Bifurcation of random maps
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3 The dynamics of one-dimensional random maps

The structure of the state space has a strong influence on the possible dynamical phenomena one can expect. For example, the Anosov map seen as a map on the plane shows rather trivial behavior. As a map on the torus, it is very complicated see (33). Generally speaking, the higher the dimension of the state space, the more difficult it is to describe the possible dynamical phenomena. In realistic models, the state space is usually of high dimension. It might even be of infinite dimension, as in the case of fluid dynamics, in which the states are vector fields or stochastic systems, in which the states are measures. However, realistic models are dissipative because there is friction which causes energy loss. It has been observed, and in many cases proven, that the attractors of these models are actually of lower dimension, much smaller than the dimension of the state space. In this chapter we describe one-dimensional dynamics of random maps and in particular show that the set of stable (in the sense of definition 2.2.1) random differentiable maps \( R^1(M) \) is open and dense. In view of the above discussion, one-dimensional dynamics should be the simplest possible; the state space is just a circle or an interval with the simplest topology one can imagine. However, there are three surprising reasons for the importance of the study of one-dimensional dynamics. The first reason is that the observed dynamics of one-dimensional systems is very rich. Most dynamical phenomena observed until now in higher-dimensional dynamics have their counterpart in one-dimensional dynamics. Second, the extensive technology developed to study one-dimensional dynamics turns out to be very useful for the study of higher-dimensional systems. And most surprisingly, predictions from the one-dimensional theory turn out to hold for realistic models of very high dimension. There is a rather precisely understood phenomenon, the so-called homoclinic bifurcations, which is observed in many realistic models and which explains why one-dimensional phenomena can appear in higher dimensional dissipative systems. By no means can one expect one-dimensional systems to explain everything. However, the richness of its dynamics and the characteristics it displays in realistic systems make it much more than a simple toy.

The aim of this chapter is to give a description of stochastic bifurcation as defined in Chapter 2 in one dimension state space. In Section 3.1 we establish that generic random diffeomorphisms are stable. Section 3.2 is devoted to the study of the one parameter families of random maps. We give the analogues of the elementary deterministic local bifurcation and classify possible random bifurcation in one definition in Theorem 3.2.1.
3.1 Stability of stationary measures

The most complete description of bifurcations in random differentiable maps is derived for random maps in one dimension. Consider a random differentiable map \( f(x; \omega) \) on the circle \( S^1 \). The random parameter \( \omega \) is drawn from \( \Delta = [-1, 1] \). What is proved below for random continuous maps on the circle holds with obvious modifications for random differentiable maps on a compact interval that is mapped inside itself by all differentiable maps.

A pathological example occurs if \( f(x; \omega) \) is constant in \( x \); the (unique) stationary measure is then a push forward of the measure on \( \Delta \). To avoid pathologies we assume (in addition to hypotheses (H1) and (H2)) the open and dense condition that

\[(H3)\] The critical points of each map \( x \mapsto f(x; \omega) \) have finite order.

Under this condition the following theorem states that the densities of the stationary measures for random differentiable maps are continuous. We will see in chapter (4) that the regularity of stationary measures for random diffeomorphisms is substantially more than random differentiable maps.

**Theorem 3.1.1.** The random differentiable map \( f \in R^0(S^1) \) possesses a finite number of ergodic stationary measures \( \mu_1, \ldots, \mu_m \) with mutually disjoint supports \( E_1, \ldots, E_m \). All stationary measures are linear combinations of \( \mu_1, \ldots, \mu_m \).

The support \( E_i \) of \( \mu_i \) consists of the closure of a finite number of connected open sets \( C^1_i, \ldots, C^p_i \) that are moved cyclically by \( f(\cdot; \Delta) \). The density \( \phi_i \) of \( \mu_i \) is a \( C^0 \) function on \( S^1 \).

**Proof.** The first statement of the theorem is just a repetition of theorem 2.2.1 in chapter 1.

To prove that the stationary densities are continuous, we will show that \( L \) maps \( L^1(S^1) \) into \( C^0(S^1) \). From this it follows that the density \( \phi_i \) is continuous.

