introduced in the 1930s by Kolmogorov via partial differential equations and measures in their path spaces. For more than half a century it was customary to consider solutions $X^x_t(\omega)$ at time $t > 0$ of an SDE for some fixed initial condition, $x$, and the distribution of corresponding random paths was usually of primary interest. This approach, also called the one point motion approach, to the SDEs was encouraged especially because the Markovian aspect of the solution of the SDE and the existence of a rich ergodic theory of Markov processes. Around 1980 several mathematicians discovered that solutions of SDEs can also be represented, similarly to the deterministic case, as $X^x_t(\omega) = \varphi_t(\omega, x)$ where $\varphi_t(\omega, x)$ is called a stochastic flow, see (74), and for each $t > 0$ and almost all $\omega$ it consists of homeomorphisms.

With the development of dynamical systems in the 20th century it became increasingly clear that discretising time and considering iterations of a single map is quite beneficial both as tool to study the original flow generated by an ordinary differential equation, for instance via the Poincaré return map, and as a rich source of new models which cannot appear in the continuous time framework (two-dimensional chaotic systems). The next step is an observation that the evolution of physical systems is time dependent by its nature, and so they could better be described by compositions of different maps picked according to some law rather than by repeated applications of exactly the same transformation. This leads to random transformations.

Random transformations were already discussed in 1945 by Ulam and Von Neumann in (108) in the framework of ergodic theory but they attracted only marginal interest. The appearance of fractals theory and iterated function systems, see (13), from one side and stochastic flows as solutions of SDEs from another side gave a substantial push to the subject and towards the end of the 1980s it became clear that powerful dynamical systems tools united with probabilistic machinery can produce a scope of results which is now known as random dynamical system theory (RDS). Assuming some additional structures on the random transformations and the SDEs motivated researchers to have a close look at the theory of smooth dynamical systems. This brought to this subject such notions as Lyapunov exponents, invariant manifolds, bifurcations of equilibria etc. We mention here that this theory has applications in statistical physics (96), economics and finance (101; 31), meteorology (38) and in other fields.

1.1.2 Random dynamical systems

We begin with a simple example to introduce the type of problems considered in this thesis. Below we formalize the mathematical notions needed to precise the questions and to study the problems. Now, consider a single map with an attracting fixed
point. Iterates of points in the basin of attraction converge to the attracting fixed point. With small noise added to the system, iterates of points will move around randomly near the attracting fixed point. When the amplitude of the noise is bigger there is a possibility to end up outside the basin of attraction of the fixed point and move somewhere else. In other words, when increasing the level of the noise one expects a transition in dynamical properties. This is one of the type of problems we study in the thesis. There is more. One can iterate more than one point under the same noise. With small noise added to the map with the attracting fixed point, one sees the following phenomenon: the iterates of the different points converge towards each other (while moving around randomly). This property may also change when increasing the level of noise. In this thesis we consider this bifurcation problem as well.

The set-up in this thesis is far more general than this example of noise added to a map with an attracting fixed point. The formalism of random dynamical systems as skew product systems or cocycles is needed for the study of all such problems. In the following we introduce this formalism, and provide simple examples.

**Shift space**

Standard dynamical systems theory cannot directly encompass non-autonomous systems and perturbed systems. The usual approach to non-autonomous systems is to enlarge the state space sufficiently to make the system autonomous. This is well known in the case of a periodically forced differential equation which can be transformed into an autonomous system by the addition of a dummy variable to represent time. In the context of arbitrary forcing, where the time-dependence is introduced through a stationary driving noise, this is most easily accomplished through modelling the driving process. By this we mean that we define a measurable dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) describing the evolution of the noise process.

**Definition 1.1.1.** A family \((\sigma^k)_{k \in \mathbb{T}}\) of mappings of a measure space \((\Omega, \mathcal{F}, \mathbb{P})\) into itself is called a measurable dynamical system with time \(\mathbb{T}\) if it satisfies the following conditions:

- \(\omega \in \Omega \mapsto \sigma \omega \in \Omega\) is measurable
- \(\sigma^0 = id_\Omega\)
- \(\sigma^{k+l} = \sigma^k \circ \sigma^l\) \(\forall k, l \in \mathbb{T}\).

We assume that \(\sigma^k \mathbb{P} = \mathbb{P}\) \(\forall k \in \mathbb{T}\): that is \(\mathbb{P}(\sigma^{-k}A) = \mathbb{P}(A)\) for all \((k, A) \in (\mathbb{T}, \mathcal{F})\).

The measure \(\mathbb{P}\) is said to be invariant with respect to \(\sigma\) or \(\mathbb{P}\) is \(\sigma\)-invariant. In this case the dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) is called a measure preserving dynamical system.

**Evolution driven by noise**

In the following we give a formal definition of a continuous random dynamical system.

**Definition 1.1.2.** A continuous random dynamical system on the topological space \(\mathcal{M}\) over the dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) is a map

\[
f : \mathbb{T} \times \Omega \times \mathcal{M} \to \mathcal{M}, \quad (k, \omega, x) \mapsto f^k(\omega, x)
\]
with the following properties:

- **Measurability**: $f$ is measurable in $k$ and $\omega$.

- **Continuity**: for each $\omega$ the map 
  $$f(\cdot, \omega, \cdot) : \mathbb{T} \times \mathcal{M} \to \mathcal{M}, \quad (k, x) \mapsto f^k(\omega, x),$$
  is continuous.

