2 Fundamentals of random maps

The purpose of this chapter is the introduction of the framework. We will summarize basic facts and statements which are needed for further investigation. The intention of this work is the study of bifurcation of random maps, hence we start by definition of random maps and their properties.

In the second section we rewrite the dynamical system generated by iteration of random maps on the space of probability densities. This is called the statistical solution of the generated dynamical system. We give a classical results ensuring the existence of equilibria to the statistical solution and we give a definition of stability of these equilibria. Since we want to study bifurcation of random maps, we give a definition of stability of random families depending on finitely many parameters in the remaining of this section.

The first two sections approach the iteration of random maps from a one point motion viewpoint. The last section is devoted to an alternative approach, the skew product approach. We give a general definition of a random dynamical system and their invariant measures. We give a result due to Furstenburg which tell us the relationship between stationary measure of the statistical solution and invariant measure of the random dynamical system. We give a definition of the notion of a random attractor. The remaining of this section is devoted to the notion of the random fixed points and the random periodicity.

2.1 Some notations and definitions

By smooth we always means $C^\infty$. Let $\mathcal{M}$ be a smooth $n$-dimensional compact Riemannian manifold with a normalized measure $\mathcal{L}$, $\mathcal{L}(\mathcal{M}) = 1$, induced by the Riemannian structure. When not otherwise mentioned, absolute continuity will be taken with respect to the probability measure $\mathcal{L}$. Let $\Delta$ be a compact domain in $n$-dimensional Euclidean space. Smoothness of a function $g$ on $\Delta$ is to be understood in the sense that $g$ can be extended to a smooth function on a neighborhood of $\Delta$.

Let $\Gamma = \{ f_\omega, \omega \in \Delta \}$ be a parameterized family given by the function $f : \Delta \times \mathcal{M} \mapsto \mathcal{M}$, where $f_\omega : \mathcal{M} \mapsto \mathcal{M}, x \in \mathcal{M} \mapsto f_\omega(x)$ is a map depending on a random parameter $\omega \in \Delta$ drawn from a probability measure $\nu$.

We set $T = \mathbb{N}$ or $\mathbb{Z}$. We define the perturbation space to be $\Omega = \Delta^T$. An element $\omega$ is a sequence $(\omega_i)_{i \in T}$ and is called a word. We endow the space $\Omega$ with the product topology, the product Borel $\sigma$-algebra $\mathcal{B}^T$ denoted $\mathcal{F}$, and the product probability measure $\nu^T$ which we denote $\mathbb{P}$. Since we assume that $\Delta$ is compact, Tychonoff’s theorem tell us that $\Omega$ is compact.
In this thesis we will work exclusively with maps which are at least continuous. Thus, when saying random map we mean random continuous map. The basic setup we are treating is of points being mapped into bounded domains according to some probability. The following standing assumptions will be made with this setup in mind. The random parameters will be chosen from a region $\Delta$ such that

(H 1). The measure $\nu$ on $\Delta$ is absolutely continuous with respect to Lebesgue measure. Its Radon-Nikodym derivative, $g$, will be supposed to be continuous on $\Delta$.

(H 2). $\omega \mapsto f(x; \omega)$ is an injective map for each $x \in M \setminus \partial M$.

Definition 2.1.1. A random map is a continuous map $f(\omega; x) : M \rightarrow M, x \mapsto f(\omega; x)$, depending on a random parameter such that $\omega \mapsto f(\omega; x)$ are $\mathcal{B}$-measurable maps for every $x \in M$.

A random differentiable map is a differentiable map $f(\omega; x) : M \rightarrow M, x \mapsto f(\omega; x)$, depending on a random parameter and such that $\omega \mapsto f(\omega; x)$ are $\mathcal{B}$-measurable maps for every $x \in M$.

A random diffeomorphism (homeomorphism) is a random differentiable map so that $x \mapsto f(\omega; x)$ is a diffeomorphism (homeomorphism) for each $\omega$.

When $f(x; \omega)$ is $C^k$, $k \geq 0$, in $\omega \in \Delta$, and $\nu$ has a $C^k$ density function $g$, we write $R^k(M)$ for the space of $C^k$ random maps $f$ on $M$.

Given an initial value, $x_0$, in $M$. Iterates of the random map $f(\omega; x)$ define a discrete-time system. Assume that the perturbation (forcing) at time $k \in \mathbb{T}$ is specified by a random variable $\omega_k$ chosen from $\Delta$ with respect to some probability measure $\nu$. The state of the system at time $k \in \mathbb{E}$ is denoted by $x_k \in M$ and evolves according to

$$x_{k+1} = f(\omega_k; x_k) \quad \text{(2.1)}$$

If we think of $\omega_k$ as a parameter, then we can interpret (2.1) as standard dynamical system with noise or forcing on the parameters. It is also helpful to write $f_\omega(x_k)$ instead of $f(\omega; x_k)$. This suggests the interpretation that instead of applying the same map $f$ every time, we choose a different map $f_\omega$ at each time step. The case of a single deterministic system $f$ subject to additive noise can be included in this formalism by setting $f_\omega(x_k) = f(x_k) + \omega_k$.

This notation and equation (2.1) suggests that the random iterates of the random map $f(\omega; x)$ may be written as

$$x_k = f^k_\omega(x) = f^k(\omega; x) = f_{\omega_{k-1}} \circ \ldots \circ f_{\omega_0}(x), \quad \omega \in \Omega, \quad x \in M,$$

In the most general case (assuming $M$ compact), $\Delta$ can be taken to be the space of all continuous maps on $M$, see (70). At the other extreme, $\Delta$ may be a discrete space consisting of a finite number of points, so that $f_\omega$ is chosen from a finite set of maps at each time step. This leads to the so called iterated function system (13).

In this work we shall choose $\Delta$ somewhere between these two extremes, and make the assumption that $\Delta$ is a compact domain in $\mathbb{R}^n$ with a piecewise smooth boundary. The following properties are easy to verify.

Property 2.1.1. For every fixed $k \geq 1$ it holds that
1. \((\omega; x) \in \Omega \times \mathcal{M} \mapsto f(\omega; x) \in \mathcal{M}\) is continuous (w.r.t. to the product topology);

2. If \(f\) is a differentiable random map, then \((\omega_0, \ldots, \omega_{k-1}; x) \in \Delta \times \cdots \times \Delta \times \mathcal{M} \mapsto f^k(\omega; x) \in \mathcal{M}\) is differentiable;

3. If \(f\) is a random diffeomorphism, then \(x \mapsto f^k(\omega; x) \in \mathcal{M}\) is a diffeomorphism for every \(\omega_0, \ldots, \omega_{k-1} \in \Delta\).

Given \(\omega \in \Omega\) and \(x \in \mathcal{M}\) we will call \(\{f^k(\omega; x)\}^\infty_{k=0}\) the \(\omega\)-orbit of \(x\) and we write \(O(\omega; x)\).

2.2 Stationary measures of random maps and stability

2.2.1 Random maps iterations

There are two ways to study the evolution of the process in (2.1). First, a probabilistic (statistical) approach based on the Markov semigroup generated by the system; its applies only if the randomness in the evolution process comes from a Markov process. Let \(\omega_0, \omega_1, \ldots\) be independent draws from \(\nu\), and state the useful convention that \(f_0(x_0, \omega) = x_0\). The iterates of the random map \(f_\omega\) gives rise to a discrete-time Markov process through the transition functions

\[
p(x, A) = \nu(\omega \mid f(\omega; x) \in A),
\]

(2.2)

for Borel sets \(A\). With \(h_x(\omega) = f(x; \omega)\), the measure \(p(x, \cdot)\) equals \((h_x)_*\nu\) defined by \((h_x)_*\nu(A) = \nu(h_x^{-1}(A))\). Vice versa, under some regularity conditions on the map \(x \mapsto p(x, \cdot)\), it is possible to represent Markov chains by mean of random maps defined on suitably chosen spaces of transformations, see (70; 94; 117) and Section 2.4 for a discussion.