Recall from lemma 2.2.1 the expression of the transfer operator

\[
L\phi(x) = \int_{V_y} k(y,x) \phi(y) d\mathcal{L}(y)
\]

Take \( \phi \in L^1(S^1) \). With \( h \) a small vector in \( \mathbb{R}^n \), consider

\[
L\phi(x + h) - L\phi(x) = \int_{V_x + h} k(y, x + h) \phi(y) d\mathcal{L}(y) - \int_{V_x} k(y, x) \phi(y) d\mathcal{L}(y)
\]

\[
= \int_{V_{x+h} \cap V_x} (k(y, x + h) - k(y, x)) \phi(y) d\mathcal{L}(y)
\]

\[
+ \int_{V_{x+h} \setminus (V_{x+h} \cap V_x)} k(y, x + h) \phi(y) d\mathcal{L}(y)
\]

\[
- \int_{V_x \setminus (V_{x+h} \cap V_x)} k(y, x) \phi(y) d\mathcal{L}(y). \tag{3.1}
\]

The continuity of \( k \) is a consequence of the continuity of the random map \( f \) and the density \( g \) of \( \nu \). This with integrability of \( \phi \) implies that the first term on the right
hand side is small for \( h \) small. The condition on the critical points of \( x \mapsto f(x; \omega) \) implies that \( V_x \) varies continuously with \( x \). This implies that the other two terms are small for \( h \) small. This implies continuity of invariant densities. 

\begin{proof}

\end{proof}

\textbf{Theorem 3.1.2.} Let \( \mu \) be an isolated ergodic stationary measure of \( f \in R^0(S^1) \) with density \( \phi \) with isolating neighborhood \( W \). Then each \( \tilde{f} \in R^0(S^1) \) sufficiently close to \( f \) possesses a unique ergodic stationary measure \( \tilde{\mu} \) with support in \( W \). The density \( \tilde{\phi} \) of \( \tilde{\mu} \) is \( C^0 \) close to \( \phi \).

\begin{proof}

See the more general case in Theorem 5.2.1.
\end{proof}

\textbf{Theorem 3.1.3.} The set of stable random differentiable maps in \( R^1(S^1) \) is open and dense.

\begin{proof}

Take \( f \in R^1(S^1) \). If the entire circle is the support of a stationary measure of \( f \), then \( f \) is stable by Theorem 3.1.2. Suppose that \( \mu \) is a stationary measure whose support \( E \) is a union \( \cup_{i=1}^{k} C_i \) of intervals \( C_i \) mapped cyclically by \( f(\cdot; \Delta) \) \( f(C_i; \Delta) = C_{i+1} \) (the indexes are taken modulo \( k \)). If \( \mu \) is an isolated measure, \( f \) restricted to an isolating neighborhood of \( E \) is stable.

The measure \( \mu \) is certainly isolated if for each boundary point \( x \in E \), either

\[ f^k(x; \Delta^N) \subset \text{interior } E, \]

or

\[ f^k(x; \omega_1, \ldots, \omega_k) = x, \quad f^j(x; \omega_1, \ldots, \omega_j) \in \text{interior } E \]

for some \( \omega_1, \ldots, \omega_k \in \Delta, j < k \). Indeed, invariance of \( E \) shows that in both cases \( f^k(y; \Delta^N) \in E \) for any \( y \) near \( x \).

If not all boundary points are as above, then there is a boundary point \( x \in \partial E \) so that \( f^l(x; \omega_1, \ldots, \omega_l) = x \) for \( l = k \) or \( l = 2k \) (\( l \) minimal) and \( f^j(x; \omega_1, \ldots, \omega_j) \in \partial E \) for \( 0 < j < l \). Write \( x_0 = x \) and \( x_j = f^j(x; \omega_1, \ldots, \omega_j) \) for \( j > 0 \). From \( x_j \in \partial E \), \( x_{j+1} = f(x_j; \omega_{j+1}) \in \partial E \) and \( \frac{\partial}{\partial \omega} f(\cdot; \omega) \neq 0 \), we see that \( \omega_{j+1} \in \partial \Delta \). Thus \( \omega_1, \ldots, \omega_l \) are all contained in \( \partial \Delta \). Note that \( \frac{\partial}{\partial \omega} f^n(x; \omega_1, \ldots, \omega_l) \geq 0 \) since otherwise \( x \) is an interior point of \( E \).

For \( f \in R^1(S^1) \), there are a neighborhood \( U \) of \( f \) and an integer \( N \) so that for each \( \tilde{f} \in U \), the support of the union of its stationary measures has at most \( N \) connected components. A random periodic orbit in the boundary of the support of a stationary measure of \( f \in U \) therefore has its period bounded by \( 2N \).