- **Cocycle property**: The map $f^k(\omega, \cdot) : \mathcal{M} \to \mathcal{M}$ forms a cocycle over $\sigma$:
  $$f^0(\omega, \cdot) = \text{id}_{\mathcal{M}} \quad \text{for all } \omega \in \Omega,$$
  $$f^{k+l}(\omega, \cdot) = f^l(\sigma^k \omega, f^k(\omega, \cdot)) \quad \text{for all } k, l \in \mathbb{T}, \quad \omega \in \Omega.$$

We may think of such a process as a *skew product* over the noise process. A skew-product is a dynamical system of the form

$$\begin{align*}
(Y, Z) &\rightarrow (Y, Z) \\
(y, z) &\mapsto (f(y), g(y, z))
\end{align*}$$

The mapping $f$ over $Y$ is called the *base transformation*, and the mapping $g$ over $Z$ is called the *fibre transformation*.

In the case of random dynamical systems, the noise process is the base, driving the evolution state variable in the fiber, see Figure 1.1. In this case $(Y, Z) = (\Omega, \mathcal{M})$ and the measurable dynamical system $S : \Omega \times \mathcal{M} \to \Omega \times \mathcal{M}$ for each $k \in \mathbb{T}$ is given by

$$S^k(\omega, x) = (\sigma^k \omega, f^k(\omega, x)). \quad (1.1)$$

**Lemma 1.1.1.** The skew product flow is well defined and satisfies the flow property, i.e.

$$S^0 = \text{id}_{\Omega \times \mathcal{M}} \quad \text{and} \quad S^{k+l} = S^k \circ S^l \quad \text{for all} \quad k, l \in \mathbb{T}$$

**Remark 1.1.1.**

In this thesis the noise and the variable of interest which it drives will, together, form a Markov chain in discrete time $(x_k = f^k(\omega, x))_{k \in \mathbb{T}}$. Roughly, a random sequence $(x_k)_{k \in \mathbb{T}}$ is Markovian if the statistics of $x_k$, $k > l$ given $x_l$ are independent of the knowledge of $x_j$ for $j < l$. We denote this Markov chain $(\omega_k, x_k)$ with $\omega_k \in \Delta$ being the noise and $x_k \in \mathcal{M}$ being the noise driven process. Notice that $\omega$ is the *complete path* $(\omega_k)_{k \in \mathbb{T}} \in \Omega$, so $\Omega = \Delta^\mathbb{T}$.

**Examples**

We illustrate the concept above by means of a number of simple examples.
1.1 Background

Example 1.1.1. Consider a linearly damped map driven by a sequence $\omega_k$ of independent identically distributed (i.i.d) random variables where $\omega_0$ is $\nu$-distributed for some probability distribution $\nu$.

$$x_{k+1} = ax_k + \omega_k,$$

where $0 < a < 1$. In this case $T = \mathbb{N}$, $M = \mathbb{R}$, $\Omega = \{\omega \in \mathbb{R}^N : \omega = (\omega_0, \omega_1, \ldots)\}$ and $\sigma$ is the shift on such sequences defined by $\sigma \omega = (\omega_1, \omega_2, \ldots)$. Then $\mathbb{P}$ is the measure induced on such i.i.d sequences. The fact that the dynamical system for the noise is measure preserving follows since the $\omega_k$ are i.i.d. The process $(x_k)$ is known as a discrete Ornstein-Uhlenbeck (OU) process.

In this example and others we can expand the time interval to cover the set of all integers $T = \mathbb{Z}$.

This example fits into a more general set-up. Let $\Delta$ be an appropriate space (in this thesis $\Delta$ will often be a compact interval) and let $f : \Delta \times M \to M$ be a parameterized family of maps and $\nu$ a probability measure on $\Delta$. Consider the following iteration

$$x_{k+1} = f(\omega_k, x_k), \quad x_0 = x$$

This can be formulated as a random dynamical system with $T = \mathbb{N}$, $\Omega = \{\omega \in \Delta^N : \omega = (\omega_0, \omega_1, \ldots)\}$ and $\sigma$ is the shift on such sequences defined by $\sigma \omega = (\omega_1, \omega_2, \ldots)$. Then $\mathbb{P} = \nu^N$ is the measure induced on such i.i.d sequences by $\nu$. The fact that the dynamical system for the noise is measure preserving follows since the $\omega_k$ are i.i.d.

Example 1.1.2. Let $M$ be the real line and let $f_1, f_2$ be the two following functions

$$f_1(x) = ax + 1 \quad \text{and} \quad f_2(x) = ax - 1,$$

where $0 < a < 1$. Consider the discrete-time dynamical system with state of the system at time $k$, given the initial state $x_0 = x$, given by

$$x_k = \phi_{k-1} \circ \ldots \circ \phi_0(x)$$
and the $\phi_i$'s are chosen independently with probability $\frac{1}{2}$ each from the set \{f_1, f_2\}. In this case $T = \mathbb{N}$, $\mathcal{M} = \mathbb{R}$, $\Omega = \{\omega \in \{f_1, f_2\}^\mathbb{N} : \omega = (\phi_0, \phi_1, \ldots)\}$, where $\phi_k$ is randomly chosen from the set \{f_1, f_2\} and $\sigma$ is the shift on such sequences defined by $\sigma \omega = (\phi_1, \phi_2, \ldots)$. Then $\mathbb{P}$ is the measure induced on such i.i.d sequences by the uniform probability on \{f_1, f_2\} (1/2 each). The fact that the dynamical system for the noise is measure preserving follows since the $\phi_k$ are i.i.d.

