Bifurcation of random maps
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4 Random attractors for random circle diffeomorphisms

In the previous chapter we discussed random maps from a one-point motion approach. In this context it makes sense to study bifurcations of stationary measures. Strictly ergodic random homeomorphisms, i.e. a random homeomorphisms with a unique ergodic stationary measure with full support (\(\text{supp}(m) = S^1\)), are known to have a unique stationary measure see (32), and the following bifurcation scenario occurs. For \(a\) bigger that a bifurcation value \(a_0\) random homeomorphisms \(f_{a,\omega}\) has a single stationary measure supported on all of the circle. For \(a \leq a_0\), \(f_{a,\omega}\) has a stationary measure supported on \(q\) disjoint intervals that are mapped cyclically under iteration of \(f_{a,\omega}\) or \(f_{a,\omega}\) admits \(q\) disjoint stationary measures with \(q\) different basin of attractions.

In this chapter we consider orientation-preserving random diffeomorphisms. The quantity of interest is the evolution of the normalized mass distribution \(L\), i.e., the random measure valued process \(\mu_k(\omega) = (f_k^k)\mu(\omega)\). It’s well known that the pull-back of the random measure valued process \(\mu_k(\omega) := (f_k^k)\mu(\omega)\) converges almost surely, see lemma 2.3.1, to a specific equilibrium. This chapter is directed in particular to give the structure of the support of this equilibrium. In particular we related the bifurcation of the stationary measure in the Markov context to the bifurcation of this random equilibrium. The previously mentioned explosion of stationary measures typically manifests itself in the following scenario: for \(a < a_0\) there is a unique random attracting periodic orbit of period \(k\), while for \(a > a_0\) there is a single attracting random fixed point.

4.1 Random fixed points and random periodicity

4.1.1 The random family admits a strictly ergodic stationary measure

Let \(f\) be a strictly ergodic random diffeomorphism. In this case the stationary measure \(m\) is equivalent to Lebesgue measure \(L\). Note that this occurs if one of the diffeomorphisms \(f_\omega = f_{a,\omega}\) has irrational rotation number. We call random maps which admit strictly ergodic stationary measure strictly ergodic random maps. Theorem 4.1.1 below gives a precise characterization of \(\mu_\omega\), the statistical equilibrium associated with \(m\). Although we state the result in the context of random diffeomorphisms, differentiability of \(f_\omega\) is not used in the proof; one only needs to assume that \(f_\omega\) are homeomorphisms depending continuously on \(\omega\) and the existence of stationary measures equivalent to Lebesgue for the semigroups generated by the maps \(f_\omega\) and
the maps $f_\omega^{-1}$. See (2; 79; 65; 71) for earlier results with a similar flavor. It is interesting to compare the result with characterizations for groups of homeomorphisms, as in (50; 64; 82).

Skew products with different dynamics in the base have also been frequently studied. See (51), compare also (52), for skew products over horseshoes, generalizing iterated functions systems. For a result on random fixed points in skew products over a hyperbolic torus automorphism, see (97). In the context of circle diffeomorphisms or homeomorphisms with quasiperiodic forcing (giving rise to skew product systems with a minimal system in the base), invariant measures and invariant graphs have been studied by many authors starting from (46), see e.g. (22; 106).

Property 4.1.1. Let $f$ a random diffeomorphism. We have one of the following properties, where the second property is the negation of the first and the third property is a special stronger case of the second property.

1. The random diffeomorphisms are equicontinuous: for each $\varepsilon > 0$, there exists $\delta > 0$ so that for all $\omega \in \Delta^\mathbb{N}$, so that for each interval $I \subset S^1$ with $\mathcal{L}(I) < \delta$, we have $(f_\omega^n)_* \mathcal{L}(I) < \varepsilon$.

2. The random diffeomorphisms are contractive: there exists $\varepsilon_0 > 0, \omega \in \Delta^\mathbb{N}$ and a sequence of intervals $I_n$ with $\mathcal{L}(I_n) \to 0$ so that $(f_\omega^n)_* \mathcal{L}(I_n) \geq \varepsilon_0$.

3. The random diffeomorphisms are strongly contractive: for all $\varepsilon > 0$, there exists $\omega \in \Delta^\mathbb{N}$ and a sequence of intervals $I_n$ with $\mathcal{L}(I_n) \to 0$ so that $(f_\omega^n)_* \mathcal{L}(I_n) \geq 1 - \varepsilon$.

For a proof and more discussion about these properties we refer the reader to (64).

Theorem 4.1.1. Consider a strictly ergodic random circle diffeomorphism. Exactly one of the following possibilities occurs:

1. The random diffeomorphisms are equicontinuous. There exists an absolutely continuous measure that is invariant under each $f_\omega$.