It is easy to see that the above hypothesis implies that For every \(x \in \mathcal{M}\) the probability measure \(p(x, \cdot)\) is absolutely continuous with density \(h_x: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+\) such that \(p(x, B) = \int_B h^k(x, y)d\mathcal{L}(y)\) for all \(x \in \mathcal{M}\) and \(B\) a Borel set. In this thesis we will use mutually the notation \(p_x(\cdot)\) meaning the transition probability \(p(x, \cdot)\).

The statistical approach to study the dynamical system (2.1) investigates the recurrence relation that gives \(m_{k+1}(B) := \nu(\omega \mid f(\omega; x_k) \in B)\) in terms of \(m_k(B)\), where \(B\) is a Borel subset of \(\mathcal{M}\). By Riesz representation theorem, a measure supported on \(\mathcal{M}\) is uniquely defined by the value of the integral

\[
m(\psi) = \int_{\mathcal{M}} \psi(x) dm(x),
\]

where \(\psi\) is a real valued continuous function on \(\mathcal{M}\). Thus we would like to know \(m_{k+1}(\psi)\) if \(m_k(\psi)\) is given. Since \(x_k\) is completely determined by \(x_0\) and \(\omega_0, \ldots, \omega_{k-1}\), it is clear that \(x_k\) and \(\omega_k\) are independent. Let \(\psi\) be a continuous function. Then the mathematical expectation of \(\psi(f(\omega_k; x_k))\) is just

\[
\int_{\mathcal{M}} \int_{\Delta} \psi(f(\omega; x)) d\nu(\omega) dm_k(x).
\]
However, since \( \psi(x_{k+1}) = \psi(f(\omega_k; x_k)) \) the mathematical expectation is also
\[
\int_{\mathcal{M}} \psi(x) dm_{k+1}.
\]
Equating these two expression we obtain
\[
\int_{\mathcal{M}} \psi(x) dm_{k+1} = \int_{\mathcal{M}} \int_{\Delta} \psi(f(\omega; x) d\nu(\omega) dm_k(x),
\]
or
\[
m_{k+1}(\psi) = L^* m_k(\psi),
\]
where
\[
L^* m(\psi) = \int_{\mathcal{M}} \int_{\Delta} \psi(f(\omega; x)) d\nu(\omega) dm(x).
\]
This integral may be again written as an integral of functions defined on the state space \( \mathcal{M} \). Consider again \( h_x(\omega) = f(\omega, x) \). This function maps \( \Delta \) into \( U_x = f(\Delta, x) \) and \( \nu \) into \( p_x \) and we have
\[
L^* m(\psi) = \int_{\mathcal{M}} \int_{U_x} \psi(y) d(h_x)_* \nu dm(x).
\]
\[
= \int_{\mathcal{M}} \int_{U_x} \psi(y) d p_x(y) dm(x) \quad (2.4)
\]
\[
= \int_{\mathcal{M}} \int_{U_x} k(x, y) \psi(y) d \mathcal{L}(y) dm(x). \quad (2.5)
\]
The operator \( L^* \) maps the space \( \mathcal{P}(\mathcal{M}) \) of all probability measures on \( \mathcal{M} \) into itself. Thus, for a given initial measure \( m_0 \), the sequence \( \{(L^*)^k m_0\} \) describes the evolution of measures corresponding to the dynamical system (2.1). Alternatively, some authors call \( \{(L^*)^k m_0\} \) the statistical solution of (2.1) corresponding to the initial distribution \( m_0 \).

Expression (2.3) may also be written as
\[
L^* m(\psi) = \int_{\mathcal{M}} U \psi(x) dm(x),
\]
where
\[
U \psi(x) = \int_{\Delta} \psi(f(\omega; x)) d\nu(\omega).
\]
This map (also called diffusion when working with continuous-time process) is just \( \mathbb{E}_{\nu}(\psi(x_1) | x_0 = x) \).

Of particular interest are fixed points of the operator defined by equation (2.3), i.e.
\[
m(\psi) = L^* m(\psi) \quad (2.6)
\]
This means that if \( x_0 \) is \( m \) distributed, then that all random variables \( x_k \) in equation (2.1) have the same distribution \( m \). Fixed points of \( L^* \) are called stationary measures. The set of such measures will denoted by \( \mathcal{S}_f(\mathcal{M}) \).
Remark 2.2.1.

From equation (2.3) it follows that \( m \in \mathcal{S}_f(M) \) if and only if \( m \ast \nu = m \), where \( \ast \) is the convolution operator defined by

\[
m \ast \nu(B) = \int_B m(f^{-1}_\omega(B))d\nu(\omega),
\]

For every Borel subset \( B \subset M \). This is just like the definition of invariant measures for deterministic systems but now averaged over all \( \omega \).

A stationary measure is called **ergodic** if it is not a trivial convex combination of two disjunct stationary measures.

A **strictly ergodic** stationary measure is a unique ergodic measure with full support (\( \text{supp}(m) = M \)). Random maps which admit a strictly ergodic stationary measure will be called strictly ergodic random maps.

Ergodic stationary measures are the most important objects to study in this theory. For example, for an ergodic stationary measure \( m \), the Birkhoff ergodic theorem yields

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(\omega; x)) = \int_M \psi(x)dm(x)
\]

for all integrable functions \( \phi \) on \( M \) and \( m \times \nu^\infty \) almost every point \( (x, \omega) \). Taking \( \phi = 1_A \), the characteristic function of a Borel set \( A \subset M \), it shows that the relative frequency with which typical orbits visit \( A \) is given by \( m(A) \).

The absolute continuity of \( p \) with respect to \( \mathcal{L} \) implies the absolute continuity of every stationary probability measure \( m \) since for every null \( N \in \mathcal{B} \) we have

\[
m(N) = \int_M p(x, N)dm(x) = \int_M \int_N k(x, y)d\mathcal{L}(y)dm(x) = 0
\]

and therefore every stationary measure, \( m \), has a stationary (invariant) density, \( \phi = dm/d\mathcal{L} \).

Write \( P_f(\omega) \) for the Perron-Frobenius operator, defined by

\[
\int_A P_f(\omega; z)\phi(x)d\mathcal{L}(x) = \int_{f^{-1}(\omega; A)} \phi(x)d\mathcal{L}(x)
\]

for Borel sets \( A \). That is, for a measure \( m \) with density \( \phi = dm/d\mathcal{L} \), \( f_*m \) has density \( P_f \phi = df_*m/dm \) (see e.g. (76)).

