Bifurcation of random maps
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6 Case studies

In this chapter we illustrate the theory in the previous chapters on two examples. Section 6.1 is devoted to a randomized version of standard circle diffeomorphisms. We explain how random saddle node bifurcations occur in this family. We consider rotation numbers and study their dependence on parameters. We give also corollaries for the skew-product of the standard circle map.

In section 6.2 we explain how random saddle node and random homoclinic bifurcations occur in random logistic maps. The reader is referred to (29) for the theory of deterministic circle and interval maps.

6.1 Random circle diffeomorphisms

6.1.1 The standard circle map

The standard circle map acting on \( x \in \mathbb{R}/\mathbb{Z} \) and depending on parameters \( a, \kappa \) is given by

\[
f_a(x) = x + a + \frac{\kappa}{2\pi} \sin(2\pi x) \mod 1.
\]

Consider \( f_a \) for a fixed value of \( \kappa \in (0, 1) \) for which \( f_a \) is a diffeomorphism. Introduce the lift \( F_a : \mathbb{R} \mapsto \mathbb{R} \),

\[
F_a(x) = x + a + \frac{\kappa}{2\pi} \sin(2\pi x).
\]

It is well known that the rotation number \( \rho_a \) of \( f_a \),

\[
\rho_a = \lim_{k \to \infty} \frac{F_a^k(x) - x}{k}, \tag{6.1}
\]

is well defined and independent of \( x \). The rotation number depends continuously on \( a \). The rotation number is rational precisely if \( f_a \) possesses periodic orbits. For a fixed rational number \( r \), the rotation number of \( f_a \) equals \( r \) for an interval of \( a \) values. In the interior of such an interval, \( f_a \) has exactly one hyperbolic periodic attractor and one hyperbolic periodic repeller, see (85).

In the following we consider standard circle diffeomorphisms with a random parameter:

\[
f_a(\omega; x) = x + \frac{\kappa}{2\pi} \sin(2\pi x) + a + \varepsilon \omega \tag{6.2}
\]

for \( x \in \mathbb{R}/\mathbb{Z} \) and a random parameter \( \omega \) chosen from a uniform distribution on \( \Delta = [-1, 1] \). The value of \( \varepsilon \) determines the amplitude of the noise, we assume it has a
fixed value. We consider fixed \( \kappa \in (0, 1) \) for which \( x \mapsto f_a(x; \omega) \) is a diffeomorphism. Write
\[
F_a(x; \omega) = x + a + \varepsilon \omega + \frac{\kappa}{2\pi} \sin(2\pi x)
\] (6.3)
for the lift of \( f_a(x; \omega) \). Note that \( F_a(x; \omega) - x \) is periodic in \( x \) with period one.

**Proposition 6.1.1.** For each parameter value \( a \), the random standard circle family \( f_a \) has a unique stationary measure \( \mu_a \). The density \( \phi_a \) of \( \mu_a \) is smooth and depends smoothly on \( a \). The support of \( \mu_a \) is either the entire circle or finitely many intervals strictly contained in the circle. The latter possibility is only possible if \( \rho_b \) is rational for each \( b \in [a - \sigma, a + \sigma] \). Bifurcations where the support of \( \mu_a \) changes discontinuously, are generic saddle node bifurcations. There are finitely many such bifurcations.

**Remark 6.1.1.**
Observe that \( f_a \) has a hyperbolic fixed point for \( a \in (-\frac{\kappa}{2\pi}, \frac{\kappa}{2\pi}) \). Hence, \( f_a \) has a stationary measure supported on a single interval precisely if \( |a| < \frac{\kappa}{2\pi} - \varepsilon \). This occurs for a nonempty interval of \( a \) values if \( \varepsilon < \frac{\kappa}{2\pi} \).

**Remark 6.1.2.**
According to Theorem 4.3.1 and the proposition above, the random standard circle map undergoes finitely many bifurcation from a one attracting and one repelling random fixed point to respectively one random attracting cycle and one repelling cycle, see Figure 6.1.