2. The random diffeomorphisms are contractive but not strongly contractive. Then there exists a smooth nontrivial periodic diffeomorphism $\theta, \theta^q = id$, on $S^1$ that commutes with every $f_\omega$. Moreover, for almost all $\omega$, $\mu_\omega$ is a union of $k$ delta-measures of mass $1/q$.

3. The random diffeomorphisms are strongly contractive and $\mu_\omega$ is a delta-measure for almost all $\omega$.

Proof. We begin the proof with the following assertion:

Assume property 4.1.1 (ii) is met. Then there exists $0 < \varepsilon_0 < 1$, so that for all $\varepsilon > 0$ the following holds: for almost all $\omega \in \Delta^2$, there is an interval $I$ with $|I| < \varepsilon$ so that $(f_\omega^j)_* m(I) \geq \varepsilon_0$ for some $j$. 
Take \( \varepsilon > 0 \). We provide \( \delta > 0 \), \( l \in \mathbb{N} \), \( 0 < \varepsilon_0 < 1 \), so that the following holds. For each segment \( \omega = \omega_{-n}, \ldots, \omega_{-1} \) of elements in \( \Delta \) we construct an interval \( I \) with length \( |I| < \varepsilon \) and a subset \( \Sigma \subset \Delta \) for some \( l' \leq l \) with \( \nu^l (\Sigma) > \delta \), so that for each \( (\omega_{-n-l'}, \ldots, \omega_{-n-1}) \in \Sigma \),

\[
\left( f_{\omega_{-n-l'}^{i+l'}} \circ f_{\omega_{-n-l'}^{i+1}} \circ \cdots \circ f_{\omega_{-n-l'}^{i+1}} \right)_* m(I) \geq \varepsilon_0.
\]

(4.1)

The assertion will be shown to follow from this.

Take \( \kappa > 1/\varepsilon \) and an orbit piece \( a_i = f_n^i \), \( 0 \leq i < \kappa \), consisting of \( \kappa \) different points. Take an open interval \( U \) containing \( a_0 \) so that \( U = f_n^i(U) \), \( 0 \leq i < \kappa \), are mutually disjoint. By property 4.1.1(ii) and \( \text{supp} \ (m) = \mathcal{S}^1 \), there exists an interval \( V \) with \( m(V) \geq \varepsilon_0 \) for some positive \( \varepsilon_0 \) and \( (\beta_0, \ldots, \beta_{\kappa-1}) \in \Delta \), so that

\[
f_j^0(V) \subset U.
\]

(4.2)

There is further \( 0 \leq i_* < q \) with

\[
|f_{\omega_{-n-l}}(U_{i_*})| < \varepsilon.
\]

(4.3)

This is clear as the \( \kappa \) disjoint intervals \( f_{\omega_{-n-l}}^i(U_{i_*}) \), \( 0 \leq i < q \), can not all have length \( \geq \varepsilon \) by \( \kappa > 1/\varepsilon \).

Write

\[
I = f_{\omega_{-n-l}}^n(U_{i_*}).
\]

Define \( (\omega_{-n-l'}, \ldots, \omega_{-n-1}) \) as the sequence consisting of \( \eta_0, \ldots, \eta_{\kappa-1} \) followed by \( \beta_0, \ldots, \beta_{\kappa-1} \). It is clear that there are many possibilities to write down this sequences. We choose one expression and we denote \( l \) the maximum length of the segment \( (\omega_{-n-l'}, \ldots, \omega_{-n-1}) \). Then we have \( l' \leq l \) and

\[
\left( f_{\omega_{-n}^{i+l}} \circ f_{\omega_{-n-l'}^{i+1}} \circ \cdots \circ f_{\omega_{-n-l'}^{i+1}} \right)_* m(I) = m\left((f_{\beta_0, \ldots, \beta_{\kappa-1}}^j)^{-1}(U)\right)
\geq m(V) \geq \varepsilon_0
\]

(4.4)

This proves 4.1. Note that \( i_* \) depends on \( \omega_{-n-l'}, \ldots, \omega_{-1} \). By the continuous dependence of \( f^j \) on \( \omega \), there is for each \( i_* \) an open set of sequences \( \omega_{-n-l'}, \ldots, \omega_{-n-1} \) in \( \Delta^{l_*+1} \) for which equation (4.4) holds. Let \( \delta \) be the minimum of the measures of the sets of sequences in \( \Delta^{l_*+1} \) thus constructed, and consider the set

\[
\Omega_{n+N} = \{ \omega \in \Delta^2 \mid \text{for each interval } I \text{ with } |I| < \varepsilon \text{ and each } k \leq N, \left(f_{\omega_{-(n+k)}}^k(I)\right)_* m(I) \leq \varepsilon_0 \}.
\]

It follows from claim (4.1) that \( \mathbb{P}(\Omega_{n+l}) \leq 1 - \delta^l \) and \( \mathbb{P}(\Omega_{n+l'}) \leq (1 - \delta^{l'})^l \). Take the limit as \( l \) goes to infinity, we conclude that \( \mathbb{P}(\Omega_n) = 0 \), where

\[
\Omega_{n+N} = \{ \omega \in \Delta^2 \mid \text{for each interval } I \text{ with } |I| < \varepsilon \text{ and each } k, \left(f_{\omega_{-(n+k)}}^k(I)\right)_* m(I) \leq \varepsilon_0 \}.
\]

This proves the above assertion.