Define

\[
V_x = \{ z \in M \mid x \in f(\Delta; z) \},
\]

which is the set of points in \( M \) that are mapped to \( x \) by some random map. The following lemma gives the evolution of the sequence of densities corresponding to the sequence \( (m_k)_{k \in \mathbb{N}} \) as an average over the random parameters \( \omega \) of the Perron-Frobenius operators for \( f(\cdot; \omega) \) and gives an equivalent formulation as an integral over the state space \( M \).
Lemma 2.2.1. The evolution of the densities $\phi_k$ is given by $L\phi_k = \phi_{k+1}$ where

$$L\phi(x) = \int_\Delta P_{f(\cdot;\omega)}\phi(x)d\nu(\omega), \quad (2.10)$$

or,

$$L\phi(x) = \int_{V_x} k(y,x)\phi(y)d\mathcal{L}(y). \quad (2.11)$$

Proof. Equation (2.10) is a direct consequence of equation (2.3) and definition (2.8). In fact for $\psi = 1_A$, where $A$ is a Borel set, we have

$$L^*m(A) = \int_\Delta \int_M 1_A(f(\cdot;\omega))dm(x)d\nu(\omega)$$

$$= \int_\Delta \int_{f^{-1}(A,\omega)} dm(x)d\nu(\omega)$$

$$= \int_A \int_\Delta P_{f(\cdot;\omega)}\phi(x)d\nu(\omega)d\mathcal{L}(x).$$

This implies (2.10). Alternatively

$$\int_M \int_\Delta \psi(f(x;\omega))\phi(x)d\nu(\omega)d\mathcal{L}(x) = \int_M \int_{V_x} \psi(y)\phi(x)k(y,x)d\mathcal{L}(y)d\mathcal{L}(x)$$

$$= \int_M \int_{U_x} \psi(y)\phi(x)k(y,x)d\mathcal{L}(x)d\mathcal{L}(y)$$

This proves (2.2.1).

Example 2.2.1. Consider the following system generated by the iteration of a one-dimensional continuous deterministic map $f$, and in the presence of additive noise

$$x_{k+1} = f(x_k) + \sigma \omega_k,$$

where $\omega_k$ is a Gaussian random variable with density $p(\omega)$, and $\sigma$ parameterizes the noise amplitude. A density of trajectories $\phi(x)$ evolves with time as

$$\phi_{k+1}(x) = L\phi_k(x) = \int_{\mathbb{R}} k(y,x)\phi_k(y)d\mathcal{L}(y),$$

where $k(y,x) = \delta_\sigma(x-f(y))$, and

$$\delta_\sigma(x) = \int \delta(x-\sigma\omega)p(\omega)d\omega = \frac{1}{\sigma}p\left(\frac{x}{\sigma}\right).$$

Remark 2.2.2. The transfer operator $L$ preserves integrals, as the following computation shows.

$$\int_M L\phi(y)d\mathcal{L}(y) = \int_M \int_{V_y} k(x,y)\phi(x)d\mathcal{L}(x)d\mathcal{L}(y)$$

$$= \int_M \int_{U_x} k(x,y)\phi(x)d\mathcal{L}(y)d\mathcal{L}(x)$$

$$= \int_M \phi(x)d\mathcal{L}(x),$$
because \( \int_{U_z} k(x, y) d\mathcal{L}(x) = 1 \).

Iterating \( L \) gives

\[
L^2 \phi(x) = \int_{V_x} k(z, x) L \phi(z) dz = \int_{V_x} \int_{V_z} k(z, x) k(y, z) \phi(y) dy dz = \int_{f^{-1}(V_x, \Delta)} \int_{V_y} k(z, x) k(y, z) dz \phi(y) dy
\]

which is of a similar form, namely \( \int_{f^{-1}(V_x, \Delta)} k_2(y, x) \phi(y) dy \) with

\[
k_2(y, x) = \int_{V_x} k(z, x) k(y, z) dz,
\]

as \( L \phi(x) \). Inductively similar expressions are derived for higher iterates of \( L \).

Let a random map \( f \in R^0(\mathcal{M}) \) be given. The existence of finitely many ergodic stationary measures for \( f \) presented in the following theorem, is scattered over chapter 5 in (37) (valid under more general conditions). Similar results are contained in (4).

For the reader’s convenience we give here a sketch of the proof. Later on we will add statements on the regularity of the stationary measures valid in the context of this thesis.

**Theorem 2.2.1.** The random map \( f \in R^0(\mathcal{M}) \) possesses a finite number of ergodic stationary measures \( m_1, \ldots, m_m \) with mutually disjoint supports \( E_1, \ldots, E_m \).

All stationary measures are linear combinations of \( m_1, \ldots, m_m \). The support \( E_i \) of \( m_i \) consists of the closure of a finite number of connected open sets \( C_1^i, \ldots, C_p^i \) that are moved cyclically by \( f(\Delta; \cdot) \).

**Proof.** The map \( L^* \) is continuous and preserve the convex and compact (in the weak* topology) space \( \mathcal{P}(\mathcal{M}) \). These are exactly the conditions to be satisfied to apply the Kakutani fixed point to the map \( L^* \) and thus the existence of at least one stationary measure. If there were infinitely many components, one could choose a point \( x_i \) in the support of each. By compactness, there would be an accumulation point \( x \). Using the continuity of the transition probabilities near \( x \) one can get a contradiction to the ergodicity and invariance of the measure (a neighborhood of \( x \) leaks to infinitely many components).

We introduce a topology on the space \( R^k(\mathcal{M}); k \geq 0 \) of random maps in order to be able to compare the dynamics of nearby random maps. Natural topologies on \( R^k(\mathcal{M}) \) are the uniform \( C^k \) topologies on \( C^k(\mathcal{M} \times \Delta, \mathcal{M}) \). See e.g. (57) for generalities on these topologies. We will assume \( R^k(\mathcal{M}) \) to be equipped with this topology. Note that the alternative approach through discrete Markov processes suggests a topology using the densities of the transition functions. Consider \( f \in R^0(\mathcal{M}) \). Write \( m_1, \ldots, m_m \) for the stationary measures of \( f \in R^0(\mathcal{M}) \) given by Theorem 2.2.1.
Definition 2.2.1. A random map \( f \in R^0(M) \) is stable if for all \( \tilde{f} \) sufficiently close to \( f \), the following two properties are satisfied.

- For each \( i, 1 \leq i \leq m \), the random map \( \tilde{f} \) has a stationary measure \( \tilde{m}_i \) whose density is \( C \) close to that of \( m_i \).
- The supports of \( \tilde{m}_i \) and \( m_i \) are close in the Hausdorff metric.

We speak of a bifurcation, or a bifurcating random differentiable map, if at least one of these properties is violated.

Definition 2.2.2. An ergodic stationary measure of \( f \in R^0(M) \) is called isolated or attracting, if there exists an open set \( W \) (an isolating neighborhood) containing the support \( E \) of \( m \), so that \( f(W; \Delta) \subset W \) and \( m \) is the only ergodic stationary measure of \( f \) with support in \( W \). The largest such neighborhood will be called the basin of attraction of the measure \( m \) and will denoted \( B(m) \). A measure with support included in \( B(m) \) is said to be in the basin of attraction of \( m \).

Corollary 2.2.1. Let \( \rho \) a probability measure. If \( \rho \in B(m) \) then \( \lim_{k \to \infty} (L^*)^k \rho = m \).

2.2.2 Families of random maps

Bifurcations are best studied in families depending on finitely many parameters. We will consider families of random maps depending on a single real parameter, where we have the goal to focus on bifurcations that typically occur varying one parameter.

Definition 2.2.3. A family of random maps is a collection of random maps \( \{f_a\} \) depending continuously on the parameters \( a \).

A family of random diffeomorphism is a collection smooth family of random differentiable maps where each map \( f_a(\omega; \cdots) \) is a diffeomorphism.

Remark 2.2.3. Alternatively, one can explicitly include noise densities \( g_a \) in the definition (considering pairs \( (f_a, g_a) \)) with \( g_a(\omega) \) varying continuously with \( (\omega, a) \). For convenience we consider fixed noise densities, but completely analogous results hold if the noise densities are allowed to vary with \( a \).

The two types of bifurcation distinguished in Definition 2.2.1 gives rise to a particular dynamical phenomenon associated to either intermittency or transients. Consider a family \( \{f_a\} \) of random maps in \( R(M) \), with \( a \) from an open interval \( I \). Suppose that \( a_0 \in I \) is a bifurcation value for \( \{f_a\} \) involving a stationary measure \( m \). Write \( \phi \) for the density of \( m \). Analogies with deterministic dynamics suggest the following two definitions. In Chapter 2 we will see that in typical one parameter families of random interval or circle differentiable maps bifurcations are isolated and of these two types.

