Bifurcation of random maps
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In this chapter we prove new results on the regularity (i.e. smoothness) of stationary densities of smooth diffeomorphisms valid in the context of this thesis. We give a description of the dependence of stationary measures on the random diffeomorphisms. This includes describing quantitative characteristics. Regularity results have direct implication on characteristics depending on the stationary measures (Lyapunov exponents, entropy, rotation number) and is widely used in applications, see (43) for an application to population dynamics.

In section 5.1 we establish the smoothness of stationary probability densities with respect to the state variable. Section 5.2 develops results on the stability of stationary densities. Theorem 5.2.2 proves generic stability of random diffeomorphisms. Densities of stable stationary measures are shown to be smooth and depend smoothly on auxiliary parameters, expect in bifurcation values. Section 5.3 treats parameter families of random maps. Section 5.4 develops material on conditionally stationary measures and applies this to compute expected escape times in Section 5.5. Section 5.6 treat the speed of decay of correlations and its dependency on the bifurcation parameter.

5.1 Smoothness of the stationary densities

In this chapter \(\mathcal{M}\) is a smooth \(n\)-dimensional compact Riemannian manifold, and \(\mathbb{T} = \mathbb{N}\). We consider the setup of chapter 1. Let \(D^k(M)\) be the space of \(C^k\) random diffeomorphisms \(f\) on \(M\) (with \(f(\omega;x)\) \(C^k\) jointly in \(x \in M\) and \(\omega \in \Delta\)), depending on a random parameter from \(\Delta\) through a distribution with a \(C^k\) density function \(g\) (differentiability on \(\Delta\) is always understood in the sense of differentiability on an open neighborhood of \(\Delta\)). Note that uniform bounded noise gives \(g = \frac{1}{2}\) in \(C^\infty(\Delta)\).

Finally, we assume that
\[
\omega \mapsto f_\omega(x) \text{ is an injective map for each } x. \tag{5.1}
\]

Let \(f \in D^\infty(M)\), and consider again the transfer operator
\[
L\phi(x) = \int_{V_x} k(y,x)\phi(y)d\mathcal{L}(y) \tag{5.2}
\]

defined in lemma 2.2.1. Denote by
\[
D(M) = \{\phi \in \mathcal{L}^1(M) \mid \phi \geq 0, \int_M \phi(x)d\mathcal{L}(x) = 1\}
\]
the space of probability densities on \( \mathcal{M} \). Remark 2.2.2 shows that \( L \) maps \( \mathcal{D} \) into itself. Smoothness of stationary densities is obtained by showing that \( L \) maps a space of smooth densities into itself.

**Theorem 5.1.1.** The transfer operator \( L \) maps \( C^k(\mathcal{M}) \) into itself and is a compact operator on \( C^k(\mathcal{M}) \).

The number 1 is an eigenvalue of \( L \) with equal algebraic and geometric multiplicity \( m \geq 1 \). The densities \( \phi_1, \ldots, \phi_m \) provide a basis of eigenfunctions with mutually disjoint support. Each eigenfunction \( \phi_i \) is \( C^\infty \) and its support consists of a finite number \( c_i \) of connected components.

**Proof.** Theorem 2.2.1 and the absolute continuity of the distribution of the noise gives the \( m \) invariant densities \( \phi_1, \ldots, \phi_m \). The geometric multiplicity of the eigenvalues 1 is equal to \( m \). Since \( L \) preserves the \( L^1 \) norm, the algebraic and geometric multiplicity of 1 are equal. To see this, suppose on the contrary that there is a nontrivial vector in \( \ker(L-I)^2 \setminus \ker(L-I) \). Elementary linear algebra gives the existence of a sequence of vectors \( \psi_n \) inside \( \ker(L-I)^2 \) converging to an eigenvector \( \phi \), such that \( \lim_{n\to\infty} L^n\psi_n = 2\phi \). Indeed, take \( \psi \in \ker(L-I)^2 \) with \( L\psi = \phi + \psi \) and let \( \psi_n = \frac{1}{n} \psi \). From \( L^n(\psi) = n\phi + \psi \) it follows that \( L^n(\phi + \psi_n) = 2\phi + \psi_n \), which converges to \( 2\phi \) if \( n \to \infty \). This contradicts the preservation of the \( L^1 \) norm by \( L \).

There can be no additional eigenvectors of \( L \) that do not correspond to linear combinations of densities. Namely, suppose \( \phi \) is an eigenvector taking both positive and negative values. Write \( \phi = \phi_+ - \phi_- \) for nonnegative functions \( \phi_+ = \max\{\phi,0\} \), \( \phi_- = \max\{-\phi,0\} \). If \( \phi \) is not a linear combinations of densities, the supports of \( \phi_+, \phi_- \) cannot be invariant. Since \( L \) is positive and preserves the \( L^1 \) norm, \( (L\phi)_+ < L\phi_+ \) so that \( \phi \) cannot be an eigenvector.

We will show that \( L \) maps \( L^1(\mathcal{M}) \) into \( C^0(\mathcal{M}) \) and \( C^i(\mathcal{M}) \) into \( C^{i+1}(\mathcal{M}) \). From this it follows that \( \phi_i \in C^\infty(\mathcal{M}) \). Take \( \psi \in L^1(\mathcal{M}) \). Use \( \psi \) to chart a neighborhood of \( x \in \mathcal{M} \) with an open set in \( \mathbb{R}^n \). With \( h \) a small vector in \( \mathbb{R}^n \), consider

\[
L\psi(x+h) - L\psi(x) = \int_{V_{x+h}} k(y,x)\psi(y)d\mathcal{L}(y) - \int_{V_x} k(y,x)\psi(y)d\mathcal{L}(y)
\]

\[
= \int_{V_{x+h} \setminus V_x} (k(y,x+h) - k(y,x))\psi(y)d\mathcal{L}(y)
+ \int_{V_{x+h} \cap (V_x \setminus V_{x+h})} k(y,x+h)\psi(y)d\mathcal{L}(y)
- \int_{V_x \setminus (V_{x+h} \cap V_x)} k(y,x)\psi(y)d\mathcal{L}(y).
\] (5.3)

The first term on the right hand side is small for \( h \) small by continuity of \( k \) and integrability of \( \psi \). The other two terms are small for \( h \) small by the continuous dependence of \( V_x \) on \( x \). Continuity of \( L\psi \) follows. Suppose next that \( \psi \in C^0(\mathcal{M}) \) and consider \( \frac{1}{|h|}(L\psi(x+h) - L\psi(x)) \) This equals the right hand side of (5.3) divided by \( |h| \). Note that

\[
\lim_{h \to 0} \frac{1}{|h|} \int_{V_{x+h} \cap V_x} (k(y,x+h) - k(y,x))\psi(y)d\mathcal{L}(y) = \int_{V_x} \frac{\partial}{\partial x} k(y,x) \frac{h}{|h|} \psi(y)d\mathcal{L}(y)
\]
is a continuous function of $x$. To check continuity of the remaining two terms it suffices to do a local calculation by covering the boundary of $V_x$ by finitely many balls and using a partition of unity. Without loss of generality we may assume that near a smooth part of the boundary of $V_x$, $V_x$ is bounded from below by the graph of a continuously differentiable function $H_x : [0, 1]^{n-1} \to \mathbb{R}$. We may also assume that $m$ equals Lebesgue measure on $\mathbb{R}^n$. Then

$$
\lim_{h \to 0} \frac{1}{|h|} \int_{V_x \setminus (V_{x+h} \cap V_x)} k(y, x)\psi(y)dy = \lim_{h \to 0} \frac{1}{|h|} \int_{[0,1]^{n-1}} \int_{H_{x}(y_1)}^{H_{x+k}(y_1)} k(y_1, y_2, x)\psi(y)dy_2dy_1
$$

$$
= \int_{[0,1]^{n-1}} \frac{\partial}{\partial x} H_x(y_1) \frac{1}{|h|} k(y_1, H_x(y_1), x)\psi(y_1, H_x(y_1))dy_1
$$

is a continuous function of $x$. The contribution near the finitely many points where $V_x$ is not smooth vanishes in the limit $h \to 0$. Summarizing, $D(L\psi)$ has an expression of the form

$$
D(L\psi)(x) = \int_{V_x} \frac{\partial}{\partial x} k(y, x)\psi(y)d\mathcal{L}(y) + \int_{\partial V_x} n(y, x)k(y, x)\psi(y)dS(y) \quad (5.4)
$$

where $n(y, x)$ measures the change of $\partial V_x$ in the direction of the unit normal vector to $V_x$ and $dS$ is the volume on $\partial V_x$. We remark that the formula is a variant of the transport theorem 7.1.12 and the Gauss theorem 7.2.9 in (1). It follows that $L\psi$ is continuously differentiable if $\psi$ is continuous. Higher order derivatives are computed inductively. This gives that $L\psi \in C^{k+1}(\mathcal{M})$ for $\psi \in C^k(\mathcal{M})$.