From above we conclude that there exist a subset \( \Omega_0 \) of full probability such that for every \( \omega \in \Omega_0 \), there exist an interval \( I \subset \mathcal{S}^1 \) with \( |I| \leq \varepsilon \) such that \( \mu_\omega(I) \geq \varepsilon_0 \). That is; \( \mu_\omega \) is an atomic measure, i.e., There exist a random number defined on \( \mathcal{S}^1 \), \( a(\omega) \),

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such that \( \mu_\omega(a(\omega)) > 0 \), \( \mathbb{P} \)-almost surely.

Next we show; using the ergodicity of the shift, \( \sigma \), and the skew product, \( S \), that \( \mu_\omega \) is the sum of finitely many Dirac measures with equal weights. We proceed by noting that, for any \( q \in \mathbb{N} \), the set

\[
F_k = \{ \omega \in \Delta^\mathbb{Z} | \ \mu_\omega \text{ has } q \text{ atoms in } S^1 \}
\]

is a measurable \( \sigma \)-invariant set because \((f_\omega)_* \mu_\omega = \mu_{\sigma \omega} \). Ergodicity of \( \sigma \) implies that for each \( k \) we have, \( \mathbb{P} \)-a.s., \( \mathbb{P}(F_k) = 0 \) or \( 1 \). Consequently the number \( k \) of atoms of \( \mu_\omega \) and their masses do not depend on \( \omega \).

Let \( m \) one of these masses. Let \( G_m = \{ (\omega, x) | \ \mu_\omega(x) = m \} \). Again because \((f_\omega)_* \mu_\omega = \mu_{\sigma \omega} \), we verify that \( G_m \) is \( S \)-invariant for any positive \( m \). We have thus \( \mathbb{P}(G_m) = 1 \). This implies that all atoms must have the same mass. The sum of these masses is necessarily equal to one. We can write

\[
\mu_\omega = \frac{1}{q} \sum_{i=1}^{q} \delta_{a_i(\omega)},
\]

where the \( a_i(\omega) : \Delta^\mathbb{Z} \rightarrow \mathbb{S}^1 \), are random variables on \( \mathbb{S}^1 \) with mass \( 1/q \) each (for some \( q \geq 1 \)). We denotes the support of \( \mu_\omega \) by \( A_\omega \) and \( k = \text{Card}(A_\omega) \) the cardinality of \( A_\omega \). The measure \( \mu_\omega \) is called a point measure.

In the following paragraphs we show that \( k > 1 \) occurs only if there is a nontrivial periodic diffeomorphism that commutes with each \( f_\omega \). If \( J \) is a small interval around \( a_i(\omega) \), disjoint from \( a_j(\omega) \) for \( j \neq i \), then from Lemma 2.3.1 one deduces that \( m(f_\omega^{-n}(J)) \) converges to \( 1/k \) as \( n \rightarrow \infty \).

Consider the skew product system \( S_*(\omega, x) = (\sigma^{-1} \omega, (f_\omega)^{-1}(x)) \) on \( \mathbb{Z}^- \times \mathbb{S}^1 \).

Write \( \nu_*^\infty \) for the product measure on \( \Delta^\mathbb{Z}^- \). There is an invariant measure \( \nu_*^\infty \times \mathcal{L} \) for \( S_* \) with \( m_* \) equivalent to Lebesgue measure. As this invariant measure for \( S_* \) is mixing see proposition 5.18 and \( m_* \) is equivalent to Lebesgue measure, \( S_*(\omega, x) \) has a dense forward orbit for \( \nu_*^\infty \times \mathcal{L} \) almost every point \( (\omega, x) \). Then also for almost all fibers \( \omega \times \mathbb{S}^1 \), the forward orbit of all points from these fibers under \( S_* \) is dense. We may therefore assume that this holds for the boundary points of \( f_\omega^{-n}(J) \).

Define \( \theta : \mathbb{S}^1 \rightarrow \mathbb{S}^1 \) by \( \theta(x) = y \) with \( m(x, y) = 1/q \). Smoothness of the density function of \( m \) implies that \( \theta \) is a diffeomorphism. Writing \( J = (c, d) \), if \( c \rightarrow d = f_\omega^{-n}(c) \) converges to \( x \) then \( d \rightarrow d_n = f_\omega^{-n}(d) \) converges to \( \theta(x) \). If \( \theta \) does not commute with some \( f_\omega \), then by continuity \( \theta f_\omega(x) \neq f_\omega \theta(x) \) from some open set \( U \) in \( \Delta^\mathbb{Z} \times \mathbb{S}^1 \). For some \( n \) large, \( (f_\omega^{-n}(x), \sigma^{-n} \omega) \) lies in \( U \). Write \( (c_n, d_n) = f_\omega^{-n}(J) \). Lemma 2.3.1 however implies that \( f_\omega \circ \theta(c_n) = \theta \circ f_\omega(d_n) \), leading to a contradiction.