Definition 2.2.4. The bifurcation at \( a_0 \) is called an intermittency bifurcation if there is a stationary density \( \phi_a \) for \( \{f_a\} \) with \( \phi_{a_0} = \phi \) and depending continuously on \( a \), so that the support \( E_a \) of \( \phi_a \) varies with \( a \), for \( a \) near \( a_0 \), as follows.
• $E_a$ varies continuously for $a$ from one side of $a_0$. Without loss of generality, we assume this to be the case for $a < a_0$.

• $E_a$ is discontinuous at $a = a_0$ and $E_a$ contains an open set disjoint from $E_{a_0}$ for $a > a_0$.

An orbit piece outside a small neighborhood $W$ of $E_{a_0}$ is called a burst. Out of the substantial literature on intermittency in dynamics, we point to references (92; 40; 58; 53; 59; 60).

**Definition 2.2.5.** The bifurcation at $a_0$ is called a *transient bifurcation* if there is a stationary density $\phi_a$ for $\{f_a\}$ with $\phi_{a_0} = \phi$ for $a$ close to $a_0$ from one side of $a_0$ (without loss of generality, we assume this to be the case for $a < a_0$), so that

• $\phi_a$ and its support $E_a$ vary continuously with $a$, for $a \leq a_0$.

• there is no stationary density near $\phi_{a_0}$ for $a$ close to $a_0$ and $a > a_0$.

![Figure 2.1: Typical times series for intermittent dynamics on the left and transient dynamics on the right. The time series are for bifurcation values after the bifurcation took place. The intermittency bifurcation involves interval diffeomorphism with a single stationary measure; the support consisting of two intervals for a period two cycle bifurcates to form a single interval. In the transient bifurcation one stationary measure out of two stationary measures existing previous to the bifurcation disappears.](image)

If a dynamical system undergoes a bifurcation at the parameter value $a = a_0$ as defined in definition 2.2.4, the system will be observed in a state corresponding to an attracting stationary measure when $a < a_0$. However the behavior for $a > a_0$ cannot be determined only from asymptotic considerations. To see this, consider again the picture on the right of Figure 2.1. Before bifurcation we have two stationary measure $m_a^1$ and $m_a^2$ with support respectively contained in the intervals $(-1.5, 0)$ and $(1.5, 3)$. At a bifurcation value of the parameter one stationary measure, $m_a^1$ say, with support contained in the interval $(0, -1.5)$ disappears. Nevertheless a typical orbit $O(\omega; x)$
with \( x \in (-1.5, 0) \) will then spend much time in a “transient regime” near the support of \( m^1_a \), which is still almost stationary. This notion of transience is of considerable importance in computational purposes, the applications might be to estimate an average \( \int f \, dm \), or to bound the \( m \)-probability of some set of unlikely states, or simply to generate typical realizations from the stationary probability measure \( m \). The number of steps required by a particular algorithm as a function of the dynamical system will depend on the “time to stationarity”, i.e. the number of steps until the distribution approaches stationarity.

Formally, let \( \{ f_a \} \) be a family of random diffeomorphisms \( \mathcal{R}^0(\mathcal{M}) \), with \( a \) from an open interval \( I \). Let \( m^1_{a_0}, m^2_{a_0}, \ldots, m^n_{a_0} \) be the stationary measures, given by Theorem 2.2.1, of \( f_a \) for some \( a_0 \in I \) and let \( W \) be an open neighborhood of the support \( E^1_{a_0} \) of \( m^1_{a_0} \) such that no other stationary measure has support intersecting \( W \). If \( f_a \) is stable, then there is a unique stationary measure \( m_a \) that is the continuation of \( m_{a_0} \).

The stationary measure \( m_a \) has its support in \( W \) for \( a \) near \( a_0 \). If \( a_0 \) is a bifurcation value for \( \{ f_a \} \), iterates \( f^k_a(x; \omega) \), for certain \( x \in W \) and \( a \) near \( a_0 \), may leave \( W \).

For \( x \in W \) and \( \omega \in \Delta_N \), define the escape time

\[
\tau_a(x, \omega) = \min\{ k \mid f^k_a(x; \omega) \notin W \}. \tag{2.12}
\]

Later on we show how the average escape time from a neighborhood of the support of a bifurcating stationary measure is more than polynomially large in an unfolding parameter. This makes it difficult to accurately establish the bifurcation parameter value using finite data, even in numerical simulations. It explains the occurrence of very long transients near a transient bifurcation and the very irregular occurrence of bursts in intermittent time series. The proof relies on the construction of conditionally stationary measures, see Chapter 5.

### 2.3 The skew product approach

If we look attentively to the set up defined in the previous section, we will remark that besides being basically restricted to the Markovian case this approach has some dynamical drawbacks.

First, the most objects we study in this approach are stationary measures and densities. These quantities measure, via the Birkhoff ergodic theorem (2.7), the asymptotic proportion of time spent by a typical orbit \( O(\omega; x) \) in a given set \( A \). Moreover, because stationary measures are objects that are determined by the one-point motions (of some initial condition) of the random map and can thus in principle not be related to the collective representation of these orbits for all initial conditions in the state space which is called the phase portrait. This portrait provides a global qualitative picture of the dynamics. Hence dynamical questions such as, e.g. (dynamical) bifurcation, attractors or repellers are concerned, the Markovian model cannot exhibit these phenomena.

These questions are addressed within the skew product (SP) or random dynamical systems theory (RDS) approach which considers noisy systems from a different standpoint. That is, the RDS theory analyzes the dynamics of different initial conditions under the same noise realization. In this way, for each noise realization, RDS studies the dynamics of a non-autonomous dynamical system. In principle, we have as
many dynamical systems as noise realizations: to each noise realization there corresponds a different non-autonomous dynamical system. We thus have a family of non-autonomous dynamical systems. The random selection of the noise realization is in fact interpreted as the random selection of one system from this family. The noise realization can take a variety of forms, so that it may seem an impossible task to analyze all systems within the family. However, remarkably, under wide conditions, for almost all noise realizations, the dynamics of the nonautomated systems within the family will strongly resemble one another. This similarity makes it possible to describe typical dynamics for the family. For a large discussion of this subject see (5).

2.3.1 Some definitions and a few general facts

We endow the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) by the filtration \(\{\mathcal{F}_m; m \leq n \in \mathbb{Z}\}\) generated by the cylinder sets, \(\mathcal{C}_{i_0,...,i_n} = \{\omega \in \Omega : \omega_m \in B_{i_m}, \ldots, \omega_n \in B_{i_n}\}\) where \(B_{i_j}\) are Borel subsets of \(\Delta\) and \(j = m, \ldots, n\) and \(m, n \in \mathbb{Z}\).

We extend this notation to the case in which \(m = -\infty\) and/or \(n = +\infty\) by setting \(\mathcal{F}_{-\infty} = \sigma\{\mathcal{F}_m; m \leq n\}\), the sigma-algebra generated by all \(\mathcal{F}_m\), and similarly for \(\mathcal{F}_{+\infty}\) and \(\mathcal{F}_{-\infty}\). A straightforward verification shows that \(\sigma^k \mathcal{F}_m = \mathcal{F}_{m+k}\) for all \(m \leq n\) and \(k\).

The sigma-algebras \(\mathcal{F}^- = \mathcal{F}_{-\infty}\) and \(\mathcal{F}^+ = \mathcal{F}_0^{+\infty}\) are called the past and the future of \(f^t\).