We prove compactness on $C^k(\mathcal{M})$ by modifying the argument in (81). Let $B^k(\mathcal{M})$ be the unit sphere in $C^k(\mathcal{M})$. Consider first $k = 0$. By the Arzela-Ascoli theorem, compactness of $L$ on $C^0(\mathcal{M})$ follows from the following two properties (compare (114)),

- for all $x \in \mathcal{M}$, $\{\|L\psi(x)\| \mid \psi \in B^0(\mathcal{M})\}$ is bounded,
- $LF$ is equicontinuous.

For $\psi \in B^0(\mathcal{M})$, $|L\psi(x)| \leq \int_{\mathcal{M}} k(y, x)d\mathcal{L}(y)$. This is a continuous function of $x$ and hence bounded. This proves the first item. The above computations showing that $L\psi$ is continuously differentiable also show that $\|D(L\psi)(x)\|$ is uniformly bounded on $B^0(\mathcal{M})$. This proves that $LF$ is equicontinuous. Compactness in $C^k(\mathcal{M})$ follows similarly by noting that

- for all $x \in \mathcal{M}$, $i \leq k$, $\{\|D^i(L\psi)(x)\| \mid \psi \in B^k(\mathcal{M})\}$ is bounded,
- $\|D^{k+1}(L\psi)(x)\|$ is uniformly bounded on $B^k(\mathcal{M})$.

$$
\square
$$

Remark 5.1.1.

The transfer operator $L$ is compact on the space $L^2(\mathcal{M})$ of quadratic integrable functions on $\mathcal{M}$, see for example (110, Section X.2). The proof of Theorem 5.1.1, demonstrating that the transfer operator increases regularity of functions, implies that the spectrum of $L$ on $L^2(\mathcal{M})$ equals that of $L$ on $C^k(\mathcal{M})$. 
5.2 Stable random diffeomorphisms

The main result of this section is Theorem 5.2.1 on stability of isolated stationary measures and Theorem 5.2.2 establishing generic stability of random diffeomorphisms. Recall that a measure is called isolated if there exists an open set $W$ (an isolating neighborhood) containing the support $E$ of $m$, so that $f(W; \Delta) \subset W$ and $m$ is the only ergodic stationary measure of $f$ with support in $W$.

For the restriction of random maps to an isolating neighborhood $W$ we consider the transfer operator acting on functions vanishing outside $W$ and at the boundary of $W$. Write

\[ C^k_0(W) = \{ f \in C^k(\mathcal{M}) \mid \text{the support of } f \text{ is contained in } \bar{W} \}. \]

Then $L$ acting on $C^k_0(W)$ is well defined. The results in the previous section hold for $L$ acting on $C^k_0(W)$.
5.2 Stable random diffeomorphisms

**Theorem 5.2.1.** Let \( m \) be an isolated ergodic stationary measure of \( f \in D^\infty(M) \) with density \( \phi \) with isolating neighborhood \( W \). Then each \( f \in D^\infty(M) \) sufficiently close to \( f \) possesses a unique ergodic stationary measure \( \tilde{m} \) with support in \( W \). The density \( \tilde{m} \) of \( \tilde{m} \) is \( C^\infty \) close to \( \phi \).

**Proof.** Recall that the closure of \( f(W; \Delta) \) is contained in the isolating neighborhood \( W \). This property extends to random diffeomorphisms sufficiently close to \( f \). Restrict the map \( x \mapsto f(\omega; x) \) to \( W \) and consider the transfer operator \( L \) acting on \( C^k_0(W) \). Then \( L \) has a single eigenvalue 1. Since the spectrum of the transfer operator varies continuously with the random diffeomorphism at \( f \), the transfer operator corresponding to each nearby random diffeomorphism possesses a single eigenvalue 1. The corresponding eigenvector is near \( \phi \). \( \square \)

**Lemma 5.2.1.** Write \( L_f \) for the transfer operator on \( C^k(M) \) for \( f \in D^\infty(M) \). Densities of stationary measures vary continuously with \( f \in D^\infty(M) \) at a random diffeomorphism \( \tilde{f} \) precisely if the multiplicity of the eigenvalue 1 for \( L_f \) is locally constant in \( f \) for \( f \) near \( \tilde{f} \).

**Proof.** Consider \( \tilde{f} \in D^\infty(M) \) with an eigenvalue 1 of multiplicity \( m \). Let \( \tilde{m}_1, \ldots, \tilde{m}_m \) be the ergodic stationary measures with densities \( \tilde{\phi}_1, \ldots, \tilde{\phi}_m \). Write \( F_{\tilde{f}} \) be the direct sum of the lines spanned by \( \tilde{\phi}_1, \ldots, \tilde{\phi}_m \). By Remark 5.1.3, the transfer operator for any \( f \in D^\infty(M) \) sufficiently close to \( \tilde{f} \) possesses a \( m \)-dimensional invariant linear space \( F_f \) that is the continuation of \( F_{\tilde{f}} \).

The spectrum of \( L_f \) restricted to \( F_f \) is in general close to 1. Suppose now that all eigenvalues equal 1. Then \( \psi \in F_f \) implies \( L_f \phi = \psi \). Write \( \phi = \phi^+ - \phi^- \) with \( \phi^+ = \max\{0, \phi\} \) and \( \phi^- = \max\{0, -\phi\} \) the positive and negative parts of \( \phi \). Because \( L_f \) preserves the \( \mathcal{L}^1 \) norm, \( (L_f \phi)^+ < L_f \phi^+ \) precisely if \( \phi^- \) and \( \phi^+ \) are not invariant. Thus \( \phi^+ \) and \( \phi^- \) are necessarily invariant. It follows that invariant densities are obtained by taking positive parts of invariant eigenfunctions. This way \( m \) invariant densities for \( f \) near those of \( \tilde{f} \) can be obtained, proving the lemma. \( \square \)

**Theorem 5.2.2.** The set of stable random diffeomorphisms in \( D^\infty(M) \) contains a countable intersection of open and dense sets.

**Proof.** Consider diffeomorphisms on an open neighborhood \( U \) of a random diffeomorphism \( \tilde{f} \in D^\infty(M) \). Write \( m \) for the multiplicity of the eigenvalue 1 for the transfer operator corresponding to \( \tilde{f} \). There is a neighborhood \( D \) of 1 in the complex plane, so that for \( U \) small enough, each \( f \in U \) has \( m \) eigenvalues counting multiplicity in \( D \). Let \( F \) denote the \( m \) dimensional invariant linear space corresponding to these eigenvalues. Consider the map that assigns to \( f \in U \) the union of the support of all functions in \( F \). By the continuous dependence of \( F \) on \( f \), this is a lower semicontinuous valued mapping and therefore continuous on a set \( B_2 \subset U \) of Baire second category (44).

Consider the map that assigns to random diffeomorphisms \( f \in D^\infty(M) \) the multiplicity \( m(f) \) of the eigenvalue 1 for the corresponding transfer map. By the continuous dependence of eigenvalues of the transfer map on \( f \), the map \( m \) is upper semicontinuous. Since \( m \) takes on finitely many values, it is continuous on an open and dense subset of \( D^\infty(M) \). Indeed, consider \( A_n = \{ f \in D^\infty(M) \mid m(f) < n \} \). The set of
points of continuity of \( m \), in the vicinity of some map in \( D^\infty(M) \), equals the intersection of a finite collection of open and dense sets \( A_n \cup (D^\infty(M) \setminus A_n) \), namely with \( n \) ranging over a finite set of positive integers.

With reference to Lemma 5.2.1, the two above items combined prove the theorem. 

\[ \square \]

### 5.3 Auxiliary parameters

Consider a smooth one parameter family of random diffeomorphisms \( x \mapsto f_a(\omega; x) \) depending on \( a \) from an open interval \( I \) in \( \mathbb{R} \). The probability transition map \( p \) and its density \( k \) depend on \( a \), we write \( p_a \) and \( k_a \). The support \( U_{x,a} \) of \( k_a \) is assumed to vary smoothly with \( x \) and \( a \). The density \( k_a(x,y) \) is a smooth function of \((a, x, y) \in \cup_{\omega} \{a\} \times \omega \times U_{x,a} \) in the sense that it can be extended to a smooth function on an open neighborhood. Let \( L_a \) denote the transfer operator for \( f_a \), given by

\[
L_a \phi(x) = \int_{V_{x,a}} k_a(y, x) \phi(y) d\mathcal{L}(y).
\]  

(5.5)

The domain of integration \( V_{x,a} = \{ y \in M \mid y \in f_a(x; \Delta) \} \) depends smoothly on \((x, a)\).