Note that by dividing out the action of \( \theta \), one obtains a random diffeomorphism that acts strongly contractive.

Finally, Statement (i) is proved in (32, Lemma 5.4). Note that we have the following property, which is the assumption used in (32, Lemma 5.4): for each \( \varepsilon > 0 \), \( \omega \in \Omega \), there exists \( \delta > 0 \) so that for each interval \( I \subset \mathbb{S}^1 \) with \( \mathcal{L}(I) < \delta \), we have \( \mathcal{L}(f_\omega(I)) < \varepsilon \). (Namely, as we have seen under property 4.1.1(ii), we also have the existence of \( \varepsilon_0 > 0 \), \( \omega \in \Delta^\mathbb{Z} \) and a sequence of intervals \( I_n \) with \( \mathcal{L}(I_n) \rightarrow 0 \) so that \( \mathcal{L}(f_\omega(I_n)) \geq \varepsilon_0 \). The above is the negation of this statement.)
Proposition 4.1.1. Under the assumptions of Theorem 4.1.1, if the random diffeomorphism is contractive then there a unique attractor $A_\omega$ consisting of a point in each fiber: for any $\xi \in S^1$ one has
\[ d(f^k_\omega(\xi), A_\sigma \omega) \to 0 \quad \text{as} \ k \to \infty \]
$\mathbb{P}$-almost surely.

Proof. We follow (71). Suppose for simplicity that the random diffeomorphism acts strongly contractive. By the same construction as in the Theorem above a random fixed point measure at $r(\omega)$ for the inverse maps is obtained. Thus for almost all $\omega$ one has for $y, z \neq r(\omega)$,
\[ d(f^r_\omega(y), f^r_\omega(z)) \to 0 \]
as $k \to \infty$. The distribution of the points $r(\omega)$ is absolutely continuous. Also the random fixed points $a(\omega)$ and $r(\omega)$ are independent; $a(\omega)$ depends only on the past of $\omega$ while $r(\omega)$ depends only on the future of $\omega$. Therefore $r(\omega) \neq a(\omega)$ for almost all $\omega$. It follows that for almost all $\xi \in S^1$, $f^k_\omega(\xi)$ converges to the point $a(\sigma^n \omega)$. \hfill \Box

The typical situation for random circle diffeomorphisms with a stationary measure equivalent to Lebesgue measure is thus to possess a unique random attracting and a unique random repelling fixed point. The next lemma emphasizes this property. It illustrates how property 4.1.1(iii) is satisfied.

Lemma 4.1.1. For a generic random circle diffeomorphism $f$ such that the rotation numbers $\rho(f_{-1})$ and $\rho(f_1)$ are different, there exists a map $f^\nu_{i_1,\ldots,\nu_m}$ with precisely one hyperbolic attracting and one hyperbolic repelling fixed point.

Proof. We can take $\nu_1, \nu_2 \in \Omega$ such that $f_{\nu_1}$ has rational rotation number, say $p/q$, and $f_{\nu_2}$ has irrational rotation number. By the genericity assumption, we may assume that $f_{\nu_1}^m$ has a finite number of fixed points, all hyperbolic. In the coordinate in $[0,1)$ on the circle, write $a_1,\ldots,a_m$ for the attracting fixed points, in order of increasing angle on the circle. We take indices $\mod m$, so that $a_{m+1} = a_1$. Write $r_1,\ldots,r_m$ for the repelling fixed points with $r_i \in [a_i, a_{i+1}]$. Generically the distances between neighboring attracting fixed points $a_i, a_{i+1}$ are different for different $i$. We assume this to be the case. Similarly we assume that the distances between repelling fixed points are all different. By relabelling the fixed points we may assume that the minimal distance between $a_i$ and $r_1$ is assumed for $i = 1$. Finally, by (29, Theorem 6.1) we may assume that $f_{\nu_2}$ is a rotation $x \mapsto x + \theta$.

We will construct a map $f_{\nu_1}^{p_1} \circ f_{\nu_2}^{p_2} \circ f_{\nu_1}^{p_3}$ that has $m-1$ attracting fixed points and $m-1$ repelling fixed points. The proposition then follows by induction.