Let \(\mathcal{P}_\mathcal{F}(\Omega \times \mathcal{M})\) be the set of probability measures on \((\Omega \times \mathcal{M}, \mathcal{F} \times \mathcal{B})\) whose projections on \(\Omega\) coincide with \(\mathbb{P}\). It is well known (see (5, Section 1.4)) that each measure \(\mu \in \mathcal{P}_\mathcal{F}(\Omega \times \mathcal{M})\) admits a unique disintegration \(\omega \mapsto \mu_\omega\), which is a random variable ranging in the space of measures such that

\[
\mu(A) = \int_{\Omega} \int_{\mathcal{M}} 1_A(\omega; x) d\mu_\omega(x) d\mathbb{P}(\omega), \quad \forall A \in \mathcal{F} \times \mathcal{B},
\]

where \(1_A\) is the indicator function of \(A\).

Recall from chapter 1 that a measure \(\mu \in \mathcal{P}_\mathcal{F}(\Omega \times \mathcal{M})\) is said to be invariant for the skew product \(S\) if \((S)_* \mu = \mu\) (that is \(\mu(S^{-1}(A)) = \mu(A)\) for every \(A \in \mathcal{F} \times \mathcal{B}\)). By (5, Theorem 1.4.5), a measure \(\mu \in \mathcal{P}_\mathcal{F}(\Omega \times \mathcal{M})\) is invariant if and only if its integration \(\mu_\omega\) satisfies the following relation for \(\mathbb{P}\)-almost all \(\omega \in \Omega\):

\[
(f^k)_* \mu_\omega = \mu_{s^k \omega} \quad \text{for all} \ k \geq 0.
\]

The set of all invariant measures for \(S\) will denoted by \(\mathcal{I}_f\).

For a random map with one side time \(T = N\) we have the following simple one to one correspondence (discovered by Ohno (89)) between \(S\)-invariant product measures \(m = \mathbb{P} \times m\) on \(\Omega \times \mathcal{M}\) and stationary measure \(m \in \mathcal{I}_f(\mathcal{M})\).

**Proposition 2.3.1.** Let \(f\) a random map with one side time \(T = N\). Then
1. A measure \( \mu = \mathbb{P} \times m \in \mathcal{F}(f) \) is \( S \)-invariant if and only if \( m \in \mathcal{S}(f) \).

2. A \( S \)-invariant measure \( \mu = \mathbb{P} \times m \) is ergodic if and only if \( m \) is ergodic.

**Proof.** See (5, Theorem 2.1.7) and (70, Theorem 2.1).

**Definition 2.3.1.** An invariant measure \( \mu \in \mathcal{P}_0(\Omega \times \mathcal{M}) \) is said to be Markov if its disintegration \( \mu_\omega \) is measurable with respect to the past \( \mathcal{F}^- \). The set of such measures will denoted by \( \mathcal{S}_p,\mathcal{F}^- \).

The way of iteration given so far is called the forward evolution of the system. The pull-back evolution arises from the map

\[
f^k_{\sigma-k}\omega(x) = f_{\sigma-1}\omega \circ \ldots \circ f_{\sigma-k}\omega(x), \quad \omega \in \Omega, \quad x \in \mathcal{M}.
\]  

(2.14)

Writing the random variable \( \omega \) as an double infinite sequence \((\ldots, \omega^{-k}, \ldots, \omega^{-1}, \omega_0, \omega_1, \ldots, \omega_k, \ldots)\) the pull-back iteration can be also written as

\[
f^k_{\sigma-k}\omega(x) = f_{\omega^{-1}} \circ \ldots \circ f_{\omega^{-k}}(x), \quad \omega \in \Omega, \quad x \in \mathcal{M}.
\]  

(2.15)

Here we observe the system at time zero, starting back in the past at time \(-k\). The pull back approach enjoys some proprieties that are invisible in the usual push-forward approach. This is illustrates in the following example and lemma 2.3.1.

**Example 2.3.1.** Consider the Ornstein-Uhlenbeck process

\[
\frac{dx_t}{dt} = -ax_t + \xi_t, \quad a > 0, \ \xi_t \text{ is a white noise}.
\]  

(2.16)

The general solution starting at time \( t_0 \) from a given initial point \( x \) is

\[
x_t(t_0, x, \omega) = e^{-a(t-t_0)}x + \int_{t_0}^{t} e^{-a(t-s)}\xi_s ds.
\]

The stationary solution for the push-forward is

\[
\lim_{t \to \infty} x_t(0, x, \omega) = \lim_{t \to \infty} e^{-at}x + \lim_{t \to \infty} \int_{0}^{t} e^{as}\xi_s ds
\]

\[
= \lim_{t \to \infty} \int_{0}^{t} e^{as}\xi_s ds
\]

\[
= \lim_{t \to \infty} x_t(\omega) \quad \text{← this is a stochastic process}.
\]

The stationary solution for the pull-back viewpoint is

\[
\lim_{t_0 \to -\infty} x_t(t_0, x, \omega) = \lim_{t \to -\infty} e^{at}x + \lim_{t \to -\infty} \int_{t_0}^{0} e^{as}\xi_s ds
\]

\[
= \int_{-\infty}^{0} e^{as}\xi_s ds
\]

\[
= x_0(\omega) \quad \text{← random variable}.
\]
In a sentence, we could say that the pull-back is able to take a picture at time zero of certain objects (sets, measures, …) that continue to fluctuate in the forward viewpoints.

The following lemma gives the correspondence between Markov $S$-invariant measures and stationary measures for the Markov family. This result is established (for different situations) in (47, Theorem 7.5), (79, Lemme 1) or (5, Theorem 1.7.2). For the reader’s convenience we give here a sketch of the proof.

**Lemma 2.3.1.** Let $f$ a random map. Then there is a one to one correspondence between Markov invariant measures $\mathcal{F}_{\sigma^k}^{-}(f)$ for the skew product $S$ and the stationary measures $\mathcal{I}(M)$ for the associated Markov family. Namely, if $m \in \mathcal{I}(M)$, then the limit

$$
\mu_\omega = \lim_{k \to \infty} (f_{\sigma^{-k}} \omega)_* m
$$

exists in the weak$^*$ topology almost surely and gives the disintegration of a Markov invariant measure $\mu$. Conversely, if $\mu \in \mathcal{F}_{\sigma}^{-}(f)$ is a Markov measure and $\mu_\omega$ is its disintegration, then $m = E_{\mu} \mu_\omega$ is a stationary measure for the Markov family.

**Proof.** The proof of this lemma is divided in two parts. First, the measure valued stochastic process $(\mu_k^\omega = (f_{\sigma^{-k}} \omega)_* m, k \in \mathbb{N})$ is shown to be a martingale with respect to the filtration $\mathcal{F}_{\sigma^{-k}}(\omega)$. Then, using the martingale convergence theorem we show that this sequence converge almost surely.

We have for a real valued continuous function $\psi$

$$
E(\mu_{k+1}^\omega(\psi)|\mathcal{F}_{\sigma^{-k}}^0) = E\left(\int \psi(f_{\sigma^{-k}} \omega)(x)dm(x)|\mathcal{F}_{\sigma^{-k}}^0\right)(\omega)
$$

$$
= E\left(\int \psi(f_{\sigma^{-k}} \omega)(x)dm(x)|\mathcal{F}_{\sigma^{-k}}^0\right)(\omega)
$$

$$
= E\left(\int \psi(f_{\sigma^{-k}} \omega)(y)[(f_{\sigma^{-k}} \omega)_* m]d\mathcal{L}(y)|\mathcal{F}_{\sigma^{-k}}^0\right)(\omega)
$$

$$
= \int \psi(f_{\sigma^{-k}} \omega)(y)E[(f_{\sigma^{-k}} \omega)_* m]d\mathcal{L}(y)
$$

$$
= \int \psi(f_{\sigma^{-k}} \omega)(y)dm(x)
$$

$$
= \mu_\omega(\psi).
$$

By the martingale convergence theorem we can show that the above sequence converge almost surely to a random variable. Furthermore, this limit is a Borel probability measure.