Example 1.1.3. Random dynamical systems from Markov chains

Consider a Markov chain whose state space is $(0, 1)$, the open unit interval. If the chain is at $x$, it picks one of the two intervals $(0, x)$ or $(x, 1)$ with equal probability $1/2$, and then moves to a random $y$ in the chosen interval. After some initial floundering, we see that the state of the chain could be represented as the iteration of random functions

$$f_u(x) = ux \quad g_u(x) = x + u(1 - x),$$

with $u$ chosen uniformly on $(0, 1)$ and $f$ and $g$ chosen with probability 1/2 each. In this case $T = \mathbb{N}$, $\mathcal{M} = (0, 1)$, $\Omega = \{\omega \in \{f_u, g_u\}^\mathbb{N} : \omega = (\phi_0, \phi_1, \ldots)\}$, where $\phi_k$ is randomly chosen from the set \{f_u, g_u\} and $\sigma$ is the shift on such sequences defined by $\sigma \omega = (\phi_1, \phi_2, \ldots)$. Then $\mathbb{P}$ is the measure induced on such i.i.d sequences by the uniform probability on \{f_u, g_u\} (1/2 each). The fact that the dynamical system for the noise is measure preserving follows since the $\phi_k$ are i.i.d.

This example is a special case of a more general result on the representation of Markov chains. Let $p(x, \cdot)$ be the transition probability function of a Markov chain defined on $\mathcal{M}$ ($p(x, B)$ specifies the probability of moving from the point $x$ into the set $B$). A representation of $p$ is a collection $\mathcal{C}$ of maps from $\mathcal{M}$ to itself and a probability measure $\nu$ on them such that

$$p(x, B) = \nu\{f \in \mathcal{C} : f(x) \in B\}, \quad x \in \mathcal{M}, \quad B \subset \mathcal{M}.$$\n
Then, as described in (70), see also Section 2.4, the Markov chain can be reconstructed by making each transition the result of picking a map from $\mathcal{C}$ with probability distribution $\nu$. In (70) Kifer goes on to show the following:

If $\mathcal{M}$ is a Borel subset of a complete metric space, then any Markov chain on $\mathcal{M}$ can be represented by a collection of measurable maps. Under some further conditions on $p(x, \cdot)$ the Markov chain may be represented by a collection of smooth maps, see (39; 94; 117).

The advantage of this random map setting is that we may associate to each random orbit (from the Markov chain) a sequence of maps which are iterated, enabling us to use the properties (continuity, differentiability, etc) of the random maps in $\mathcal{C}$ to derive additional properties (which are not directly clear from the Markov chain setting only) of the random orbits.

Example 1.1.4. Random perturbation of flows

In this thesis we do not consider continuous-time random processes. For completeness we briefly describe a continuous-time example following the above set-up. For details we refer to (5). Let $dx = f(t, x)dt$, and $x_0 = x \in \mathcal{M}$ be an ordinary differential equation. We assume for simplicity that $\mathcal{M} = \mathbb{R}$. A well defined and good understood
random version of this dynamical system can be made basically through a diffusion. The perturbation is given by a Wiener process "added" to the ordinary differential equation:

$$dx_t = f(t, x_t)dt + g(t, x_t)dW_t, \quad t \in \mathbb{R}, \quad x_0 = x.$$ (1.3)

A solution of this equation is well defined and is a stochastic process $\varphi(t, \omega): \mathbb{R} \times \Omega \to \mathbb{R}$, $(t, \omega) \mapsto \varphi(t, \omega)$ for some probability space $\Omega$ and a given Wiener process $(W_t)_{t \in \mathbb{R}}$. The precise meaning of the above process can be formulated in the language of stochastic integrals in the following way:

$$\varphi_t(x) = x + \int_0^t f(s, \varphi_s(x))ds + \int_0^t g(t, \varphi_s(x))dW_s$$

Here the last term is the Itô integral. Under some regularity conditions on $f$ and $g$, there exist a unique solution with continuous paths.

From a probabilistic point of view the set of processes $\varphi_t(x)$ for all possible initial points $x$ forms a Markov family with respect to the $\sigma$-algebra generated by the Wiener process up to time $t$.

The density function of the process $\varphi_t(x)$ is defined as the function $\phi(t, x)$ that satisfies

$$\mathbb{P}\{\varphi_t(x) \in B\} = \int_B \phi(t, y)dy$$

and satisfies the initial-value problem:

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} - f(x) \frac{\partial \phi}{\partial x}, \quad t > 0, x \in \mathbb{R},$$ (1.4)

where $u$ is the density of the initial condition $x$. Equation (1.4) is known as the Fokker-Planck equation or the Kolmogorov forward equation.

The solution of the Fokker-Planck equation is usually seen as a flow of densities governed by a semigroup of Markov operators. Stationary densities, which are the main objects to investigate to understand the asymptotic behavior of such processes, are the eigenvectors (eigenmodes) of these Markov operators.