Write $B_i = (r_{i-1}, r_i)$ for the basin of attraction of $a_i$. Take compact intervals $I_i \subset B_i$ with small symmetric difference $I_i \triangle B_i$. For $p_1$ large, $f_{\nu_1}^{p_1}$ is a contraction on all intervals $I_i$ and maps $I_i$ into a small neighborhood of the point $a_i$. As $f_{\nu_2}$ is an irrational rotation, we can take $p_2 \in \mathbb{N}$ such that $f_{\nu_2}^{p_2}(a_1) \subset B_2$ and $f_{\nu_2}^{p_2}(a_1) \subset B_i$ for all other $i$. By taking $p_1$ large enough, the same holds with $a_i$ replaced by $f_{\nu_1}^{p_1}(I_i)$. Write $I_{1,2}$ for the convex hull of $I_1$ and $I_2$ inside the interior of $\overline{B_1} \cup \overline{B_2}$. Take $p_3$ large enough so that $f_{\nu_1}^{p_3}$ is a contraction on $I_{1,2}$. Then $f_{\nu_1}^{p_1} \circ f_{\nu_2}^{p_2} \circ f_{\nu_1}^{p_3}$ has $m-1$ hyperbolic attracting fixed points in $I_{1,2}, I_3,\ldots,I_m$.\hfill \Box
The proof is finished by establishing that the \( m - 1 \) intervals in \( S^1 \setminus \{ I_{1,2} \cup I_3 \cup \cdots \cup I_m \} \) each contain only a hyperbolic repelling fixed point, for \( p_1, p_3 \) large enough. Take compact intervals \( J_i \) inside \((a_i, a_{i+1})\) (the basin of attraction of \( r_i \) for \( f_i^{-q} \)) for \( i \neq 1 \), so that \( J_i \bigtriangleup (a_i, a_{i+1}) \) is small. The intervals \( I_{1,2}, I_3, \ldots, I_m \) and \( J_2, \ldots, J_m \) cover \( S^1 \).

For \( p_3 \) large enough, \( f_i^{-q} \) is a contraction on the intervals \( J_2, \ldots, J_m \) and maps them into small neighborhoods of \( r_2, \ldots, r_m \). By construction, \( f_i^{-q} \circ f_i^{-p_2} \) maps \( J_i \) inside \((a_i, a_{i+1}), i = 2, \ldots, m \). For \( p_1 \) large enough, \( f_i^{-p_1} \circ f_i^{-p_2} \circ f_i^{-q} \) is a contraction on \( J_2, \ldots, J_m \). This finishes the proof.

### 4.1.2 The random family has cyclic intervals

In this subsection we suppose that the random family admits a unique stationary measure supported on a union \( E \) of \( \kappa \) disjoint intervals. In this case, it is well known that the \( k \) intervals are moved cyclically by \( f(., \Delta) \), i.e. \( E \) can be decomposed into finitely many disjoint intervals \( I_i \in E, i \in \{1, 2, \ldots, q\}, \) such that:

\[
p(x, I_i+1_{\text{mod} k}) = \begin{cases} 
1, & x \in I_i; \\
0, & x \in E \setminus I_i.
\end{cases}
\]

This implies that each interval \( I_i \) is invariant under \( f_i^q \). We will first analyze the case when \( q = 1 \), a generalization of the results when \( q > 1 \) is straightforward.

**Theorem 4.1.2.** Suppose the random diffeomorphism \( f \) admits a stationary measure \( m \) which is ergodic and supported on an interval \( I \). Then the limit measure \( \mu_\omega \) is Dirac.

**Corollary 4.1.1.** Suppose the random diffeomorphism \( f \) admits a stationary measure \( m \) which is ergodic and supported on a union \( E \) of \( q \) intervals. Then the limit measure \( \mu_\omega \) is a point measure with support \( \mathbb{P} \)-almost surely finite set with cardinality equal to \( q \).

**Proof of theorem 4.1.2.** We will use the same argument as in (5, Theorem 1.8.4). The argument given there works on continuous-time RDS defined on \( \mathbb{R} \). For discrete-time one dimensional systems similar argument may be used under the condition that the dynamical system is order preserving which is the case in our set up.

Write \( I = (x_-, x_+) \) and let \( a(\omega) \) be the smallest median of \( \mu_\omega \), i.e. the infimum of all point \( x \) for which

\[
\mu_\omega([x, x_+]) \geq \frac{1}{2} \leq \mu_\omega([x_-, x])
\]

Consider \( I_\omega = (x_-, a(\omega)] \) for which \( \mu_\omega(I_\omega) \geq \frac{1}{2} \) by definition. We claim that \( I_\omega \) is invariant set.

Indeed, the order preserving property of the random family implies that \( a(\omega) \) is the median of \( \mu_\omega \) if and only if \( f_\omega(a(\omega)) \) is a median of \( f_\omega \). Since \( \mu_\omega \) is \( f_\omega \)-invariant we have

\[
a(\sigma(\omega)) = f_\omega(a(\omega))
\]

implying that \( f_\omega(I_\omega) \subset I_{\sigma(\omega)} \). Since \( \mu_\omega \) is ergodic, \( \mu_\omega(I_\omega) = 1, \mathbb{P} \)-almost surely.