**Proposition 5.3.1.** For \( r \geq 0 \), the transfer operator \( \phi \mapsto L_a \phi \) as a map from \( C^{k+r}(M) \) into \( C^k(M) \) is a \( C^{r+1} \) map of \( a \) and \( \phi \).

**Proof.** By Proposition 5.1.1, \( L_a \) depends continuously on \( a \). For the derivative of \( L_a \phi \) with respect to \( a \) we find an expression similar to (5.4),

\[
\frac{\partial}{\partial a} L_a \phi(x) = \int_{V_{x,a}} \frac{\partial}{\partial a} k_a(y, x) \phi(y) d\mathcal{L}(y) + \int_{\partial V_{x,a}} s_a(y, x) k_a(y, x) \phi(y) dS(y) \quad (5.6)
\]

for some smooth function \( s_a \). It follows that for \( \phi \in C^k(M) \), \( \frac{\partial}{\partial a} L_a \phi \in C^k(M) \). This implies differentiability of \((\phi, a) \mapsto L_a \phi \) for \( \phi \in C^k(M) \). Higher differentiability is treated similarly. 

The operator \( L_a : C^k(M) \to C^k(M) \) does not depend \( C^2 \) on \( a \), since \( \frac{\partial^2}{\partial a^2} L_a \phi \) may not exist if \( \phi \in C^0(M) \) and \( \frac{\partial^2}{\partial a^2} L_a \phi \) is a \( C^{k-1} \) function if \( \phi \in C^k(M) \). What does hold is that \((x, a) \mapsto L_a \phi_a(x) \) is \( C^{k+1} \) if \((x, a) \mapsto \phi_a(x) \) is \( C^k \).

Consider again a parameter value \( a_0 \) and an ergodic stationary measure \( m_{a_0} \) with support \( E_{a_0} \). The following result extends Theorem 5.2.1, providing an analogous statement in the context of families. If \( m_{a_0} \) is an isolated ergodic stationary measure then there are ergodic invariant measures \( m \) for \( a \) near \( a_0 \) with nearby densities.

**Theorem 5.3.1.** Suppose \( m_{a_0} \) is an isolated ergodic stationary measure. Then the stationary density \( x \mapsto \phi_a(x) \) of \( m \) depends \( C^\infty \) on \((x, a)\).

**Proof.** Let \( W \) be the isolating neighborhood for \( m_{a_0} \). For \( a \) near \( a_0 \), \( f_a(W; \Delta) \) is strictly contained in \( W \) and \( f_a \) has a unique stationary measure with support in \( W \). Restrict \( f_a \) to \( W \) for such values of \( a_0 \).
Consider the transfer operator $L_a$ for $f_a$ acting on $C^k_0(W)$. Write $F$ for the line in $C^k_0(W)$ spanned by $\phi_{a_0}$. Then $C^k_0(W) = F \oplus H^k_0(W)$ with $H^k_0(W)$ consisting of $C^k$ functions with vanishing integral;

$$H^k_0(W) = \{ \phi \in C^k_0(W) \mid \int_M \phi(x) d\mathcal{L}(x) = 0 \}.$$ 

Write $\phi_a$ for the eigenvectors of $L_a$ continuing $\phi_{a_0}$ provided by Theorem 5.2.1. Decompose $\phi_a = \phi_{a_0} + \psi_a$ with $\psi_a \in H^k_0(W)$. Then $\psi_a$ is a solution of $L_a \psi_a = \psi_a + \phi_{a_0} - L_a \phi_{a_0}$. Note that $(x, a) \mapsto \phi_{a_0}(x) - L_a \phi_{a_0}(x)$ is $C^\infty$. The spectrum of $L_{a_0} |_{H^k_0(W)}$ is away from 1. Proposition 5.7.1 below implies the result. 

Remark 5.3.1.

1. The support $E_a$ of $m_a$ can still vary discontinuously in the Hausdorff metric with $a$. The number of components of the support of the stationary measure can also change, while the stationary density varies smoothly.

2. Consider a smooth function $\phi$ with support on an isolating neighborhood $W$ for $m_{a_0}$ and compute averages of $\phi$ along orbits $f^k_a(\omega, x)$. By the Birkhoff ergodic theorem, for typical initial points $x \in W$ and noise sequences $\omega$, the averages lie on a smooth function of $a$ for a near $a_0$.

5.4 Conditionally (almost) stationary measures

To study average escape times from open sets we make use of conditionally stationary measures, which are measures for which on average a fixed percentage of mass escapes under an iterate. We recall the notion of conditionally invariant measure, see (91; 90; 24; 59; 30) for its use in deterministic dynamics. Let a map $f : \mathcal{M} \to \mathcal{M}$ be given and restrict $f$ to a domain $W \subset \mathcal{M}$. Let $V \subset W$ be the set of points in $W$ that are mapped into $W$, points in the complement of $V$ in $W$ are mapped outside $W$. Consider $f : V \to W$. A conditionally invariant measure for $f$ on $W$ is a measure $m$ on $\mathcal{M}$ so that $m(A) = m(f^{-1}(A))/m(f^{-1}(W))$ for Borel sets $A \subset W$.

Definition 5.4.1. Let $f \in D^\infty(\mathcal{M})$. Let $W$ be an open domain in $\mathcal{M}$. A measure $\bar{m}$ on $\bar{W}$ is a conditionally stationary measure if

$$\bar{m}(A) = \frac{\int_W p(x, A) \bar{m}(x)}{\int_W p(x, W) \bar{m}(x)}$$

for Borel sets $A \subset W$.

See (77) where this notion is called a quasistationary measure. Note that a conditionally stationary measure is a stationary measure if $\int_W p(x, W) \bar{m}(x) = 1$, that is, if the support of the conditionally stationary measure lies inside $\bar{W}$.

Lemma 5.4.1. A measure $\bar{m}$ on $\bar{W}$ is a conditionally stationary measure for $f$ if and only if $\mathbb{P} \times \bar{m}$ is a conditionally invariant measure for the skew product $S$ on $\Omega \times \bar{W}$. 

Proof. Write \( \Gamma(x) = \{ \omega \in \Omega \mid f(\omega, x) \in W \} \). Consider \( S : \cup_{x \in W} \{ \{x\} \times \Gamma(x) \} \rightarrow \Omega \times W \), \( S(\omega, x) = (\sigma \omega; f_\omega(x)) \). We must show that the following two statements are equivalent.

(i) \( \mathbb{P} \times \bar{m}(S^{-1}(A)) / \mathbb{P} \times \bar{m}(S^{-1}(\Omega \times W)) = \mathbb{P} \times \bar{m}(A) \) for Borel sets \( A \subset \Omega \times W \).

(ii) \( \int_W \int_{\Delta} 1_V(f(\omega; x))d\bar{m}(x)d\nu(\omega)/ \int_W \int_{\Delta} 1_W(f(\omega; x))d\bar{m}(x)d\nu(\omega) = \int_W 1_V(x)d\bar{m}(x) \) for Borel sets \( U \subset W \).

Take a Borel set \( V \times U \) with \( V \subset \Omega \) and \( U \subset W \) and compute

\[
\mathbb{P} \times \bar{m}(S^{-1}(U \times V)) = \mathbb{P} \times \bar{m}\left( \bigcup_{\omega \in \Delta} f^{-1}(\omega; U) \times \{\omega\} \times V \right)
= \bar{m} \times \nu\left( \bigcup_{\omega \in \Delta} f^{-1}(\omega; U) \times \{\omega\} \right) \mathbb{P}(V)
= \int_W \int_{\Delta} 1_U(f(\omega; x))d\bar{m}(x)d\nu(\omega)\mathbb{P}(V) \quad (5.7)
\]

Further

\[
\bar{m} \times \mathbb{P}(U \times V) = \int_W 1_U(x)d\bar{m}(x)\mathbb{P}(V) \quad (5.8)
\]

Equations (5.7) and (5.8) contain the implication \((i) \Rightarrow (ii)\) when applied for \( \Omega \times U \) for the enumerator and for \( \Omega \times W \) for the denominator.

To show that \((ii) \Rightarrow (i)\), note that (5.7) and (5.8) show that \((i)\) holds for Borel sets \( A \) of the form \( V \times U \) if \((ii)\) is assumed. Therefore it holds for all Borel sets in \( \Omega \times W \).

We continue with the introduction of transfer operators whose fixed points are the densities of conditionally stationary measures. The transfer operator \( \bar{L} \), defined for functions in \( L^1(W) \) with integral 1, is given by

\[
\bar{L}(\phi) = \frac{1}{\bar{W}}L\phi / \int_W L\phi(x)d\mathcal{L}(x).
\]

Write \( \bar{L} \phi = 1_{\bar{W}}L\phi \).