From a dynamical point of view the application $x \mapsto \varphi_t(x, x)$ defines a map $\varphi_t(\omega): \mathcal{M} \to \mathcal{M}$ which can be shown, see (5; 74), to be a diffeomorphism under reasonable regularity conditions on $f$ and $g$. These maps satisfy the following cocycle property:

$$\varphi_0(\omega) = id_{\mathcal{M}} \quad \text{(identity map of } \mathcal{M}),$$

$$\varphi_{t+s}(\omega) = \varphi_t(\sigma_s(\omega)) \circ \varphi_s(\omega),$$

for $s, t \geq 0$ and $\omega \in \Omega$, and for a family (flow) of preserving transformations $\sigma_s: \Omega \to \Omega$, $\sigma_0 = id_{\Omega}$, $\sigma_{s+t} = \sigma_s \circ \sigma_t$ and $\sigma_t^2 = \mathbb{P}$ $\forall t \in \mathbb{R}$.

The two previous dynamical systems, $\sigma$ and $\varphi$ induce a dynamical system $S$ defined on the Cartesian product of the space $\Omega$ and the manifold $\mathcal{M}$,

$$S: \Omega \times \mathcal{M} \to \Omega \times \mathcal{M}.$$
$S_t(\omega, x) = (\sigma_t(\omega), \varphi_t(\omega, x)), \quad t \in \mathbb{R}.$

To this large-dimensional dynamical system many of standard methods of dynamical systems and ergodic theory can be applied which can be again interpreted in random terms (on the fiber space).

### 1.1.3 Equilibria, stationary and random invariant measures

One of the keys of the theory of dynamical systems is the determination of equilibrium points and their stability properties.

In contrast to deterministic systems where fixed points and periodic orbits are the invariant objects to be investigated first, stationary and invariant measures are the fundamental invariant objects for random systems and the building blocks for sophisticated analysis involving e.g. stability, Lyapunov exponents, decay of correlations, escape time from/to subsets of the state space, to name just a few frequent notions of dynamical and probabilistic/statistical nature. For clarity, a stationary measure is a concept from Markov processes theory and an invariant measure is a concept from (skew product) dynamical systems theory. Existence of fixed points and periodic orbits for RDS occur only in highly non-typical cases.

#### Stationary measures

Stationary measures are a statistical notion related to the asymptotic distribution of the Markov chain induced by a random dynamical system as defined above. Roughly, a probability measure $m$ is stationary if $\mathbb{P}\{x_k \in A\} \to m(A)$ as $k \to \infty$ for each Borel set $A \subset \mathcal{M}$ and each value $x_0$.

**Example 1.1.5.** Consider Example 1.1.2. Given an initial value $x_0 = x$, we consider sequences $x_k = \phi_{k-1} \circ \ldots \circ \phi_0(x)$, ($\phi_i \in \{f_1, f_2\}$) obtained through random iteration, $k \geq 1$.

This random iteration induces a Markov chain

$$x_{k+1} = ax_k + \omega_k,$$

where $\omega_k = \pm 1$ with probability 1/2. The stationary distribution has an explicit representation, as the law of

$$X_\infty = \omega_0 + a\omega_1 + a^2\omega_2 + \ldots \tag{1.5}$$

The random sequences on the right converge to a finite random limit because $0 < a < 1$.

Clearly, the distribution of $X_\infty$ is unchanged if $X_\infty$ is multiplied by $a$ and then a new $\omega$ is added: that is stationarity. The sequences representation (1.5) can therefore be used to study the stationary measure, however, many mysteries remain, even for this simple case, see section 1.1.4.

Stationary measures do not give sufficient dynamical information for some purposes. In the previous example the distribution of almost every path $(x_k, \omega_k)_{k \in \mathbb{N}}$ will converge to the same probability measure, say $m$. But from this fact we cannot
conclude about the collective motion of all initial conditions. Do different paths with different initial conditions converge to the same path with asymptotic distribution $m$? If not, is there an attracting (repelling) random set? These questions are addressed within the framework of (random) invariant measures, discussed below.

### Invariant measures

The simplest way to introduce the concept of invariant measures is through the skew-product picture. Define $\pi_\Omega : \Omega \times M \to \Omega$ by $\pi_\Omega(\omega, x) = \omega$ (the projection on the base). Notice that the structure of the skew-product means that

$$\pi_\Omega \circ S = \sigma \circ \pi_\Omega.$$

If $\mu$ is invariant for $S$, so that $S\mu = \mu$ (that is $\mu(S^{-1}(A)) = \mu(A)$ for every $A \subset \Omega \times M$), then we see that

$$\pi_\Omega \mu = (\pi_\Omega \circ S)\mu = (\sigma \circ \pi_\Omega)\mu$$

so that $\pi_\Omega \mu$ is $\sigma$-invariant. Recalling, from definition, that $\mathbb{P}$ is invariant for $\sigma$ these considerations motivate the following definition:

**Definition 1.1.3.** Given a random dynamical system $f$, a probability measure $\mu$ is invariant for $f$ if

1. $S^k\mu = \mu$ for all $k$ $\in \mathbb{T}$,
2. $\pi_\Omega \mu = \mathbb{P}$.