Using the same argument for \( J_\omega = [a(\omega), x_+] \) we obtain for \( \{a(\omega)\} = I_\omega \cap J_\omega \), and \( \mu_\omega(\{a(\omega)\}) = 1 \). Conclusion: \( \mu_\omega = \delta_{a(\omega)} \mathbb{P} \)-a.s.
Proof of corollary 4.1.1. The argument in Theorem (4.1.2) can be extended to the more general case. Replace \( f_\omega \) by \( f_\omega q \), then for each \( i \in \{1, 2, \ldots, q\} \), \( I_i \) is \( f_\omega q \)-invariant. For each \( i \in \{1, 2, \ldots, q\} \) apply the Theorem (4.1.2) to \( f_\omega i \). Then there exist on each \( I_i \) a random variable \( a : \Omega \to S^1 \) such that \( f_\omega i(a(\omega)) = a(\sigma^i\omega) \). Define for \( i = 1, \ldots, q \), \( a_i(\omega) := f_\omega \sigma^{-i-1}(a(\sigma^{-1}\omega)) \) and \( \nu_\omega = \frac{1}{q} \sum_{i=1}^{q} \delta_{a_i(\omega)} \).

We claim that \( A_\omega = \{a_1(\omega), a_2(\omega), \ldots, a_q(\omega)\} \) is a random invariant set and that \( \nu_\omega \) is an invariant ergodic measure for \( f_\omega \).

Indeed, it’s not difficult to see that \( f_\omega(a_i(\omega)) = a_{i+1}(\sigma\omega) \) for \( i = 1, \ldots, k-1 \) and \( f_\omega(a_q(\omega)) = a_1(\sigma\omega) \).

To see that \( \nu_\omega \) is ergodic, let \( I_\omega \) an invariant interval, i.e. \( f_\omega(I_\omega) = I_\omega \). We have

\[
\nu_\omega(I_\omega) = \frac{1}{q} \sum_{i=1}^{q} \delta_{a_i(\omega)}(I_\omega)
\]

Let

\[
A_i := \{\omega : a_i(\omega) \in I_\omega\}.
\]

Then for \( i = 1, \ldots, q \) we have \( \sigma(A_i) = A_{i+1} \) and \( \sigma^q A_i = A_i \). But \( \sigma^q \) is ergodic (to see this notes that the bernoulli shift is mixing hence \( \sigma^q \) is also mixing en in particular ergodic). Invariance of \( \sigma \) with respect to \( P \) and ergodicity of \( \sigma^q \) imply \( P(A_i) = 0 \) for all \( i \) or \( P(A_i) = 1 \) for all \( i \). This gives \( \nu_\omega(I_\omega) = 0 \) or \( 1 \) and thus \( \nu_\omega \) is ergodic.

Uniqueness and ergodicity of the limit measure gives \( \mu_\omega = \nu_\omega \) and the limit measure is thus a point measure. \( \square \)

Remark 4.1.1.

The cyclic behavior fails if there exist an element \( f_\omega \) with irrational rotation number because in this case it is well known that the action of \( f_\omega \) is strictly ergodic, see (55).

4.2 Random saddle node bifurcation

The material in the previous two sections shows that generically the following picture holds true. A stationary measure supported on \( q \) intervals implies that existence of a unique random attracting periodic orbit of period \( q \). A stationary measure supported on the circle implies the existence of a unique random attracting fixed point. One can expect that in a bifurcation where the support of a stationary measure explodes from \( q \) intervals to the circle, a random periodic orbit of period \( q \) bifurcates to a random fixed point. This picture is confirmed in Theorem 4.3.1 below.

Definition 4.2.1. The smooth one parameter family of random diffeomorphisms \( f_a \) on the circle undergoes a random saddle node bifurcation at \( a = a_0 \), if there exists \( \bar{x} \) in the boundary of the support of a stationary measure such that

\[
f_{a_0,\omega_1,\ldots,\omega_q}^k(\bar{x}) = \bar{x}, \quad \frac{d}{dx} f_{a_0,\omega_1,\ldots,\omega_q}^k(\bar{x}) = 1, \quad (4.6)
\]

for some \( \omega_1, \ldots, \omega_q \in \partial \Omega \).

The random saddle node bifurcation is said to unfold generically, if

\[
\left( \frac{d}{dx} \right)^2 f_{a,\omega_1,\ldots,\omega_q}^q(\bar{x}) \neq 0, \quad \frac{\partial}{\partial a} f_{a,\omega_1,\ldots,\omega_q}^q(\bar{x}) \neq 0 \quad (4.7)
\]
at \( a = a_0 \).

It is shown in (118) that, in the context of families of random circle diffeomorphisms, a (generically unfolding) random saddle node bifurcation is the only possible codimension one bifurcation of stationary densities.