**Remark 2.3.1.**

1. The almost sure convergence is the previous lemma is not valid if we replace the pull-back iteration by the forward iteration. In this last case we have only convergence in distribution.
2. Markov measures are physical measures whose the support is, under some condition see Chapter (5), a random attractor.

3. We have $\mu = (f_k)^* \mu$ (invariance of the limit measure). The sequence $(\mu_k)_{k \in \mathbb{N}}$ is a stationary Markov chain on the space of probability measure on $\mathcal{M}$. The distribution of $\mu$ is called the statistical equilibrium associated to $m$ see (79).

**SBR properties of the random invariant Markov measure**

There are very few examples where stationary measures for random maps are explicitly computed, see for example (23). Furthermore, there are examples where stationary measures are, dynamically, not very interesting. See chapter 4 for an example. In these cases, it would be of interest to find sufficient conditions for the pull-back iterates of another measure $\rho$, not necessarily stationary, to converge. This is a property of Sinai-Bowen-Ruelle type.

We know from lemma 2.2.1 that if $\rho \in \mathcal{B}(m)$ then $\lim_{k \to \infty} (L^*)^k \rho = m$. But we do not know how to conclude from this fact. We can conclude if we assume more, on the two-point motion

$$L^*_2 (m \times m)(\psi) = \int_{\mathcal{M} \times \mathcal{M}} U\psi(x, y)dm(x)dm(y),$$

and

$$U\psi(x, y) = \int_{\Delta} \psi(f(\omega, x), f(\omega, y)) d\nu(\omega).$$

The function $\psi$ is a real valued continuous function on $\mathcal{M} \times \mathcal{M}$. All convergence below are understood in the weak sense. The next result can be found in (15).

**Lemma 2.3.2.** Let $\bar{\mu}$ be the probability measure defined on $\mathcal{M} \times \mathcal{M}$ defined as

$$\bar{\mu} = E[\mu \times \mu].$$

Then

- $(L^*_2)^k (m \times m) \to \bar{\mu}$
- $\bar{\mu}$ is invariant for $L^*_2$.

The following fact has been proven in (36).

**Proposition 2.3.2.** Let $\rho$ a probability measure on $\mathcal{M}$. Assume the following ergodic property of the two-point motion:

$$(L^*_2)^k (\nu_1 \times \nu_2) \to \bar{\mu}$$

for every $\nu_1, \nu_2 \in \{\rho, m\}$. Then

$$E \left[ |(f_{\sigma-k}\mu - \mu)\psi|^2 \right] \to 0$$
for every real valued continuous function $\psi$. Assume in addition that the convergence (2.18) is exponential, in the sense that there exist a constant $\lambda$ such that for every $\nu_1, \nu_2 \in \{\rho, m\}$ and every real valued continuous function $\psi$ on $M \times M$ one has

$$|(L_2^k (\nu_1 \times \nu_2)(\psi) - \bar{\mu}(\psi))| \leq Ce^{-\lambda k}, \quad k \in \mathbb{N}$$

for some constant $C > 0$ depending only on $\psi$. Then for every $\nu \in \{\rho, m\}$, every continuous $\psi$ defined on $M$, every $\lambda' < \lambda$ and $\mathbb{P}$-almost every $\omega \in \Omega$ one has

$$|(f_{\sigma^{-k}\omega} \rho(\psi) - \mu_{\omega}(\psi)| \leq C'(\omega)e^{-\lambda' k} \quad k \in \mathbb{N}$$

for some random variable $C'(\omega)$.

Random attractors

Let $f$ be a random map. A family of subsets $A_\omega$, $\omega \in \Omega$ is called a random compact (closed) set if $A_\omega$ is compact (closed) for almost all $\omega$ and $\Omega_U = \{\omega \in \Omega : A_\omega \cap U \neq \emptyset\} \in \mathcal{F}$ for every open set $U \subset M$. A random compact set $A_\omega$ is said to be measurable with respect to a sub-$\sigma$-algebra $\mathcal{F}' \subset \mathcal{F}$ if $\Omega_U \in \mathcal{F}'$ for any open set $U \subset M$.

Given two sets $A, B$ and points $x, y$ we set and a distance $d$ on $M$, we set

$$d(x, B) = \inf_{y \in B} d(x, y), \quad d(A, B) = \sup_{x \in A} d(x, B)$$

**Definition 2.3.2.** A random compact set $A_\omega$ is said to be (forward) invariant if

$$f_\omega A_\omega \subseteq A_{\sigma \omega}$$

i.e. if $d(f_\omega A_\omega, A_{\sigma \omega}) = 0$ for $\mathbb{P}$-almost all $\omega$. It is exactly invariant if

$$f_\omega A_\omega = A_{\sigma \omega}.$$  

For a deterministic system, we say that the trajectory $x(t)$ converges to an invariant set if $d(x(t), A) \to 0$ as $t \to \infty$. For random dynamical systems, one can only expect this to occur in a noise-dependent way, and so there are several ways to generalize convergence depending on which noise trajectories are disregarded.

**Definition 2.3.3.** A random compact set $A_\omega$ is called a random point attractor (in the almost sure convergence sense) if for each $\xi \in M$ and $\omega \in \Omega_0$, where $\Omega_0 \in \mathcal{F}$ is a set of full measure independent of $\xi$, the sequence of random variable $r_k(\omega) = d(f^k_\omega(\xi), A_{\sigma^k \omega})$ converges almost surely to zero.

Similarly one may look at the sequence $r_k(\omega) = d(f^k_{\sigma^{-k}\omega}(\xi), A_\omega)$). This type (pull-back) of convergence of a trajectory to a random set is also used to define random attractors. We will use the "forward" definition which seems more natural.

**Example 2.3.2.** Consider again example 1.1.1, i.e. the perturbed difference equation $x_k = ax_k + \omega_k$ and $0 < a < 1$. We have

$$x_{k+1} = a^k x_1 + \sum_{i=0}^{k-1} a^i \omega_{k+i-1-i}. \quad (2.19)$$
For this example we show that the random attractor is a (random) point. To see the forward convergence we set $l = 0, x_0 = y$ to obtain

$$x_k = a^k y + \sum_{i=0}^{k-1} a^i \omega_{k-1-i}.$$ 

Letting $k \to \infty$ gives a natural guess for the random attractor, namely

$$A_{\sigma^k \omega} = \sum_{i=0}^{\infty} a^i \omega_{k-1-i}.$$

This sequence converges to a finite random limit (point) $X_\infty(\omega)$ because $0 < a < 1$. Furthermore we have

$$|x_k - A_{\sigma^k \omega}| = |a^k y + \sum_{i=0}^{\infty} a^i \omega_{k-1-i}|$$

$$= |a^k| |y - \sum_{i=k}^{\infty} a^{-i} \omega_{k-1-i}|$$

$$= |a^k| |y - A_{\omega}|$$

$$\to 0$$

as $k \to \infty$ with probability 1. Thus, the forward convergence of the perturbed difference equation to a random point attractor is almost sure.

To illustrate the pull-back convergence we set $k + l = 0$ and $x_l = y$ in equation 2.19 to obtain

$$x_0(\omega) = a^k y + \sum_{i=0}^{k-1} a^i \omega_{1-i}.$$ 

The right hand side of this expression is exactly the pull-back iteration of the perturbed difference equation starting at time $-k$. Letting $k \to \infty$, and let

$$A_{\omega} = \sum_{i=0}^{\infty} a^i \omega_{1-i}$$

This infinite sum converge to a finite random (point) limit $X_\infty(\omega)$ because $0 < a < 1$. The random attractor is this random point.