### 1.1.4 Stability and instability: bifurcation theory

Bifurcation theory is a subject with classical mathematical origins dated back to before the work of L. Euler in the 18th century see, (42), however, the modern development of the subject starts with Poincaré and the theory of dynamical systems. In the last half century, this theory has undergone a drastic development with the infusion and the interaction of ideas and methods from dynamical systems theory, topology, geometry, measure theory, group theory and numerical study of dynamics. As a result, it is difficult to draw the boundary with any confidence. The characterization offered long ago by V. I. Arnold (1972) at least reflects how broad the subject has become:

"The word bifurcation, meaning some sort of branching process, is widely used to describe any situations in which the quantitative, topological picture of the object we are studying alters with a change of the parameters on which the objects depend. The objects in question can be extremely diverse: for example, real or complex curves or surface functions or maps, manifolds or fibrations, vector fields, differential or integral equations."

In this thesis the ‘objects in question’ will be random dynamical systems in the form of perturbed difference equations, we will call these systems also random maps.
These discrete-time systems, have obtained much less attention, despite their potential importance in applications. In this setting, we assume that the forcing (perturbation) at time $k \in \mathbb{Z}$ is specified by a random variable $\omega_k$ chosen with respect to some probability measure from an appropriate space. The state of the system at time $k \in \mathbb{Z}$ is denoted by $x_k \in \mathcal{M}$ and evolves according to

$$x_{k+1} = f(x_k, \omega_k) = f_{\omega_{k-1}} \circ \ldots \circ f_{\omega_0}(x), \quad \omega \in \Omega, \quad x \in \mathcal{M},$$

(1.6)

where $\mathcal{M}$ is the state space of the system, which we will assume to be a smooth compact manifold.

The picture made by the dynamics of this process may be seen mainly in two different ways, the one-point motion and the $n$-point motion as presented below.

**The one point motion**

The classical (statistical) way is to follow the realization of the random process with some initial condition $x_0$ and look at the distribution of the random path. Under some conditions a “stationary regime” exists and may be described by a stationary probability distribution $m$ such that if the process starts in this regime (that is, if $x_0$ has distribution $m$) then it remains in the regime. Moreover if the process starts with some other distribution then it may converge in some probabilistic sense with $m$ as limiting distribution. This limiting probability distribution depends on parameters that enter the equations of motion or boundary conditions. If one varies these parameters the stationary measure may deform (in some sense) slightly without altering its qualitative features, or sometimes the dynamics may be modified significantly, producing a qualitative change in the stationary probability distribution. To understand the dynamics of this induced process (evolution of measures) on the Banach space, $\mathcal{S}(\mathcal{M})$, of finite signed measures on $\mathcal{M}$ equipped with the total variation norm (see (67, Chapter 1) and below), we associate classically a linear transition operator $L^* : \mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathcal{M})$ defined as follow.

For each $m \in \mathcal{S}(\mathcal{M})$

$$L^*m(A) = \int_{\mathcal{M}} p(x, A)dm(x), \quad A \subset \mathcal{M}.$$  

(1.7)

This operator, called *stochastic kernel, transfer operator, Markov operator, Frobenius-Perron operator* and sometimes *Ruelle operator*, allows us to describe the evolution of the system. This operator acts linearly on the space of all probability measures on the state space. Eigenvectors corresponding to the eigenvalue 1 of this operator are exactly the stationary measures of the process in (1.6). Using this relationship, the Markov chains theory was developed in the previous century by means of linear operator theory, see Doeblin (35), Yosida and Kakutani (109), Doob (37), Neveu (86, chapter 5), Revuz (95, chapter 6), to name but a few. Ergodic properties of the process (1.6) follow then from the structure of the spectrum of the transfer operator. The transfer operator may also be used to investigate the stability (instability) of the process (1.6). One choice of definitions of stability, frequently used in the theory of Markov processes, will be discussed now. Let us mention that later on we will use different (refined) notions in this thesis, better adapted to descriptions of dynamics.
A process as in (1.6) with operator $L^*$ (see Chapter 2 for a relationship between the process of (1.6) and $L^*$) is said to be stable if this map is continuous under suitable choice of metric. For a formal definition, let $\|L^*\| = \sup\{\|L^*m\|, \|m\|_{TV} \leq 1\}$ the operator norm, where $\|m\|_{TV} := \sup_{A \subset \mathcal{M}} \|m(A)\|$ is the total variation norm on the space of finite signed measures on $\mathcal{M}$ (note that the $\|m\|_{TV} = 1$ if $m$ is a probability measure). For the definition and convergence properties of this norm see for example (67, Chapter 1).

**Definition 1.1.4.** A Markov chain with transition transfer operator $L^*$ is said to be stable if there is some $\epsilon$ and a neighborhood $N(L^*, \epsilon) = \{Q : \|L^* - Q\| \leq \epsilon\}$ of $L^*$ such that

(a) Every $Q \in N(L^*, \epsilon)$ has a unique stationary probability measure $\nu = Q\nu$.

(b) For every sequences $\{Q_n\}$ of transfer operators in $N(L^*, \epsilon)$ for which $\|Q_n - L^*\| \to 0$, one has $\|\nu_n - m\|_{TV} \to 0$, where $\nu_n$ is the stationary probability measures (s.p.m) for $Q_n$, and $m$ is the s.p.m for $L^*$.