Under the conditions of the above theorem, the density function of the stationary measure \( m_a \) is smooth and depends smoothly on the parameter \( a \) even though the support of \( m_a \) varies discontinuously in the Hausdorff topology, see Chapter5.

4.3 Convergence to a random attractor

In chapter 1 we have given a definition of random attractor. These are objects which are difficult to compute directly from the definition. In general we have the following relation between Markov invariant measures and random attractor:

\[ \text{supp}(\mu_\omega) \subset A_\omega \text{ a.s.}, \]

where \( A_\omega \) is an arbitrary random attractor; see (28).

The main result of this section is proposition 4.3.1 and corollary 4.3.1, which state that the support of the unique random invariant measure, \( A_\omega \), for the random map \( f \) is the minimal attractor.

**Proposition 4.3.1.** Suppose the assumptions of Corollary 4.1.1 hold. Then \( \mathbb{P} \)-almost surely there exists a random set \( R_\omega = \{r_1(\omega), \ldots, r_q(\omega)\} \) such that

\[ d(f^k_\omega(\xi), A_{\sigma^k_\omega}) \to 0 \quad \text{as } k \to \infty \]

for any \( \xi \not\in R_\omega \). Moreover

\[ \forall x \in S^1 \quad \mathbb{P}\{\omega| x \in R_\omega\} = 0. \]

**Corollary 4.3.1.** Suppose the assumptions of Corollary 4.1.1 hold. Then for any \( \xi \in S^1 \) one has

\[ d(f^k_\omega(\xi), A_{\sigma^k_\omega}) \to 0 \quad \text{as } k \to \infty \]

\( \mathbb{P} \)-almost surely.

**Proof of proposition 4.3.1.** Write \( f^{-k}_{\sigma^k_\omega} = (f^k_\omega)^{-1} \). It follows from corollary 4.1.1 that for almost all \( \omega \) there exists a random set \( R_\omega = \{r_1(\omega), \ldots, r_k(\omega)\} \) such that

\[ (f^{-k}_{\sigma^k_\omega})_* \mathcal{L} \to \frac{1}{q} \sum_{i=1}^q \delta_{r_i(\omega)} \quad \text{as } k \to \infty. \quad (4.8) \]

Because the stationary measure for the inverse random diffeomorphisms are absolutely continuous w.r.t. Lebesgue measure, we have \( \mathbb{P}\{\omega| r_i(\omega) = x\} = 0 \) for each \( x \in S^1 \) and \( i \in \{1, \ldots, q\} \), this implies that

\[ \forall x \in S^1; \quad \mathbb{P}\{\omega| x \in R_\omega\} = 0. \]
4.3 Convergence to a random attractor

Hence, for almost all $\omega$, $\mathcal{R}_\omega$ is disjoint from $\mathcal{A}_\omega$. It is further easily seen that between two points of one set there is one point of the other set.

Equation (4.8) gives that the entire circle with the exception of $q$ small intervals end up, under iterates of the inverse random diffeomorphism, in a small neighborhood of $\mathcal{R}$. For almost $\omega$ one has that $\mathcal{A}$ lies outside a small neighborhood of $\mathcal{R}$. Considering the random maps $f$ again, it follows that for almost all $\omega$ one has

$$\lim_{k \to \infty} d(f^k_\omega(\xi), \mathcal{A}_{\sigma^k\omega}) = \lim_{k \to \infty} d(f^{-k}_{\sigma^k\omega}(\xi), \mathcal{A}_{\sigma^k\omega}) = 0$$

for $\xi \notin \mathcal{R}_\omega$.

**Proof of corollary 4.3.1.** For any $\xi \in S^1$, $\xi \notin \mathcal{R}_\omega$ almost surely and hence,

$$d(f^k_\omega(\xi), \mathcal{A}_{\sigma^k\omega}) \to 0 \text{ as } k \to \infty$$

$\mathbb{P}$-almost surely.

We have thus proved that the random invariant set $\mathcal{A}_\omega$ attracts for almost all $\omega$ all elements of the circle. Recall from Chapter 2 that the random set $\mathcal{A}_\omega$ has been called a random periodic attractor.

**Theorem 4.3.1.** Suppose that $f_a$ is a generic family of random diffeomorphisms with a random saddle node bifurcation at $a = a_0$, so that the support $E_a$ of the stationary measure $\mu_a$ has $q$ connected components for $a < a_0$ and equals the circle for $a > a_0$. For each $a \leq a_0$ and a close to $a_0$, $f_a$ has one attracting random cycle of period $q$ and one repelling random cycle of period $q$. For $a > a_0$ and a close to $a_0$, $f_a$ has one attracting and one repelling random fixed point.

**Proof.** For each $a > a_0$ sufficiently close to $a_0$, the random diffeomorphism $f_a$ satisfies Property 4.1.1 (3). Apply Theorem 4.1.1. For $a \leq a_0$, we are in the situation of section (4.1.2). Corollary 4.1.1 and equation (4.5) give the existence of a periodic random cycle. Corollary 4.3.1 shows that this random cycle is attracting. The existence of the repelling random cycle is an immediate consequence taking the inverse random map.