**Proposition 5.4.1.** \( \bar{L} \) maps \( C^0(\bar{W}) \) into itself and is a compact operator on it.

**Proof.** If \( k(x, y) \) denotes the density of the stochastic transition function \( P(x, \cdot) \), then

\[
\bar{L}\phi(x) = \int_{\bar{W} \cap V_x} k(y, x)\phi(y)d\mathcal{L}(y).
\]

Note that \( \bar{W} \cap V_x \) depends continuously on \( x \). Recall from the proof Theorem 5.1.1 that we must show that
• for all \( x \in \overline{W} \), \( \{|\tilde{L}\psi(x)| \mid \psi \in B^0(\overline{W})\} \) is bounded,
• \( \tilde{L}F \) is equicontinuous.

Here \( B^0(\overline{W}) \) is the unit sphere in \( C^0(\overline{W}) \). The first item follows as before: \( |\tilde{L}\psi(x)| \leq \int_{\overline{W}} k(y, x) d\mathcal{L}(y) \) is bounded by a continuous function and thus bounded. For the second item we must show that for each \( \epsilon > 0 \) there is \( \delta > 0 \) so that for all \( x \in \overline{W}, \psi \in B^0(\overline{W}) \), \( |\tilde{L}\psi(x + h) - \tilde{L}\psi(x)| < \epsilon \) if \( |h| < \delta \). Recall, see (5.3),

\[
\tilde{L}\psi(x + h) - \tilde{L}\psi(x) = \int_{W \cap V_{x+h} \cap V_x} (k(y, x + h) - k(y, x)) \psi(y) d\mathcal{L}(y)
+ \int_{W \cap V_{x+h} \setminus (W \cap V_{x+h} \cap V_x)} k(y, x + h) \psi(y) d\mathcal{L}(y)
- \int_{W \cap V_x \setminus (W \cap V_{x+h} \cap V_x)} k(y, x) \psi(y) d\mathcal{L}(y).
\]

Now

\[
|\int_{W \cap V_{x+h} \cap V_x} (k(y, x+h) - k(y, x)) \psi(y) d\mathcal{L}(y)| \leq \int_{W \cap V_{x+h} \cap V_x} |k(y, x+h) - k(y, x)| d\mathcal{L}(y),
\]

which is small for \( |h| \) small by uniform continuity of \( k \). And

\[
|\int_{W \cap V_{x+h} \setminus (W \cap V_{x+h} \cap V_x)} k(y, x + h) \psi(y) d\mathcal{L}(y)|
\leq \int_{W \cap V_{x+h} \setminus (W \cap V_{x+h} \cap V_x)} |k(y, x + h)| d\mathcal{L}(y)
\]

is small for \( |h| \) small by boundedness of \( k \) and uniform continuity of the volume of \( V_x \) in \( x \). Similarly for the third term. This proves equicontinuity. \( \square \)

Recall from Proposition 5.1.1 that \( L \) depends continuously on the random diffeomorphism. The same argument shows that \( \tilde{L} \) depends continuously on the random diffeomorphism.

**Proposition 5.4.2.** The transfer operator \( \tilde{L} \) as a linear map on \( C^0(\mathcal{M}) \) depends continuously on \( f \in D^k(\mathcal{M}) \).

We obtain conditionally stationary measures by a perturbation argument, perturbing from an invariant measure. We do not develop general existence results for conditionally stationary measures, as such general results are not needed for our purposes. Let \( \{f_a\} \) be a family of random diffeomorphisms depending on a real parameter \( a \in \mathcal{I} \). Consider, for \( a_0 \in \mathcal{I} \), a stationary density \( \phi_{a_0} \) with support \( E_{a_0} \). Let \( W \) be a neighborhood of \( E_{a_0} \) disjoint from the supports of possible other stationary densities of \( f_{a_0} \).

**Proposition 5.4.3.** For a close to \( a_0 \), \( f_a \) possesses a conditionally stationary density \( \tilde{\phi}_a \) on \( W \), with \( \tilde{\phi}_{a_0} = \phi_{a_0} \) and \( (x, a) \mapsto \tilde{\phi}_a(x) \) continuous in \((x, a)\). One has

\[
\tilde{L}\tilde{\phi}_a = \alpha(a)\tilde{\phi}_a,
\]

where \( \alpha(a) = \int_W \tilde{L}\tilde{\phi}_a(x) d\mathcal{L}(x) \) depends continuously on \( a \).
Proof. The operator $\tilde{L}_a$ varies continuously with $a$ and therefore possesses a single eigenvalue close to 1 for $a$ close to $a_0$. The function $\tilde{\phi}_a$ is the corresponding eigenfunction.

Recall the definition of the escape time $\chi_a(\omega, x)$ for $x \in W$ and $\omega \in \Omega$:

$$\chi_a(\omega, x) = \min\{k \mid f_a^k(\omega, x) \notin W\}.$$ 

Lemma 5.4.2.

$$\int_{\Omega} \int_{W} \chi_a(\omega, x) d\tilde{\phi}_a(x) dL(x) dP(\omega) = \frac{1}{1 - \alpha(a)}.$$ 

Proof. Let $S_a(\omega, x) = (f_a(\omega; x), \sigma \omega)$. Write

$$\Gamma_n^a = \{(x; \omega) \mid f_a^n(\omega_1, \ldots, \omega_n; x) \in W\}$$

for the set of points in $\Omega \times W$ that remain in $\Omega \times W$ for $n$ iterates of $S_a$. The exit set $E_n^a$ of points that leave $\Omega \times W$ in $n$ iterates equals $\Gamma_n^a \setminus \Gamma_{n-1}^a$. Thus $\chi_a(\omega, x) = n$ on $E_n^a$. Write $\tilde{m}_a$ for the conditionally stationary measure with density $\tilde{\phi}_a$. From (5.7) with $A = \Delta^N \times W$ we get

$$\alpha(a) = \int_W p(x, W) d\tilde{m}_a(x)$$

$$= \int_\Delta \int_W 1_W(f(\omega; x)) d\nu(\omega) d\tilde{m}_a(x)$$

$$= \mathbb{P} \times \tilde{m}_a(S_a^{-1}(\Delta^N \times W)).$$

It follows that $\mathbb{P} \times \tilde{m}_a \times = \alpha^{k-1} - \alpha^k = \alpha^{k-1}(1 - \alpha)$. Calculate

$$\int_{\Omega} \int_{W} \chi_a(\omega, x) dP(\omega) d\tilde{m}_a(x) = \sum_{k=1}^{\infty} \mathbb{P}(E_n^a)$$

$$= \sum_{k=1}^{\infty} k \alpha^{k-1}(1 - \alpha)$$

$$= \frac{1}{1 - \alpha}.$$ 

As a corollary we obtain that the average escape time from $W$ goes to infinity as $a \to a_0$. More precise estimates are derived in the following section.

Proposition 5.4.4. $\int_{\Omega} \int_{W} \chi_a(\omega, x) dP(\omega) dL(x)$ converges to $\infty$ as $a \to a_0$. 

Proof. Let $E_a$ be the interior of the support of $\tilde{\phi}_a$.

$$
\int_{\Omega} \int_W \chi_0(\omega, x)dP(\omega)dL(x) \geq \int_{\Omega} \int_{E_a} \frac{\chi_0(\omega, x)}{\phi_a(x)}dP(\omega)d\tilde{\phi}_a(x)dL(x)
\geq C \int_{\Omega} \int_{E_a} \chi_0(\omega, x)d\tilde{\phi}_a(x)dL(x)
= \frac{C}{1 - \alpha(a)},
$$

for $C = 1/\max\{x \in W \mid \tilde{\phi}_a(x)\}$. For $x$ from the support $E_{a_0}$, the image $f_{a_0}(x; \Delta)$ is contained in $E_{a_0}$ by invariance of $E_{a_0}$. By continuity of $f_a$, $f_a(x; \Delta) \subset W$ for a close enough to $a_0$. Since $\tilde{\phi}_a$ depends continuously on $a$, $1 - \alpha(a) \leq \int_{W \setminus E_{a_0}} \tilde{\phi}_a(x)dL(x)$ converges to 0 as $a \rightarrow a_0$. The proposition follows.

5.5 Escape times

In this section estimates for the average escape time from small neighborhoods of the support of a stationary measure that undergoes a bifurcation are derived, as function of the unfolding parameter.