A random point attractor $A_{\omega}$ is said to be minimal if for any other random point attractor $A'_{\omega}$, we have $A_{\omega} \subset A'_{\omega}$ for almost all $\omega$.

The following proposition is a straightforward consequence of Theorem 3.4 and Theorem 4.3 and Remark 3.5 in (28).

**Proposition 2.3.3.**

(i) Let $f$ a random map. Suppose there exists a random compact set $K_{\omega}$ attracting the trajectories of $f$ in the following sense. There exist a full measure set $\Omega_0 \in \mathcal{F}$ and

$$\lim_{k \to t} d(f_{\sigma^{-k} \omega}(\xi), K_{\omega}) = 0 \quad \text{for any } \omega \in \Omega_0, \xi \in \mathcal{M}.$$
Then $f_\omega$ possesses a random attractor $A_\omega$ that is measurable with respect to the past $\mathcal{F}^-$.  

(ii) For each Markov invariant measure $\mu \in \mathcal{M}_{\mathcal{F}^-}(f)$, its disintegration $\mu_\omega$ is supported by each random attractor $A_\omega$; i.e., $\mu_\omega(A_\omega) = 1$ almost surely.

**Definition 2.3.4.** A random fixed point of a random map $f_\omega$ is a random variable $x^* : \Omega \to M$ such that 

$$x^*(\sigma_\omega) = f_\omega(x^*(\omega)) \quad \text{for all} \quad \omega \in \Omega. \quad (2.20)$$

A random fixed point $x^*$ is called globally attracting in a family of sets $(U(\omega))_{\omega \in \Omega}$ if for all $\omega \in \Omega$ and for all $\xi \in U(\omega)$

$$\lim_{k \to \infty} d(f^k_\omega^{-\omega}(\xi), x^*(\omega)) = 0.$$  

Equation (2.20) implies $x^*(\sigma^k_\omega) = f_\omega(x^*(\omega))$ for all $k \in \mathbb{Z}$. Hence a random fixed point is a stochastic process which satisfies the random difference equation (2.1) and whose state is determined only by the dynamical system modelling the noise (the map $\sigma$).

Stability of a random fixed point requires that for all $\omega \in \Omega$ the sample-path of all initial values in some sets $U(\omega)$ converges to (and therefore eventually moves as) the sample path $k \mapsto x^*(\sigma^k_\omega)$ of the random fixed point. $x^*(\omega)$ is the initial state corresponding to this sample-path. An alternative way to characterize random fixed points is using the skew product system $S$. Define the skew-product flow. $x^*(\omega)$ is random fixed point if and only if the graph of $x^*(\omega)$ is invariant under $S$.

### Random Periodicity

Periodicity (and in general the notion of recurrence) plays an important role in the theory of deterministic dynamical systems and in Markov chains theory and it would be interesting to say a word about periodicity in random dynamical systems in this section. To our knowledge, the only study in this direction has been done in (72). The author in (72) emphasizes that, in contrast to deterministic system, one has to distinguish between random periodic orbits, random periodic points and random periodic cycles.

A finite random invariant set $A$ with $\#A_\omega = k$ $\mathbb{P}$-a.s. has been called in (72) a random periodic orbit of period $k$.

If $\sigma^k$ is ergodic, a random variable $a(\omega)$ has been called a random periodic point of period $k$ if $f^k_\omega(a(\omega)) = a(\sigma^k_\omega)$ $\mathbb{P}$-a.s.

If the random maps $f_\omega$ act on the line (circle and assuming that $\sigma^k$ is ergodic, the $k$ random points $a_i(\omega)$, where $a_1 < a_2 < \ldots < a_k$ $\mathbb{P}$-a.s., are called in (72) a random cycle of period $k$, if there exist a deterministic $k$-permutation $\pi$ such that $f_\omega a_i(\omega) = a_{\pi(i)}(\sigma_\omega)$.

It turns out that these notions are not quite the same, in particular, the existence of a random periodic orbit does not imply, in general, the existence of a random periodic point and the latter does not imply, in general, the existence of a random cycle. This difference is clearly demonstrated in section (4.1.1) and section (4.1.2). In section (4.1.1), we will prove the existence of an $\mathbb{P}$-a.s. finite invariant random set $A_\omega$, i.e.
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\[ f_\omega A_\omega = A_\sigma \omega. \]

Furthermore, \( A_\sigma \omega = \{ a_1(\sigma \omega), \ldots, a_k(\sigma \omega) \} \) is a permutation (depends on \( \omega \)) of \( \{ f_\omega(a_1(\omega)), \ldots, f_\omega(a_k(\omega)) \} \). This random permutation is defined as

\[ \pi : \Omega \times \{ 1, \ldots, k \} \to \{ 1, \ldots, k \} \quad \pi_\omega(i) = j \text{ if } f_\omega(a_i(\omega)) = a_j(\sigma \omega). \quad (2.21) \]

Contrary to section (4.1.1), the system in section (4.1.2) admits a finite invariant random which the elements deterministically permuted under the action of the random map. To be precise and because of the cyclic propriety of the system and expression (4.5), one has \( f_\omega(a_i(\omega)) = a_{i+1}(\sigma \omega) \) for \( i = 1, \ldots, k \) and \( f_\omega(a_k(\omega)) = a_1(\sigma \omega) \). Hence the permutation in equation (2.21) is deterministic \( \pi = \begin{pmatrix} 1 & 2 & \ldots & k-2 & k-1 & k \\ 2 & 3 & \ldots & k-1 & k & 1 \end{pmatrix} \).

2.4 Representations of discrete Markov processes

In this section we explore the relation between random maps and discrete Markov processes given by stochastic transition functions. The random maps considered in this paper depend on random parameters, where the number of random parameters equals the dimension \( n \) of the state space \( \mathcal{M} \). Proposition 2.4.1 gives a wide class of Markov processes that can be represented by random maps by \( n \) random parameters. The Markov process given by random maps depending on a larger number of random parameters (or even given by some measure on the space of maps) can be represented by random maps with \( n \) random parameters.

Iterating a random map involves more random parameters obtained by independent draws at each iterate. By means of an example we explain how random maps with a smaller number of random parameters may be brought into the context of this paper. Consider the delayed logistic map \( x_{n+1} = \mu x_n(1 - x_n - 1) \). Let \( y_{n+1} = x_n \). This defines a dynamical system \((x_{n+1}, y_{n+1}) = (\mu x_n(1 - y_n), x_n)\). Assume now that \( \mu \) is a random parameter varying in some interval with some distribution. This yields a random map

\[ f(x, y; \mu) = (\mu x(1 - y), x). \]

The derivative \( Df \) is singular along \( x = 0 \). As \( \mu \) is a single random parameter, this random diffeomorphism does not fit into the context considered in this paper. Considering two iterates gives two independent draws \((\mu, \nu)\) of the random parameter (that is, random parameters taken from a square) and yields the random map

\[ f^2(x, y; \mu, \nu) = (\mu x(1 - y), \nu \mu x(1 - x)(1 - y)). \]

If \( x \) and \( y \) stay away from 0 and 1, the map and the dependence of \((\mu, \nu)\) are injective. The second iterate of the delayed logistic map with bounded parametric noise fulfills the assumptions in this paper.