**Remark 1.1.2.**

- The convergence in total variation norm in definition 1.1.4 is equivalent to the convergence in $L^1$ of the densities $\phi$ and $\psi_n$ of $m$ and $\nu_n$ when they exist. We have, see (76, pg.401),

$$\|\nu_n - m\|_{TV} \to 0 \iff \int_X |\psi_n - \phi|dx \to 0.$$ 

- This notion of convergence is closely related to the local stability of deterministic dynamical systems. In fact it is the local stability of deterministic dynamical systems, after all the process in equation 1.7 is an infinite dimensional linear deterministic system.

The definition above depends on the used norm. One could require in addition to the definition above that $d(supp\nu_n, supp\nu m) \to 0$, where

$$d(A, B) := \sup(x \in \mathcal{A}d(x, B), supx \in Bd(x, A))$$

is the Hausdorff metric and $supp\nu m$ is the support of the measure $m$. Indeed, in this thesis we will study carefully this type of stability in Hausdorff metric.

Definition 1.1.4 seems not to be very relevant for systems which are uniformly ergodic (see e.g. (78, Chapter 9) for a definition and other properties of this type systems) because of the following theorem.

**Theorem 1.1.1.** (Kartashov (67, Th. 1.6)) A Markov chain is stable in the norm $\|\cdot\|_{TV}$ if and only if it is uniformly ergodic.

A large category of perturbed dynamical systems are uniformly ergodic, yet they can exhibit different behavior when perturbing the dynamics of the system (for example by varying a parameter). The change in the behavior of the stationary distribution is usually made by a qualitative change in the behavior of the deterministic part of the system. The example we have in mind is a discrete-time system perturbed by
“additive Gaussian” noise and defined on compact metric spaces (such systems have a unique stationary density), see example 1.1.6 below. These systems are uniformly ergodic when the deterministic part is well behaved. Other examples are diffusion processes which admit unique stationary probability distributions (these are positively recurrent diffusion processes). These processes are always uniformly ergodic, and consequently stable in the above sense.

In such examples the stationary density can show a transition from a unimodal (one peak) to a bimodal (two peaks) or a crater like form. This behavior indicates that the system has made a transition from a state to another one, i.e. a bifurcation, see example 1.1.6 and Figure 1.2 below. Horsthemke and Lefever (63), Arnold and Boxler (7), and Sri Namachchivaya (105), have carefully studied the number and the location of extrema of the stationary density $\phi_a$ as a function of a parameter $a$. This concept has been formalized based on the ideas of Zeeman (114; 116) according to which two probability densities $\phi$ and $\psi$ are equivalent if there are two diffeomorphisms $h, k$ such that $\phi = h \circ \psi \circ k$. This gives rise to a notion of structural stability and thus of a point of structural instability of families of densities $\phi_a$ which captures the above mentioned observations. For example, the transition point $a = a_0$ from a unimodal density to a bimodal density is such a point of structural instability. This concept is called ‘noise induced transitions’ by physicists and later phenomenological, or P-bifurcation by mathematicians.

Example 1.1.6. Consider the following discrete-time dynamical system:

$$x_{k+1} = x_k \exp(a - x_k) + \sigma \omega_k$$

(1.8)

where $\omega_k$ is a Gaussian random variable, $\sigma$ is a constant (amplitude of the noise) and $a > 0$ is a parameter.

In the deterministic case, i.e. $\sigma = 0$, the asymptotic behavior of this system is governed by the parameter $a$. If $0 < a < 2$ there is a stable fixed point or attractor $p = a$ and all trajectories converge to the fixed point. At $a = 2$ there is a bifurcation into an attracting periodic cycle of period 2. If $a > a_1 \approx 2.526$ the bifurcation will continue and produce, successively, attracting periodic cycles of periods 4, 8, 16, ...

It turns out that equation (1.8), as a Markov process, is uniformly ergodic for all $a > 0$ and thus no instability (in the sense of definition 1.1.4) behavior will be detected when varying the parameter $a$. However, since the deterministic version of equation (1.8) undergoes a bifurcation at $a = 2$ and the “stationary” density shows a transition at $a \approx 2$ from a unimodal density to a bimodal density, a phenomenological definition of structural instability will be more appropriate to describe the change in the dynamics of the process in equation (1.8).

Remark 1.1.3. The P-bifurcation is not ‘scale invariant’, i.e., if $X$ is a random variable whose probability density $\phi_X$ has two maxima, there is a strictly monotone transformation $h : \mathcal{M} \rightarrow \mathcal{M}$ such that $\phi_{h(X)}$ has only one maximum.

Example 1.1.7. Let $X$ be a real random variable with density

$$\phi = \frac{2}{5} \mathbb{1}_{[0,1]} + \frac{1}{5} \mathbb{1}_{[1,2]} + \frac{2}{5} \mathbb{1}_{[2,3]}$$
Figure 1.2: The “stationary” density of the system in equation (1.8) with $a = 1.9$ (left) and $a = 2.1$ (right). For $a = 1.9$ the deterministic dynamical system has a stable fixed point. For $a = 2.1$ the deterministic system has a stable 2-periodic cycle.

which has two “peaks”. Let $h: \mathbb{R} \to \mathbb{R}$ defined by

\[ h(x) = \begin{cases} 
10x & \infty < x < 1, \\
x + 9 & 1 \leq x < 2, \\
10x - 9 & 2 \leq x < \infty 
\end{cases} \]

Then

\[ \phi_h(x) = \frac{1}{25} \mathbb{I}_{[0,10]} + \frac{1}{5} \mathbb{I}_{[11,12]} + \frac{1}{25} \mathbb{I}_{[11,21]} \]

has one peak. In general the equivalence classes are preserved only if volume-preserving diffeomorphisms are admitted as coordinate changes, for which $|\det Dh(x)| = 1$ for all $x$. For more examples see (115).