Let $\{(f_a, g_a)\}, a \in I$, be a smooth one parameter family of random diffeomorphisms on $M$. Suppose that $a_0 \in I$ is a bifurcation value for $\{(f_a, g_a)\}$. Let $m$ be a stationary measure for $f_{a_0}$ involved in a bifurcation. Let $W$ be a small neighborhood of $E$. By Proposition 5.4.3, for $W$ sufficiently close to $E$ and $a$ near $a_0$, $\{(f_a, g_a)\}$ possesses a unique conditionally stationary measure $\bar{m}_a$ with support in $W$. Write $\phi_a$ for the density of $\bar{m}_a$. The transfer operator $L_a$ acting on $C^0(W)$, has $\phi_a$ as a unique fixed point.

Let $X^0$ be the set of points in $\overline{W}$ with $\partial V_{x,a} \cap \partial W \neq \emptyset$ for $x \in X^0$. For $i \geq 0$, define $X^{i+1} = f(X^i; \partial \Delta)$. We suppress the dependence of $X^i$ on $a$ from the notation.

Lemma 5.5.1. If $x_i \in X^i$, then for $x_{i+1} \in f(x_i; \partial \Delta)$ one has $x_i \in \partial V_{x_{i+1},a}$.

Proof. This is clear from the definition.

At $x \in X^0$, the boundary of $V_{x,a} \cap W$ varies continuously but not smoothly with $(x, a)$. It follows that $L_a(\phi)$ cannot be expected to be more than continuous on $X^0$ even for smooth $\phi$.

Lemma 5.5.2. Suppose that $(x, a) \mapsto \phi_a(x)$ is $C^k$ outside $X^0 \cup \cdots \cup X^{k-1}$, such that derivatives up to order $k$ are bounded and their restrictions to a component of $W \setminus (X^0 \cup \cdots \cup X^{k-1})$ extend continuously to the boundary of the component. Then $(x, a) \mapsto L_a \phi_a(x)$ is $C^{k+1}$ outside $X^0 \cup \cdots \cup X^k$. Likewise, derivatives up to order $k + 1$ are bounded and their restrictions to a component of $W \setminus (X^0 \cup \cdots \cup X^k)$ extend continuously to the boundary of the component.

Proof. For $x \not\in X^0$, the derivative of $L_a \phi$ is of the form

$$
D(L_a \phi)(x) = \int_{V_{x,a} \cap W} \frac{\partial}{\partial x} k_a(y, x)\phi(y) dL(y) + \int_{\partial(V_{x,a} \cap W)} n_a(y, x)k_a(y, x)\phi(y) dS(y).
$$

(5.10)
for a smooth function $k_a$ and a piecewise smooth function $n_a$ (smooth outside the intersection of $\partial W$ with $\partial V_{x,a}$). This identity shows that $L_a \phi$ is $C^1$ with bounded derivatives outside $X^0$, for any $\phi \in C^0(W)$. The same holds for $L_a(\phi)$. Higher order derivatives are treated inductively. Similar to (5.10) one has

$$\frac{\partial}{\partial a} L_a \phi(x) = \int_{V_{x,a} \cap W} \frac{\partial}{\partial a} k_a(y,x) \phi(y) d\mathcal{L}(y) + \int_{\partial(V_{x,a} \cap W)} s_a(y,x) k_a(y,x) \phi(y)dS(y)$$

(5.11)

for some piecewise smooth function $s_a$. This shows that $(x,a) \mapsto L_a \phi_a(x)$ is $C^1$ outside $X^0$ for continuous functions $(x,a) \mapsto \phi_a(x)$. The derivatives of $(x,a) \mapsto L_a \phi_a(x)$ on $W \setminus X^0$ are bounded; moreover the derivatives on a component of $W \setminus X^0$ extend continuously to the boundary of the component.

Higher order derivatives are treated inductively. Suppose that $(x,a) \mapsto \phi_a(x)$ is $C^k$ outside $X^0 \cup \cdots \cup X^{k-1}$, such that derivatives up to order $k$ are bounded and their restrictions to a component of $W \setminus (X^0 \cup \cdots \cup X^{k-1})$ extend continuously to the boundary of the component. Then $(x,a) \mapsto L_a \phi_a(x)$ is $C^{k+1}$ outside $X^0 \cup \cdots \cup X^k$. Likewise, derivatives up to order $k + 1$ are bounded and their restrictions to a component of $W \setminus (X^0 \cup \cdots \cup X^k)$ extend continuously to the boundary of the component.

The transfer operator $L_a$ is the composition of the linear map $\hat{L}_a$ and the projection

$$\Pi(\phi) = \phi / \int_W \phi(x) d\mathcal{L}(x).$$

The projection $\Pi$ is a smooth map which is well defined near $\phi_{ao}$ in $C^0(W)$, a direct computation shows

$$D \Pi(\phi) h = \frac{1}{\int_W \phi(x) d\mathcal{L}(x)} h - \frac{\phi}{(\int_W \phi(x) d\mathcal{L}(x))^2} \int_W h(x) d\mathcal{L}(x).$$

Also,

$$\frac{\partial^i}{\partial a^i} \int_W \phi_a(x) d\mathcal{L}(x) = \int_W \frac{\partial^i}{\partial a^i} \phi_a(x) d\mathcal{L}(x).$$

This implies that also $(x,a) \mapsto L_a(\phi_a)(x)$ is $C^{k+1}$ outside $X^0 \cup \cdots \cup X^k$ and has bounded derivatives.

**Proposition 5.5.1.** For each $k \geq 1$, $\bar{\phi}_a$ is $C^k$ outside $X^0 \cup \cdots \cup X^{k-1}$ jointly in $(x,a)$, the derivatives up to order $k$ are uniformly bounded.

**Proof.** Proposition 5.4.3 gives that $\bar{\phi}_a(x)$ is continuous in $(x,a)$. Recall that the support $E$ of $m$ consists of finitely many, say $k$, connected components, permuted cyclically by the random diffeomorphism. An iterate of $f_{an}$ thus maps each component into itself. The restriction of the transfer operator $L^k_{ao}$ to a small neighborhood of $E$ has a single eigenvalue 1 and a remaining spectrum strictly inside the unit circle (49). There is therefore no loss in generality to assume that the support of $m$ consists of a single connected component $E$, which we will assume for the remainder of the proof.

Write $H^k(W) = \{ \psi \in C^k(W) \mid \int_W \psi = 0 \}$. Define the operator $T_a : H^0(W) \to H^0(W)$ by

$$T_a(\psi) = \bar{L}_a(\phi_{ao} + \psi) - \phi_{ao}.$$  

(5.12)
Decompose $\phi_a = \phi_{a_0} + \bar{\psi}_a$, so that $T_a(\bar{\psi}_a) = \bar{\psi}_a$. From the proof of Lemma 5.5.2, we get that $T_a$ a smooth map on $H^0(\mathbb{W})$;

$$DT_a(\psi) = D\Pi(\tilde{L}_a(\phi_{a_0} + \psi))\tilde{L}_a.$$  

However, $\tilde{L}_a$ maps continuously differentiable functions to continuous functions, so that $T_a$ does not define a map from $H^k(\mathbb{W})$, $k \geq 1$, to itself. As a consequence we cannot obtain smoothness properties of $\phi_a$ by applying the implicit function theorem. To get smooth dependence of $\phi_a$ outside sets $X^1$ we reason as follows. We derive equations the derivatives of $\phi_a$ must satisfy, establish that the equations can be solved, and show that the solutions are the derivatives of $\phi_a$. The reasoning follows the lines of the proof of Proposition 5.7.1 in Section 5.7.

To prove that $\bar{\psi}_a$ varies $C^1$ with $a$ in points outside $X^0$, note that $\frac{\partial}{\partial a} \bar{\psi}_a(x)$ should be a solution $M_a(x)$ to

$$\frac{\partial}{\partial a} T_a(\bar{\psi}_a)(x) + DT_a(\bar{\psi}_a)M_a(x) = M_a(x) \quad (5.13)$$