By transversality techniques a number of arbitrary small perturbations of \( f \) are carried through. The perturbations affect \( f(\cdot; \omega) \) for \( \omega \in \partial \Delta \) and can be extended to other values of \( \omega \) using test functions. We will not present the detailed perturbations, but refer to (29, Section III.2) for a description of the techniques. By a small perturbation of \( f \) we may assume that the graph of each map \( f^i(\cdot; (\partial \Delta)^c) \), \( 1 < i \leq 2N \), intersects the diagonal in \( S^1 \times S^1 \) transversally. That is,

\begin{enumerate}
    \item [(H4)] The periodic orbits of period \( i \leq 2N \) for \( f(\cdot; \partial \Delta) \) are hyperbolic.
\end{enumerate}

There is then a bounded number of random periodic orbits with period bounded by \( 2N \). A further small perturbation ensures that
(H5). Each periodic point \( x \) of period \( i \leq 2N \) is periodic for only one sequence \( \omega_1, \ldots, \omega_i \in \partial \Delta \).

Write \( \mathcal{P} \) for the points in these periodic orbits. Recall that the number of critical points of \( f(\cdot; \partial \Delta) \) is finite. A final small perturbation ensures that

(H6). The critical values of \( f(\cdot; \partial \Delta) \) are disjunct from \( \mathcal{P} \).

Conditions (H4), (H5), (H6) are clearly open and thus describe an open and dense subset of \( \mathcal{R}^3(\mathbb{S}^1) \).

Consider \( f \) from this open and dense set. Let \( \mu \) be a stationary measure of \( f \) with support \( E \). Let \( x \) be a boundary point of \( E \) belonging to a periodic orbit in \( \partial E \). By (H4), \( x \) belongs to a hyperbolic periodic orbit. By (H5), there is a unique graph \( f^j(\cdot, \omega_1, \ldots, \omega_l) \) with \( \omega_1, \ldots, \omega_l \in \partial \Delta \) through \( x \). It is not possible that \( \frac{d}{dx} f^j(x; \omega_1, \ldots, \omega_l) > 1 \), since other orbits would then be repelled and \( x \) would not be in the boundary of \( E \). Hence \( 0 < \frac{d}{dx} f^j(x; \omega_1, \ldots, \omega_l) < 1 \): the random periodic orbit through \( x \) is an attracting periodic orbit for \( f^j(\cdot; \omega_1, \ldots, \omega_l) \). By (H6), there are no interior points in \( E \) being mapped onto \( x \) under iterates of \( f \). As a consequence, \( \mu \) is isolated. Therefore \( f \) is stable. \( \square \)

### 3.2 Bifurcations

As a next step we consider one parameter families of random maps and show that bifurcations typically occur at isolated parameter families. Theorem 3.2.1 below moreover describes the possible codimension one bifurcations. To be able to describe these bifurcations we consider \( R^k(\mathbb{S}^1) \) random maps, with \( k \geq 2 \). Furthermore we assume that the one parameter family is \( C^k \) in the parameter. The space of families of these random maps \( x \mapsto f_a(x; \omega) \), \( a \in I \), will be given the uniform \( C^1 \) topology as maps \( (x, \omega, a) \mapsto f_a(x; \omega) \) on \( \mathbb{S}^1 \times \Delta \times I \). We start with a description of three types of bifurcations caused by violation of one of the conditions (H4), (H5), (H6). These are proved to be the only codimension one bifurcations.

**Definition 3.2.1.** Let \( (x, \omega, a) \mapsto f_\omega(x; \omega) \) a \( R^3(\mathbb{S}^1) \times C^4(I) \) a one parameter family of random maps, with \( k \geq 2 \). We say that the family \( f_\omega \) undergoes a **random saddle node bifurcation** at \( a = a_0 \), if there exists \( \bar{x} \) in the boundary of the support of a stationary measure such that

\[
\begin{align*}
    f^{k}_a(\bar{x}; \omega_1, \ldots, \omega_k) &= \bar{x}, \\
    \left( \frac{d}{dx} \right)^2 f^{k}_a(\bar{x}; \omega_1, \ldots, \omega_k) &= 1
\end{align*}
\]

for some \( \omega_1, \ldots, \omega_k \in \partial \Delta \). The random saddle node bifurcation is said to *unfold generically*, if

\[
\left( \frac{d}{dx} \right)^2 f^{k}_a(\bar{x}; \omega_1, \ldots, \omega_k) \neq 0, \quad \frac{\partial}{\partial a} f^{k}_a(\bar{x}; \omega_1, \ldots, \omega_k) \neq 0
\]

at \( a = a_0 \).

**Definition 3.2.2.** Let \( (x, \omega, a) \mapsto f_\omega(x; \omega) \) a \( R^3(\mathbb{S}^1) \times C^4(I) \) a one parameter family of random maps. We say that the family \( f_\omega \) undergoes a **random homoclinic bifurcation** at \( a = a_0 \), if there exists
Figure 3.1: Consider a random map $f_{a_0}$ for which points are mapped randomly into the region bounded by the two graphs. Depicted on the left are the graphs a random map $f(\cdot; \omega)$, $\omega \in \partial \Delta$, with a random saddle node bifurcation. The support of the stationary density is the interval between the hyperbolic fixed point of the lower map and the saddle node fixed point of the upper map. The right picture shows graphs of a random map with a random homoclinic bifurcation. Here the support of the stationary density stretches from the left hyperbolic fixed point of the lower map to the critical value of the upper map.