The $n$-points motion

The second way to approach the perturbed dynamical system is the following: the realization of the process (1.6) with initial condition $x_0$ defines the orbit of $x_0$. The collective representation of these orbits for all initial conditions in the state space comprises the phase portrait (which depends on the realization of the random sequence $(\omega_0, \omega_1, \ldots)$). This portrait provides a global qualitative (i.e. topological) picture of the dynamics, and this picture depends again on the parameters of the system. Like in case of the stationary probability measure, this phase portrait may drastically change in changing the parameters. This approach is a natural generalization of the deterministic concept and in particular to the notion of flow.

In deterministic systems, the general term “bifurcation” is used to describe ”qualitative changes” in the phase portrait of a parameterized family $f_a$ of dynamical systems which occurs when the parameter $a$ is varied. The vague notion of “qualitative changes” has been successfully formalized by using the concepts of topological
equivalence and structural stability. A parameter value $a_0$ is called a bifurcation point of the family $f_a$ if the family $f_a$ is not structurally stable at $a_0$, i.e. if any neighborhood of $a_0$ there are parameter values $a$ such that $f_a$ is not topologically equivalent to $f_{a_0}$. When the topological equivalence is replaced by local topological equivalence (for example, in the neighborhood of a particular point), the parameter value $a_0$ is called a local bifurcation point. Classical local bifurcations are saddle node, transcritical, pitchfork, and Hopf bifurcations.

In analogy to the deterministic case, Arnold (5) introduced the notions of structural stability and topological equivalence of continuous time RDS. These notions and others are based on results from the theory of skew product dynamical systems. These results are reinterpreted in a random dynamics framework. He called a parameter value $a_0$ a (local) bifurcation point of the family $f_a$ if this family is not structurally stable at $a_0$, i.e. if in any neighborhood of $a_0$ there are parameters values $a$ such that $f_a$ and $f_{a_0}$ are not (locally) topologically equivalent. This definition of structural stability is a fine definition and very difficult to verify when studying random dynamical systems. Later we give a more appropriate definition of dynamical bifurcation based on the invariant measures.

At the present time the modest aim of dynamical stochastic bifurcation theory is to discover the stochastic analogues of the elementary deterministic local bifurcation. In the last 20 years there were some attempts to reach this goal. It starts with a case study of the one dimensional perturbed version of the deterministic normal forms of the transcritical, pitchfork and saddle-node bifurcations by Arnold and Boxler (8). More precisely, they considered the bifurcation parameter as noisy, i.e. they carefully add a white noise process to the parameter so that they obtained a family of Stratonovich stochastic differential equations which is explicitly solvable. The explicit solutions to the stochastic differential equations makes the analysis clear. They show for example that the saddle node-bifurcation disappears after adding white noise. They notice that the destruction could be avoided if one use a bounded noise with an appropriate amplitude. They also show that, choosing specific multiplicative noise, the stochastic dynamical bifurcation scenario for the pitchfork case is analogous to the deterministic scenario. Later on, Crauel and Flandoli (26) showed that the same pitchfork bifurcation will be destroyed using an additive noise. This destruction is intuitively comprehensible, by saying that even if the diffusion parameter $\sigma$ in the SDE is small, the perturbation $\sigma dW_t$ may become "big" and will overwhelm the global dynamics.

Crauel, Imkeller and Steinkamp (27) consider general parameterized families of one-dimensional stochastic equations

$$dx_t = f_a(x_t)dt + g_a(x_t)dW_t,$$

such that zero is a fixed point and the absolute value of the diffusion coefficient is positive on $\mathbb{R} \setminus \{0\}$, where $a \in \mathbb{R}$ and $f,g$ satisfy some smoothness assumptions. They describe, dependent on the invariant measure for the Markov semigroup, dynamical bifurcation scenarios, i.e. they give under technical assumptions necessary conditions for a stochastic pitchfork and transcritical bifurcations.

In multidimensional case there have been some attempts and general criteria in this direction, e.g. Arnold and Xu (10), Baxendale and Imkeller (14) and many case
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studies c.f. Ariaratnam (3), Arnold, Bleckert and Schenk-Hoppé (6), Arnold and Imkeller (9), Keller and Ochs (69), Schenk-Hoppé (100), amongst others. We notice here that for the most of these studies the dynamical bifurcation is based on the pattern of sign changes of Lyapunov exponents which should announce the appearance of new branches of bifurcating invariant measures. Such bifurcations are called the D-Bifurcations.

To compare the previous two concepts of bifurcation (the $P$ and the $D$-bifurcation), Baxendale (16) gives an example in which the density of the invariant measure of the semigroup does not depend on the bifurcation parameter, while the top Lyapunov exponent change sign. In (26), Crauel and Flandoli exhibit an example, where the density changes from one peak to a two peak function at a parameter value, while the invariant measure remains a random point with negative Lyapunov exponent. No nontrivial example was available where the $P$ and the $D$-bifurcation coincide in the same parameter value.