We claim that this equation is uniquely solvable. The spectral radius of $DT_a(0) = L_{a_0}$ is smaller than 1. As a consequence of the continuous dependence of $\tilde{L}_a$ on $a$ (see Proposition 5.4.2), also $DT_a(\bar{\psi}_a)$ varies continuously with $a$. For $a$ sufficiently close to $a_0$, the spectral radius of $DT_a(\bar{\psi}_a)$ is therefore also smaller than 1. Hence

$$(I - DT_a(\bar{\psi}_a))^{-1} = I + \sum_{i=1}^{\infty} (DT_a(\bar{\psi}_a))^i, \quad (5.14)$$

see (68). This formula can be applied for $DT_a(\bar{\psi}_a)$ acting on $L^2$ functions. Indeed, $DT_a$ is compact on $L^2(\mathbb{W}) \subset L^1(\mathbb{W})$, see Remark 5.1.1, and has spectrum strictly inside the unit circle in $\mathbb{C}$. From

$$M_a(x) = (I - DT_a(\bar{\psi}_a))^{-1} \frac{\partial}{\partial a} T_a(\bar{\psi}_a)(x)$$

$$= (I + DT_a(\bar{\psi}_a) + (DT_a(\bar{\psi}_a))^2 + (DT_a(\bar{\psi}_a))^3 + \cdots) \frac{\partial}{\partial a} T_a(\bar{\psi}_a)(x) \quad (5.15)$$

we get that $M_a$ is continuous outside $X^0$ since it equals the sum of $\frac{\partial}{\partial a} T_a(\bar{\psi}_a)$ and a uniform limit of continuous functions (compare Lemma 5.5.2). In particular $M_a$ is uniformly bounded and has continuous extensions to the closure of components of $W\setminus X^0$. We must show that

$$|\bar{\psi}_{a+h}(x) - \bar{\psi}_a(x) - M_a(x)h| = o(|h|)$$

for $x \notin X^0$, as $h \to 0$. Consider $\gamma_a(x) = \bar{\psi}_{a+h}(x) - \bar{\psi}_a(x)$ for $x \notin X^0$. Then

$$\gamma_a(x) = T_{a+h}(\bar{\psi}_a + \gamma_a(x)) - T_a(\bar{\psi}_a)(x)$$

$$= DT_a(\bar{\psi}_a)\gamma_a(x) + \frac{\partial}{\partial a} T_a(\bar{\psi}_a)(x)h + R(x), \quad (5.16)$$

where

$$R(x) = T_{a+h}(\bar{\psi}_a + \gamma_a(x)) - T_a(\bar{\psi}_a)(x) - DT_a(\bar{\psi}_a)\gamma_a(x) - \frac{\partial}{\partial a} T_a(\bar{\psi}_a)(x)h.$$
We claim that for any $\epsilon > 0$ there is $\delta > 0$ so that $|R| < \epsilon(|\gamma_a| + |h|)$, if $|h|$ and $|\gamma_a|$ are smaller than $\delta$. Since $\gamma_a(x)$ is continuous in $h$ we may further restrict $\delta$ in this estimate so that $|R| < \epsilon(|\gamma_a| + |h|)$ holds for $|h|$ smaller than $\delta$. Further, $(I - DT_a(\bar{\psi}_a)) \gamma_a(x) = \frac{\partial}{\partial a}T_a(\bar{\psi}_a)(x)h + R(x)$. Using (5.14) and the bound on $|R|$ gives $|\gamma_a| \leq k|h|$ for some $k$ if $|h| < \delta$. Therefore $|R| < \epsilon(1 + k)|h|$ for some $k > 0$, if $|h| < \delta$. Now (5.13) and (5.16) give
\[
\gamma_a(x) - M_a(x)h = (I - DT_a(\bar{\psi}_a))^{-1}R(x).
\]
Using (5.14) it follows that $|\gamma_a - M_a h| = o(|h|)$, $h \to 0$. This proves that $M_a$ equals the partial derivative $\frac{\partial}{\partial a}\bar{\psi}_a$.

Higher orders of differentiability are proved by induction. Assume that $(x, a) \mapsto \bar{\psi}_a(x)$ has been shown to be $C^k$ outside $X^0 \cup \cdots \cup X^{k-1}$. Recall from Lemma 5.5.2 that for $C^k$ maps $(x, a) \mapsto \bar{\psi}_a(x)$, $\frac{\partial}{\partial a}L_a(\bar{\psi}_a)$ is $C^k$ outside $X^0 \cup \cdots \cup X^k$. The right hand side of (5.15) is therefore $C^k$ outside $X^0 \cup \cdots \cup X^k$. The above reasoning shows that $M_a = \frac{\partial}{\partial a}\bar{\psi}_a$ outside $X^0 \cup \cdots \cup X^k$. Therefore $\frac{\partial}{\partial a}\bar{\psi}_a$ is $C^k$ outside $X^0 \cup \cdots \cup X^k$. Also $D\bar{\psi}_a = D(T_a(\bar{\psi}_a))$ is $C^k$ outside $X^0 \cup \cdots \cup X^k$, so that $(x, a) \mapsto \tilde{\psi}_a(x)$ is $C^{k+1}$ outside $X^0 \cup \cdots \cup X^k$. The same clearly holds for $(x, a) \mapsto \phi_a(x)$.

The following result shows how the average escape time from a neighborhood of the support of a bifurcating stationary measure is more than polynomially large in an unfolding parameter. This makes it difficult to accurately establish the bifurcation parameter value using finite data, even in numerical simulations. It explains the occurrence of very long transients near a transient bifurcation and the very irregular occurrence of bursts in intermittent time series. The proof, in Section 5.5, relies on the construction of conditionally stationary measures in Section 5.4.

**Theorem 5.5.1.** Let $f_a$ be a family of random diffeomorphisms in $D^\infty(M)$, with the parameter $a$ from an open interval $I$. Let $m_a$ be a stationary measure of $f_{a_0}$ for some $a_0 \in I$ and let $W$ be an open neighborhood of the support $E_{a_0}$ of $m_{a_0}$ such that no other stationary measure has support intersecting $\overline{W}$. For each $k > 0$ there is a constant $C_k > 0$ so that
\[
\int_\Omega \int_W \chi_a(\omega, x) d\mathbb{P}(\omega) dm(x) \geq C_k |a - a_0|^{-k}.
\]

**Proof.** We repeat the computation in the proof of Proposition 5.4.4. Let $E_a$ be the interior of the support of $\phi_a$. Applying Lemma 5.4.2,
\[
\int_\Omega \int_W \chi_a(\omega, x) d\mathbb{P}(\omega) d\mathcal{L}(x) \geq \int_{E_a} \int_\Omega \chi_a(\omega, x) d\mathbb{P}(\omega) \tilde{\phi}_a(x) d\mathcal{L}(x)
\]
\[
\geq C \int_{E_a} \int_\Omega \chi_a(\omega, x) \phi_a(x) d\mathcal{L}(x)
\]
\[
= C \frac{1}{1 - \alpha(a)}.
\]
for $C = 1/ \max\{x \in \overline{W} \mid \tilde{\phi}_a(x)\}$. By Proposition 5.5.1, $(x, a) \mapsto \phi_a(x)$ is $C^k$ almost everywhere and has uniformly bounded derivatives. For each integer $k$ there is a constant $C_k$ with $|\phi_a| \leq C_k |a - a_0|^k$ on $\overline{W} \setminus E_{a_0}$. As in the proof of Proposition 5.4.4 we get that for each $k$ there is a constant $C_k > 0$, so that $1 - \alpha(a) \leq C_k |a - a_0|^k$. \[\Box\]
5.6 Decay of correlations

Rates of decay of correlations are intimately related to the speed of convergence to equilibrium which intuitively corresponds to the speed with which the spatial distribution of a typical finite pieces of trajectory approach the stationary distribution. This is relevant for example in numerical computations where one wants to estimate long term behavior by studying relatively short term pieces of trajectory. Rates of decay of correlations also affects so-called extreme statistics which are relevant to all kinds of applications involving the probability and frequency of rare events.

A single random map with an isolated measure supported on a single component has exponential decay of correlations as precised in the following proposition. The interest from our perspective in computing the speed of decay of correlations lies in the study of bifurcations where the support of a stationary measure has several components merging. This will be discussed below. The the sequel we prove Proposition 5.6.2 and Theorem 5.6.1, main result of this section. The reader can consult (111; 12) for background on decay of correlations.

Proposition 5.6.1. Suppose \( m \) is a unique stationary measure. The measures \( \mu, \mu_+ \) are ergodic. If the support of the stationary measure \( m \) is connected, then the stationary measure \( m \) is mixing and the invariant measures \( \mu, \mu_+ \) are mixing.

Proof. We will establish equivalence between the different mixing properties of the stationary measure \( m \) for the Markov process and the measures \( \mu, \mu_+ \) for the skew product systems with one or two sided time. That is, the following statements are equivalent.

(a) \( \lim_{n \to \infty} \int_{\Delta^n} m((f^n_{\omega_1, \ldots, \omega_n})^{-1} (A) \cap B) d\nu(\omega_1) \cdots d\nu(\omega_n) = m(A)m(B) \) for Borel sets \( A, B \subset M \).

(b) \( \lim_{n \to \infty} \mu(S^{-n}(V) \cap W) = \mu_+(V)\mu_+(W) \) for Borel sets \( V, W \subset \Delta^N \times M \).

(c) \( \lim_{n \to \infty} \mu(S^{-n}(V) \cap W) = \mu(V)\mu(W) \) for Borel sets \( V, W \subset \Delta^Z \times M \).

The proposition follows since a stationary measure of a Markov process is ergodic and mixing in case its support is connected (37; 118).