- a stationary measure $\mu$ with support $E$ with a hyperbolic periodic point $\bar{x}_a$ in the boundary of $E$ for all $a$ near $a_0$, and

- a critical point $\bar{y}_a$ for $f_a(\cdot; \omega_1)$, $\omega_1 \in \partial \Delta$, in the interior of $E$, such that

$$f'_{a_0}(\bar{y}_{a_0}; \omega_1, \ldots, \omega_l) = \bar{x}_{a_0}$$

for some $\omega_2, \ldots, \omega_l \in \partial \Delta$. The random homoclinic bifurcation unfolds generically if

$$\frac{\partial}{\partial a} \left( f'_{a}(\bar{y}_{a}; \omega_1, \ldots, \omega_l) - \bar{x}_a \right) \neq 0$$

at $a = a_0$.

**Definition 3.2.3.** Let $(x, \omega, a) \mapsto f_a(x; \omega)$ a $R^0(S^1) \times C^1(I)$ a one parameter family of random maps. We say that the family $f_a$ undergoes a random boundary bifurcation at $a = a_0$, if there exists $\bar{x}$ in the boundary of the support of a stationary measure and $(\omega_1, \ldots, \omega_k) \neq (\tilde{\omega}_1, \ldots, \tilde{\omega}_k) \in (\partial \Delta)^k$, such that

$$f'_{a_0}(\bar{x}; \omega_1, \ldots, \omega_k) = \bar{x}, \quad \frac{d}{dx} f'_{a_0}(\bar{x}; \omega_1, \ldots, \omega_k) \in (0, 1)$$

and

$$f'_{a_0}(\bar{x}; \tilde{\omega}_1, \ldots, \tilde{\omega}_k) = \bar{x}, \quad \frac{d}{dx} f'_{a_0}(\bar{x}; \tilde{\omega}_1, \ldots, \tilde{\omega}_k) \in (1, \infty)$$
Figure 3.2: Depicted are parts of the graphs of a random map $f(\cdot; \omega), \omega \in \partial \Delta$. The solid curves lie on two of the graphs of $f(f(\cdot; \omega_1); \omega_2), \omega_1, \omega_2 \in \partial \Delta$, intersecting in a point that lies on two hyperbolic period two orbits (one stable, one unstable) distinguished by different $\omega$ values. A random boundary bifurcation results if this point lies on the boundary of the support of a stationary measure.

Write $\bar{x}_a(\omega_1, \ldots, \omega_k)$ and $\tilde{x}_a(\tilde{\omega}_1, \ldots, \tilde{\omega}_k)$ for the continuations of the hyperbolic periodic points. The random boundary bifurcation is said to unfold generically, if

$$\frac{\partial}{\partial a} f^k_a(\bar{x}_a(\omega_1, \ldots, \omega_k); \omega_1, \ldots, \omega_k) \neq \frac{\partial}{\partial a} f^k_a(\tilde{x}_a(\tilde{\omega}_1, \ldots, \tilde{\omega}_k); \tilde{\omega}_1, \ldots, \tilde{\omega}_k)$$

(3.8)

at $a = a_0$.

For an open interval $I$, write $R^k(I, S^1)$ for the space of $C^k$ families of random maps in $R^k(S^1)$ depending on a parameter in $I$. Equip the space $R^k(I, S^1)$ with the $C^k$ topology.

**Theorem 3.2.1.** For $f_a$ from an open and dense subset of $R^1(I, S^1)$, $f_a$ has only finitely many bifurcations. A bifurcation point is a random saddle node bifurcation, a random homoclinic bifurcation, or a random boundary bifurcation and is generically unfolding. If the number of stationary measures is locally constant at a bifurcation point, the bifurcation is an intermittency bifurcation. Otherwise the bifurcation is a transient bifurcation.

**Proof.** For a bifurcation value for a family in $R^1(I, S^1)$, either (H 4), (H 5), or (H 6) is violated.

Similar transversality arguments as in the proof of Theorem 3.1.3 show the following. For an open and dense subset of $R^1(I, S^1)$, at most one of these conditions is violated at a bifurcation value and the resulting bifurcation unfolds generically as stated in Definition 3.2.1, 3.2.2 or 3.2.3. Since the random bifurcations are unfolding generically, they occur isolated.

$\square$