The previously mentioned attempt to understand bifurcations of stochastic dynamical systems was especially focused on stochastic differential equations and very few works on discrete systems are traceable in the literature. There are a few case studies on the effect of noise on the logistic map, we cite Gutierrez, Iglesias and Rodriguez (54), Linz and Lücke (80), Mayer and Haken (84), Schenk-Hoppé (102) and Steinsaltz (107).

Other methods and conclusion

The area of stability and instability, even in classical dynamical systems, has largely been driven by phenomena discovered in specific models (e.g., specific differential equations). Attempts to unify the observations have created stability and bifurcation “theories” which, in their turn, have found applications in systems other than the ones originally designed for. It follows from the previous modest discussion that we will observe the same trend in the area of stability in stochastic systems and certainly more. Many notions of stochastic stability may be defined and studied, some of which are more case-specific. For instance, there are comparison methods. Frequently, it is the case that one can somehow dominate the system under study by a system whose stability is known or can easily be deciphered. Quite useful for this kind of method are the stochastic ordering concepts, see, e.g., Baccelli and Brémaud (11). Another method is based on contractivity: consider again the Markov chain $x_k$ in equation (1.6) and suppose that $f$ is contractive in the first argument in the following sense: There is $x_0 \in \mathcal{M}$ such that

$$d(f(x, \omega), f(x_0, \omega)) \leq d(x, x_0) \quad \text{for all } x \in \mathcal{M} \text{ and for almost all } \omega,$$

then the Markov chain $x_k$ has a unique stationary measure $m$ and $\rho[p_k(x, .), m] \leq A_x r^k$ for constants $A_x$ and $0 < r < 1$. Here $p_k(x, dy)$ is the law of $x_k$ given that $x_0 = x$ and $\rho$ is the Prokhorov metric used for the distance between two probabilities. Moreover, under the same assumption, the “backward iteration”

$$y_k = f_{\omega_0} \circ \ldots \circ f_{\omega_{k-1}}(x)$$

converge almost surely to a random limit point, at an exponential rate, and this limit has the unique stationary distribution $m$. Such systems were called strongly stable.
1 Introduction

Foss and Denisov (45) used martingale arguments to study the transience (instability) of a class of Markov chains. Finally, large deviation techniques have also been used in stability and instability studies of various stochastic systems, see, e.g., Puhalskii and Rybko (93), and Gamarnik and Hasenbein (48).

This thesis contributes to the development the theory of bifurcation of stationary measures of random maps (transformations), which we hope itself to contribute in raising this "infant" called stochastic bifurcation. I used the word "infant" because many authors still speak of stochastic bifurcation theory being in its "infancy" although they have been dealing with this subject for more than 20 years now.

1.2 Thesis overview

The aim of this thesis is to study bifurcation theory of smooth random maps. We study bounded perturbations of deterministic maps leading to bifurcations in which, to fix thought, orbits escape from the basin of attraction of a given equilibrium of the deterministic system. Essentially two possibilities may occur: escaping orbits can or cannot return near the equilibrium. Escaping orbits lead to transient dynamics if orbits do not return and intermittent dynamics if orbits do return. This type of behavior gives rise to a definition of bifurcation which takes the explosion of the support of the stationary measures into account. We give also answers to questions like: how can such transitions occur? What are the quantitative characteristics? In particular we present a satisfactory theory for one dimensional random maps, including a classification of possible bifurcations. This extends the general theory in particular by classifying codimension one bifurcations. Typically only finitely many bifurcations occur in one parameter families of random maps, in contrast to families of deterministic maps. The most precise and complete study is presented for randomly perturbed circle diffeomorphisms. Here, we relate the bifurcation of stationary measures, occurring when the support of a stationary measure explodes from finitely many intervals to the entire circle, to the $D$-bifurcation of a random periodic orbit to a random fixed point. We will see that these two bifurcations happen at the same parameter value. We now list the topics in the different chapters. Chapter 2 is an introductory chapter, in which we present the fundamentals of the framework as needed for the further investigation. We give a new definition of stability and random bifurcation valid under mild conditions. The last section in Chapter 2 is devoted to general facts about random dynamical systems as skew-products systems and their invariants characteristics. We also give the relationship between stationary measures for random maps and random invariant measures for the generated random dynamical system.

Chapter 3 focusses on a complete description of bifurcations of random diffeomorphisms in one dimensional state space. We give the analogues of the elementary deterministic local bifurcations and classify random bifurcations in Theorem 3.2.1.

In Chapter 4 we treat the one dimensional bifurcation problem for random circle diffeomorphisms from the $n$-point motion approach. We give a characterization and the structure of the invariant random measure obtained from iteration of Lebesgue measure. We relate the bifurcation of stationary measures of the one-point motion...
to the bifurcation of the random invariant measure from a random fixed point to a random periodic orbit. In the remaining part of this chapter we give corollaries for the skew-products and their invariant graphics.

In Chapter 5 we discuss various aspects of bifurcation of random diffeomorphisms in higher dimension. We discuss stability theorems and parameter dependence. We give results on the smoothness of stationary densities as functions of the state space variable and the bifurcation parameter. Section 5.4 develops material on conditionally stationary measures and applies this to compute expected escape times in Section 5.5. Section 5.6 treats the speed of decay of correlations and its dependency on the bifurcation parameter.

In Chapter 6 we apply the developed theory to two popular examples, the randomized standard circle map and a random version of the logistic map.