To prove that (a) implies (b), it suffices to consider product sets \( V = A_1 \times B_1, W = A_2 \times B_2 \) (compare (113, Theorem 1.17)). Compute

\[
\begin{align*}
\mu_+(S^{-n}(V)) &= \mu_+(\{(\omega_1, \ldots, \omega_n)\} \times A_1 \times \bigcup_{\omega_1, \ldots, \omega_n \in \Delta} (f^n_{\omega_1, \ldots, \omega_n})^{-1}(B_1)) \\
&= \lim_{n \to \infty} \int_{\Delta^n} m((f^n_{\omega_1, \ldots, \omega_n})^{-1} (A) \cap B) d\nu(\omega_1) \cdots d\nu(\omega_n) = m(A)m(B). 
\end{align*}
\]
Hence
\[ \nu_+^\infty \times m(S^{-n}(V) \cap W) \]
\[ = \nu_+^\infty(\vartheta^{-n}(A_1) \cap A_2) \int_{A^n} m\left(\left(f^\omega_{n_1,\ldots,n_n}^{-1}(B_1) \cap B_2\right) d\nu(\omega_1) \cdots d\nu(\omega_n)\right) \]
\[ = \nu_+^\infty(A_1) \nu_+^\infty(A_2) m(B_1)m(B_2) \]
\[ = \nu_+^\infty \times m(V) \nu_+^\infty \times m(W), \]
as \( n \to \infty \). That (b) implies (a) follows from the above computation with the fact that \( \nu_+^\infty \) is mixing and \( \mu_+ \) is mixing.

For the equivalence of (b) and (c) we note that the skew product \( S \) with the invariant measure \( \mu \) is the natural extension of \( S_+ \) with the invariant measure \( \mu_+ = \nu_+^\infty \times m \) (5, Appendix A). A natural extension inherits ergodicity and mixing properties. Clearly (c) implies (b) as the system with time \( \mathbb{N} \) is a factor of the system with time \( \mathbb{Z} \). To see that (b) implies (c), we need to show
\[ \lim_{n \to \infty} \mu(S^{-n}(A_1) \cap A_2) = \mu(A_1) \mu(A_2), \]
(5.18)
for Borel sets \( A_1, A_2 \) in \( \Delta^2 \times \mathcal{M} \). Write \( O_1, O_2 \) for the coordinate projections of \( A_1, A_2 \) onto \( \Delta^2 \). For \( \varepsilon > 0 \), take two sets \( A'_1, A'_2 \) with \( \mu(A_1, \Delta A'_i) < \varepsilon \), \( i = 1, 2 \), such that the coordinate projections \( O'_1, O'_2 \subset \Delta^2 \) are cylinder sets. Then for some \( n > 0 \), \( \vartheta^{-n}(O'_i) \) defines a cylinder set in \( \Delta^N \). By the mixing property of \( \mu_+ \), (5.18) holds for \( A'_1, A'_2 \).

By approximation it is true for all Borel sets.

The disintegrations \( \mu_\omega \) are called fiber mixing if
\[ \lim_{n \to \infty} \mu_\omega((f^\omega_{n_1,\ldots,n_n}^{-1}(A) \cap B) = \mu_{\sigma^{-n}\omega}(A) \mu_\omega(B), \]
(5.19)
see (18). Even when \( \mu \) is mixing, the \( \mu_\omega \)'s need not be fiber mixing. This follows from Theorem 4.1.1; random circle diffeomorphisms that are equivariant under the action of a cyclic group can be expected to have multiple random attracting fixed points in each fiber.

Remark 5.6.1.
Mixing random dynamical systems enjoy the property that the support of the random limit measure
\[ \mu_\omega = \lim_{k \to \infty} (f^\omega_{n-k}\omega)_* m, \]
(5.20)
is a random attractor while in general we have only the inclusion \( \text{supp}(\mu_\omega) \subset \mathcal{A}(\omega) \) where \( \mathcal{A}(\omega) \) is an arbitrary random attractor, see (75).

Write
\[ U^n \psi(x) = \int_{\Delta^n} \psi \circ f^n(\omega_1, \ldots, \omega_n) d\nu(\omega_1) \cdots d\nu(\omega_n). \]

Proposition 5.6.2. Let \( f \) be a random map with an isolated stationary measure \( m \) with connected support. Let \( W \) be an isolating neighborhood for \( m \). Take \( \varphi, \psi \in \mathcal{L}^2(W) \). Then
\[ |\int_{\mathcal{M}} \varphi(x) U^n \psi(x) dm(x) - \int_{\mathcal{M}} \varphi(x) dm(x) \int_{\mathcal{M}} \psi(x) dm(x)| \leq C \eta^n \]
for some $C > 0$, $0 < \eta < 1$.

Note that the exponential decay of correlations holds for observables $\varphi, \psi \in L^2(W)$, thus including characteristic functions of open sets.

We return to a family $\{f_a\}$, $a \in I$, of random maps. Assume that $f_a$ has an isolated measure $m_a$ for all $a \in I$ with an isolating neighborhood $W$. Suppose $a_0 \in I$ is a bifurcation value for an intermittency bifurcation so that

- the support of $m_a$ consists of $k$ components for $a \leq a_0$,
- the support of $m_a$ consists of a single component for $a > a_0$.

We incorporate the dependence of $U^n$ on $a$ into the notation by writing $U^n_a$.

**Theorem 5.6.1.** Let $f_a$ be as above. Take $\varphi, \psi \in L^2(W)$. There are a constant $C > 0$ and a smooth function $a \mapsto \eta_a$ with $\eta_a = 1$ for $a < a_0$ and $\eta_a < 1$ for $a > a_0$, so that

$$\left| \int_M \varphi(x) U^n_a \psi(x) dm(x) - \int_M \varphi(x) dm(x) \int_M \psi(x) dm_a(x) \right| \leq C \eta_a^n$$

for $a > a_0$.

The smoothness properties of $\eta_a$ imply that $\eta_a - 1$ is a flat function of $a$ at $a = a_0$.

As in the discussion of escape times, that shows how slowly the bifurcation manifests itself in time series when moving the parameter $a$.

We consider a random family $\{f_a\}$ restricted to an isolating neighborhood $W$ of a stationary measure $m_a$, for all values of $a$ form an interval $I$. The transfer operator $L_a$ on $C^k_0(W)$ possesses a single eigenvalue at 1. If the support of $m_a$ consists of $r$ components, $L_a$ has eigenvalues $e^{2\pi i/j}$, $0 \leq j < r$, on the unit circle in the complex plane. These eigenvalues make up the peripheral spectrum of $L_a$, see Remark 5.1.2.

In this section we consider bifurcations in which the number of components of the support of $m_a$ changes. We will see how the rate of decay of correlations varies with the parameter $a$, providing a proof of Theorem 5.6.1.

**Proposition 5.6.3.** Let $f_a$, $a \in I$, be a family of random diffeomorphisms with an isolating neighborhood $W$. The eigenvalues and eigenvectors of the peripheral spectrum of $L_a$ on $C^k_0(W)$ vary smoothly with $a$.

**Proof.** Let $\lambda_a$ be an eigenvalue that depends continuously on $a$ and lies on the unit circle for $a = a_0$. Since $m_{a_0}$ is an isolated stationary measure, $\lambda_{a_0}$ is a simple eigenvalue (see Remark 5.1.2). Proposition 5.7.1 in Section 5.7 implies the result. \qed

Recall (2.8) and Lemma 2.2.1. Write

$$L^n_\alpha \varphi(x) = \int_{\Delta^n} P_{f_a(\omega_1, \ldots, \omega_n; x)} \varphi(x) d\nu(\omega_1) \cdots d\nu(\omega_n),$$

$$U^n_\alpha \psi(x) = \int_{\Delta^n} \psi \circ f_a(\omega_1, \ldots, \omega_n; x) d\nu(\omega_1) \cdots d\nu(\omega_n).$$
As in the computation for Lemma 2.2.1,
\[ \int_M L^n_a \varphi(x) \psi(x) d\mathcal{L}(x) = \int_M \varphi(x) U^n_a \psi(x) d\mathcal{L}(x). \]  
(5.21)

After these preparations we now prove the statements on the speed of decay of correlations. First consider a single random map \( f \).