There are other examples of maps with parametric noise that cannot be made to fulfill the assumptions used in this paper. For instance, random maps \( f(x; \omega) = x + (x - \omega)^2 \) with random \( \omega \) from an interval, fail to satisfy the injectivity assumption of \( \omega \mapsto f(x; \omega) \). If \( \omega \) is chosen from a uniform distribution, then the density of the transition function will not be bounded. Figure 2.2 indicates a random boundary bifurcation for a similar random map.
2.4 Representations of discrete Markov processes

Figure 2.2: A random map $f(x; \omega) = f(x - \omega; 0) + \omega$ with a random boundary bifurcation of the stationary measure with support between the ordinates indicated by dotted lines. If $\omega$ is chosen from a uniform distribution, then the density of the transition function will not be bounded.

Consider discrete Markov processes given by transition functions $P(x, \cdot)$. The following properties hold.

For fixed $A \in \mathcal{B}$, $x \mapsto P(x, A)$ is measurable.

For fixed $x \in \mathbb{R}^n$, $P(x, \cdot)$ is a probability measure.

Denote by $y \mapsto k(x, y)$ the density of $P(x, \cdot)$. Write $U_x$ for the support of $k(x, \cdot)$ and let $U = \bigcup_x (\{x\} \times U_x)$. Assume that $U_x$ is diffeomorphic to the closed unit ball $\Delta$ in $\mathbb{R}^n$ and varies smoothly with $x$. We will assume that $y \mapsto k(x, y)$ depends smoothly on $(x, y) \in U$, meaning that $k$ can be extended to a smooth function defined on an open neighborhood of $U$. Under these conditions we will construct a representation by a finitely parameterized family of differentiable maps. That is, we will construct a family of differentiable maps $\{f_\mu\}$ on $\mathbb{R}^n$, with parameters $\mu$ from an $n$ dimensional ball, and a measure $\nu$ on the parameter space so that $P(x, A)$ equals $\nu\{\mu \in \Delta \mid f_\mu(x) \in A\}$. A corresponding result holds for discrete Markov processes with noise from an $n$-dimensional box, see (19, Appendix D). See (70) for a discussion of the existence of representations by sets of measurable or continuous maps. The paper (94) contains a result on representations by differentiable maps, under the assumption of unbounded noise.

Proposition 2.4.1. There is a family of differentiable maps $f_\mu$, $\mu \in \Delta$, and a measure $\nu$ on $\Delta$ with smooth strictly positive density, so that

1. $(x, \mu) \mapsto f_\mu(x)$ is smooth,
2. for each $x \in \mathcal{M}$, $\mu \mapsto f_\mu(x)$ is injective,
3. $P(x, A) = \nu(\mu \in \Delta \mid f_\mu(x) \in A)$. 
Proof. We follow the arguments in (19), combined with the use of polar coordinates to map the unit ball $\Delta$ to $[0,1]^n$. Let $\psi_x : V_x \rightarrow \Delta$ be a diffeomorphism, depending smoothly on $x$, from the support $V_x$ of $y \mapsto k(x,y)$ to the unit ball. Consider polar coordinates $\chi : [0,1]^n \rightarrow \Delta$ on the unit ball,

\[ \chi(\xi_1, \ldots, \xi_n) = \xi_1 \begin{pmatrix} \cos(\pi \xi_2) \\ \sin(\pi \xi_2) \cos(\pi \xi_3) \\ \vdots \\ \sin(\pi \xi_2) \cdots \sin(\pi \xi_{n-1}) \cos(2\pi \xi_n) \\ \sin(\pi \xi_2) \cdots \sin(\pi \xi_{n-1}) \sin(2\pi \xi_n) \end{pmatrix}. \]

For $\xi \in [0,1]^n$, define sets $B_i(\xi)$, $0 \leq i \leq n$, by

\[ B_i(\xi) = \prod_{j=1}^{i} [0, \xi_j] \times \prod_{j=i+1}^{n} [0,1]. \]

Write $C_i(\xi) = \psi_x^{-1} \chi^{-1}(B_i(\xi))$ and let $\omega = (\omega_1, \ldots, \omega_n)$ be given by

\[ \omega_i = \int_{C_i(\xi)} k(x,y)dy / \int_{C_{i-1}(\xi)} k(x,y)dy. \]

Since $k > 0$, $\omega = \Theta(\xi)$ gives a 1-1 correspondence. Let $\eta_i = \int_{C_i(\xi)} dm(y) / \int_{C_{i-1}(\xi)} dm(y)$. Here $\Psi(\xi)$ is a 1-1 correspondence. The correspondence $\omega \rightarrow \eta$ is a smooth diffeomorphism as $k$ is smooth and strictly positive. Then

\[ f_\mu(x) = \psi_x \chi \Psi^{-1} \Theta \chi^{-1}(\mu). \]

gives the required smooth random maps. \qed

For discrete Markov processes on a circle there is an easy necessary and sufficient condition on the transition maps for a representation by random diffeomorphisms.

**Proposition 2.4.2.** Let $\mathcal{M}$ be the circle endowed with Lebesgue measure. Write $V_x = [l_-(x), l_+(x)]$. There is a representation by random smooth diffeomorphisms if and only if

\[ -k(x, l_-(x))l_+(x) + \int_{l_-(x)}^z \frac{\partial}{\partial x} k(x,y)dy \neq 0 \]

for $z \in V_x$.

**Proof.** The construction of the representation by random smooth maps proceeds as follows. For $\xi \in [0,1]$, write $C(\xi) = [l_-(x), l_-(x) + \xi (l_+(x) - l_-(x))]$ and let

\[ \omega = \int_{C(\xi)} k(x,y)dy. \]

Since $k > 0$, the map $\Theta$, $\Theta(\xi) = \omega$, is a diffeomorphism. The representation by random diffeomorphisms is given through

\[ f_\omega(x) = l_-(x) + \Theta^{-1}(\omega)(l_+(x) - l_-(x)). \]
Note that for fixed $\omega$,

$$
\frac{d}{dx} \int_{[\ell_-, \ell_+]} k(x, y) dy = 0.
$$

(2.22)

With, say, $\ell_- < f_\omega$, (2.22) yields

$$
-k(x, \ell_-(x))l'_-(x) + k(x, f_\omega(x))f'_\omega(x) + \int_{\ell_-(x)}^{f_\omega(x)} \frac{\partial}{\partial x} k(x, y) dy = 0.
$$

Hence $f'_\omega(x) = 0$ precisely if

$$
-k(x, \ell_-(x))l'_-(x) + k(x, f_\omega(x))f'_\omega(x) + \int_{\ell_-(x)}^{f_\omega(x)} \frac{\partial}{\partial x} k(x, y) dy = 0.
$$

Proposition 2.4.3. Suppose the stationary measure $m$ has support $E$ consisting of $q$ disjoint intervals. For any $l \in \mathbb{N}$, there are constants $C_l > 0$ so that

$$
\phi(x) \leq C_l(d(x, \partial E))^l
$$

for $x \in E$ (here $d(x, \partial E)$ denotes the distance of $x$ to the boundary of $E$).

Proof. The transfer operator has an expression

$$
L\phi(x) = \int_{\ell_-(x)}^{\ell_+(x)} k(y, x) \phi(y) dy
$$

for smooth functions $l_-, l_+$ and $k(x, y)$ positive and bounded (say $k \leq K$).

Denote $E = [a, b]$. We consider $\phi$ near $a$. Of course, $\phi(y) = 0$ for $y$ near $a$ and $y < a$. Suppose $\phi(y) \leq C_l(y - a)^l$ for $y > a$ and $y$ near $a$. Then again for $y$ close to $a$,

$$
L\phi(y) = \int_{a}^{l_+(x)} k(y, x) \phi(y) dy \leq \int_{a}^{l_+(x)} K C_l(y - a)^k dy \leq C_{l+1}(y - a)^{l+1}
$$

for some $C_{l+1}$. The proposition follows.