**Remark 4.3.1.**

The random periodic orbit $\mathcal{A}_\omega$ founded in the previous section may be seen, analogously to deterministic systems, as a global attractor (random invariant set that attracts all initial conditions). A local attractor is a subset of the global attractor that only attract a subset of initial conditions, it basin of attraction. The elements $a_i(\omega)$ of the random set $\mathcal{A}_\omega$ are such local attractors, we define the $i^{th}$ ‘random basin of forward attraction’ by

$$J^i_\omega = \{ x \in S^1 \mid d(f^k_\omega(x), f^k_\omega(a_i(\omega))) \to 0 \text{ if } k \to \infty \}.$$

In particular $J^i_\omega$ contains $a_i(\omega)$. It is not difficult to see here again that the $k$ random sets $J^i_{\sigma^k\omega}$ are a random permutation of $f_\omega(J^i_\omega)$. Again from Lemma 2.3.1 we deduce that for almost all $\omega$ we have $m(J^i_\omega) = \frac{1}{q}$, see Theorem 4.1.1.
4.4 Corollaries for the skew products

Consider again the skew product system defined in equation (1.1). For any random set $\mathcal{A}_\omega$ we define $\text{graph}(\mathcal{A}) \subset \Omega \times S^1$ as the subset

$$\mathcal{G} := \{(\omega, x) \mid x \in \mathcal{A}_\omega\}$$

**Remark 4.4.1.**

If $\mathcal{A}_\omega$ is an invariant random set then $\text{graph}(\mathcal{A})$ is not necessarily defined, as $\mathcal{A}_\omega$ is an equivalent class.

In our context, the graph of the $P$–almost surely finite set $\mathcal{A}_\omega = \{a_1(\omega), \ldots, a_k(\omega)\}$ is the union of the of the graphs of the measurable functions $a_i : \Omega \to S^1$.

**Theorem 4.4.1.** (106, Theorem 5). Let $\mathcal{A}_\omega$ be the random attractor in Theorem 4.1.1 and corollary 4.1.1, then

- The graph $\mathcal{G}$ is invariant subset of the space $\Delta^Z \times S^1$ under the skew product $S$.
- The graph $\mathcal{G}$ is the support of an invariant measure $Q$, and $(S, \mathcal{G}, Q)$ is ergodic.

**Remark 4.4.2.**

1. The graph $\mathcal{G}$ has properties that depends on the dynamics $\sigma$ as well as $f_\omega$, $\mathcal{G}$ will typically only be measurable in $\omega$.

2. We have seen in corollary (4.3.1) that the random set $\mathcal{A}_\omega$ is fibre wise "global" attractor. It’s thus justified to call $\mathcal{G}$ attracting in the sense that for $P$-a.e. $\omega \in \Delta^Z$, $\lim_{k \to \infty} d(\pi_2 \mathcal{G}, \pi_2 S^k(\omega, x)) = 0$ for every $x \in S^1$, where $\pi_2$ is the natural projection on the second space.

3. The graph $\mathcal{G}$ is chaotic invariant set because its dynamics under the map $S$ is chaotic. This is a consequence of the expansivity propriety of the Bernoulli shift.

**Lemma 4.4.1.** Suppose $\text{supp } m = S^1$. For almost all $\omega \in \Delta^Z$, the orbit of $(\omega, x_1(\omega))$ under $S$ lies dense in $\Delta^Z \times S^1$.

**Proof.** The statement is a consequence of the following observation: the set of points $(\omega, x) \in \Delta^Z \times S^1$ with dense orbits under the action of the skew product $S$, has full $\nu^\infty \times \mathcal{L}$-measure.

For this, recall first that with one sided time, $S_+$ has a mixing invariant measure $\nu_+^\infty \times m$. Hence, $\nu_+^\infty \times m$ almost every point has a dense forward orbit. Note that $m$ is equivalent to Lebesgue measure, so that $\nu_+^\infty \times \mathcal{L}$ almost every point has a dense forward orbit under $S_+$.

One can likewise consider the one sided time skew product defined by iterating the inverse diffeomorphisms $f_\omega^{-1}$, for which the same statement holds.

Together this implies the observation and thus the lemma. $\square$
We call the set $\mathcal{G}$ a *Strange Chaotic Attractor*. In many sources the terms 'strange' and 'chaotic' are thought to be synonymous when applied to attractors. While it is true to say that most well-known examples of chaotic attractors are also strange, and the two terms are often used interchangeably, the two properties are very different. The term 'chaotic' refer to a dynamical property- that is, a propriety which determined by the behavior of trajectories as time progresses. 'Strange', on the other hand, is a term which refers solely to the geometrical structure of an object, and thus has no time dependence.