Proof of Proposition 5.6.2. Let \( \phi \) be the stationary density. Write \( L^n \varphi = (\int_M \varphi(y) d\mathcal{L}(y)) \phi + D^n \varphi \). Compute
\[
\int_M \varphi(x) U^n \psi(x) d\mathcal{L}(x) = \int_M L^n \varphi(x) \psi(x) d\mathcal{L}(x) = \int_M \left[ \left( \int_M \varphi(y) d\mathcal{L}(y) \right) \phi(x) + D^n \varphi(x) \right] \psi(x) d\mathcal{L}(x),
\]
so that
\[
\left| \int_M \varphi(x) U^n \psi(x) d\mathcal{L}(x) - \int_M \varphi(x) \psi(x) d\mathcal{L}(x) \right| \int_M \psi(x) d\mathcal{L}(x) \right| = \left| \int_M D^n \varphi(x) \psi(x) d\mathcal{L}(x) \right|.
\]

Note that the spectral radius of \( R \) is smaller than 1. By continuity of \( R \), there is \( N > 0 \) so that \( \|D^N\| < 1 \) for all \( a \) near \( a_0 \). Hence for \( n \in \mathbb{N} \), \( \|D^n\| = C \eta^n \) for some \( C > 0 \), \( \eta < 1 \). The proposition follows from
\[
\left| \int_M D^n \varphi(x) \psi(x) d\mathcal{L}(x) \right| \leq \|D^n\|_{L^2(M)} \|\psi\|_{L^2(M)} \leq C \eta^n \|\varphi\|_{L^2(M)} \|\psi\|_{L^2(M)}.
\]

Proof of Theorem 5.6.1. This is proved by following the computation in the proof of Proposition 5.6.2 above and noting that \( L_a \) has for \( a > a_0 \) a single eigenvalue 1 and \( k-1 \) eigenvalues that have moved smoothly into the unit circle. Write \( \eta_a \) for the largest radius of the eigenvalues of \( L_a \) that lie inside the unit circle. As a consequence of the smooth dependence of the eigenvalues near the unit circle, see Proposition 5.6.3, \( \eta_a \) is a smooth function of \( a \).

We claim that there exists \( C > 0 \) so that for all \( a \) near \( a_0 \), \( \|D^n\| \leq C \eta_a^n \). For \( a = a_0 \), let \( E \) be the union of the eigenspaces for the eigenvalues in the peripheral spectrum. For \( a \) near \( a_0 \), let \( E_a \) be the continuation of \( E_{a_0} = E \). As \( E_a \) is finite dimensional and has a basis depending smoothly on \( a \), it is clear that there exists \( C > 0 \) so that for \( a > a_0, n \in \mathbb{N} \),
\[
\|D^n_{|E_a}\| \leq C \eta_a^n. \quad (5.22)
\]

Let \( F \) be a subspace of \( L^2(W) \) complementary to \( E \). Write \( P_a \) for the projection to \( E_a \) along \( F \). Then \( R = P_a R + (I - P_a) R \). By continuity of \( R \) and \( P_a \), there is \( N > 0 \) so that \( \|((I - P_a) R)^n\| < 1 \) for all \( a \) near \( a_0 \). Hence for \( n \in \mathbb{N} \),
\[
\|((I - P_a) R)^n\| < C \nu^n \quad (5.23)
\]
Lemma 5.7.1. If \((20, \text{Proposition 3.6.1})\) to show that eigenvectors and eigenvalues of \(L\) are considered complex valued functions. Similarly we can consider \(L\) the solutions.

Putting this into the bottom equation yields a single equation for \(\lambda\) and let \(\psi = \phi_{a0} + \psi_a\) with parameters \(a\).

The top equation can be solved for \(\psi_a\) as a function of \(\lambda\) and \(a\) for \(a\) near \(a_0\) and \(\lambda\) near \(\lambda_{a0}\). In fact, by the Fredholm alternative, \(\psi_a = ((I - P)L_a - \lambda I)^{-1}((I - P)L_a\phi_{a0})\).

Putting this into the bottom equation yields a single equation for \(\lambda\). Note that for stationary measures, \(\lambda = 1\) automatically solves this equation, compare the proof of Theorem 5.3.1 in Section 5.3.

Write the top equation as a fixed point equation \(T_\alpha \psi_a = \psi_a\) with parameters \(\alpha\). The map \(T_\alpha\) is a compact linear map mapping \(F^k(M)\) into \(F^{k+1}(M)\), compare the proof of Theorem 5.1.1.

Lemma 5.7.1. If \((x, \alpha) \mapsto \psi_a(x)\) is \(C^h\), then \((x, \alpha) \mapsto T_\alpha \psi_a(x)\) is \(C^{k+1}\).

Proof. See Section 5.3. \(\square\)

Proposition 5.7.1. Consider the integral equation \(T_\alpha \psi_a = \psi_a\) with \(T_\alpha\) as above. The fixed point \(x \mapsto \psi_a(x)\) is smooth jointly in \(x, \alpha\).
Proof. Given is a unique fixed point \( \psi_\alpha \), depending continuously on \( \alpha \). Formally differentiating \( T_\alpha \psi_\alpha = \psi_\alpha \) with respect to \( \alpha \) gives
\[
\frac{\partial}{\partial \alpha} (T_\alpha \psi_\alpha(x)) = \frac{\partial}{\partial \alpha} T_\alpha \psi_\alpha(x) + T_\alpha \frac{\partial}{\partial \alpha} \psi_\alpha(x) = \frac{\partial}{\partial \alpha} \psi_\alpha(x).
\]
So \( \frac{\partial}{\partial \alpha} \psi_\alpha \) should be the solution \( M_\alpha \) of \( \frac{\partial}{\partial \alpha} T_\alpha \psi_\alpha(x) + T_\alpha M_\alpha(x) = M_\alpha(x) \). That is,
\[
M_\alpha = (I - T_\alpha)^{-1} \frac{\partial}{\partial \alpha} T_\alpha \psi_\alpha.
\] (5.24)

By the Fredholm alternative (73), \( I - T_\alpha \) is invertible on \( F^k(M) \). The right hand side of (5.24) is therefore a continuous function. To establish that \( M_\alpha \) is the desired derivative \( \frac{\partial}{\partial \alpha} \psi_\alpha \), we must show \( |\psi_{\alpha+h}(x) - \psi_\alpha(x) - M_\alpha(x)h| = o(|h|) \) as \( h \to 0 \) (compare the proof of the implicit function theorem in e.g. (17) or (25)). Write \( \gamma_\alpha(x) = \psi_{\alpha+h}(x) - \psi_\alpha(x) \). Now
\[
\gamma_\alpha(x) = T_\alpha \gamma_\alpha(x) - T_\alpha \psi_\alpha(x) + T_\alpha \psi_\alpha(x)h + R(x),
\] (5.25)
where \( R(x) = T_{\alpha+h}(\psi_\alpha + \gamma_\alpha)(x) - T_\alpha \gamma_\alpha(x) - T_\alpha \gamma_\alpha(x) - \frac{\partial}{\partial \alpha} T_\alpha \psi_\alpha(x)h \). Since \( \psi_\alpha \mapsto T_\alpha(\psi) \) is differentiable, for any \( \epsilon > 0 \) there is \( \delta > 0 \) with \( |R| < \epsilon(|\alpha| + |h|) \) if \( |\gamma_\alpha|, |h| < \delta \). Since \( \gamma_\alpha \) is continuous in \( h \), we may further restrict \( \delta \) so that this estimate on \( |R| \) holds for \( |h| < \delta \). From (5.25) we get \( \gamma_\alpha = (I - T_\alpha)^{-1} \left( \frac{\partial}{\partial \alpha} T_\alpha \psi_\alpha h + R \right) \) so that \( |\gamma_\alpha| < C|h| \) for some \( C \), if \( |h| < \delta \). This implies \( |R| < \epsilon(1+C)|h| \) for \( |h| < \delta \). As
\[
(I - T_\alpha)(\gamma_\alpha - M_\alpha h) = R
\]
(from (5.24) we get \( (I - T_\alpha)M_\alpha = \frac{\partial}{\partial \alpha} T_\alpha \psi_\alpha \)), we derive \( |\gamma_\alpha(x) - M_\alpha(x)h| < K\epsilon|h| \) for some \( K > 0 \), if \( |h| < \delta \). This proves that \( M_\alpha \) equals the partial derivative \( \frac{\partial}{\partial \alpha} \psi_\alpha \).

Higher order derivatives are treated by induction. Assume that \( (x,\alpha) \mapsto \psi_\alpha(x) \) has been shown to be \( C^j \). By Lemma 5.7.1, \( (x,\alpha) \mapsto T_\alpha \psi_\alpha(x) \) is \( C^{j+1} \). So \( D \psi_\alpha = D(T_\alpha \psi_\alpha) \) is \( C^j \). www.science.uva.nl/docentensite As \( (I-T_\alpha)^{-1} \) maps \( F^j(M) \) to \( F^j(M) \), the right hand side of (5.24) is a \( C^j \) function. The above reasoning shows that \( M_\alpha = \frac{\partial}{\partial \alpha} \psi_\alpha \). Therefore \( \frac{\partial}{\partial \alpha} \psi_\alpha \) is \( C^j \), so that \( (x,\alpha) \mapsto \psi_\alpha(x) \) is \( C^{j+1} \). \( \square \)