![Bifurcation diagram](image)

**Figure 6.1:** Numerically computed random attractor. For varying parameter values, always 100 equidistributed points are iterated 100,000 times under the random diffeomorphism. The end points are plotted and connected through a line. The 100 points always end up close to each other (there is no line because there no points to connect) except for \( a \) from an interval near \( a = 0.5 \), where the stationary density has two components,

**Proof of proposition 6.1.1.** It is well known that a circle diffeomorphism with irrational rotation number has its orbits lying dense in \( \mathbb{R}/\mathbb{Z} \). It follows that if the family
of circle maps \( f_a(\omega; x) \) for varying \( \omega \in \Delta \) contains a member with irrational rotation number, there is a strictly ergodic stationary measure.

Suppose now that \( f_a(\omega; x) \) for each \( \omega \in \Delta \) has rational rotation number \( \rho_a + \sigma = \frac{p}{q} \). Write \( x_\omega \) for a periodic point from a periodic attractor of \( x \mapsto f_a(\omega; x) \) depending continuously on \( \omega \). Recall that \( x \mapsto f_a(\omega; x) \) has a unique periodic attractor. Let \( V_a = \bigcup_{\omega \in \Delta} x_\omega = [x_{-1}, x_1] \). The random standard family is increasing in \( x \) and in \( \omega \), so that for all \( x \in V_a \), and all \( \omega \in \Delta \) we have

\[
x_{-1} = F_q^a(x_{-1}; -1) \leq F_q^a(x_{-1}; \omega) \leq F_q^a(x_1; \omega) \leq F_q^a(x_{-1}; 1) = x_1.
\]

It follows that the orbit of \( V_a \) is invariant. For a fixed \( \omega \in \Delta \), all points outside the unique periodic repeller of \( f_a(\cdot; \omega) \) are attracted to its periodic attractor. This implies that there is a unique stationary measure supported on the orbit of \( V_a \).

Compute

\[
\frac{\partial}{\partial a} f_k^a(\omega; x) = \sum_{i=0}^k \frac{\partial}{\partial a} f_i(\omega; x) \frac{d}{dx} f_a^{k-i}(f_k^{i-1}(\omega; x)).
\]

As all terms in the sum are positive, a random saddle node bifurcation occurs isolated. The random family \( \{f_a\} \) therefore has only a finite number of random saddle node bifurcations.

![Figure 6.2: Numerically computed stationary densities of the random standard circle map. On the left for \(|a| + \sigma < \frac{\varepsilon}{2\pi}\), on the right for \(|a| + \sigma > \frac{\varepsilon}{2\pi}\). The explosion of the support of the stationary density follows a random saddle node bifurcation.](image)

We define the rotation number for the random standard circle map, when its exist, by

\[
\rho_a(\omega; x) = \lim_{k \to \infty} \frac{F_k^a(\omega; x) - x}{k}.
\]

The rotation number measures the average rotation per iterate of \( f_a \). Note that \( \rho_a \) is a random variable, depending also on the starting point \( x \).
A simple but useful lemma shows that \( \rho_a \) is independent of the initial condition \( x \).

**Lemma 6.1.1.** If \( \rho_a(x; \omega) \) exists for some \( x \in \mathbb{R}/\mathbb{Z}, \omega \in \Omega \), then \( x \mapsto \rho_a(\omega; x) \) exists for all \( x \in \mathbb{R}/\mathbb{Z} \) and is constant in \( x \).

**Proof.** Observe that \( F_k^a(\cdot; \omega) \) is a lift of \( f_k^a(\cdot; \omega) \), so that \( F_k^a(x; \omega) - x \) is periodic in \( x \) with period 1. Thus

\[
\max_{x \in \mathbb{R}} \{F_k^a(x; \omega) - x\} - \min_{x \in \mathbb{R}} \{F_k^a(\omega; x) - x\} < 1.
\]

Compute

\[
|F_k^a(\omega; x) - F_k^a(\omega; y)| \leq |(F_k^a(\omega; x) - x) - (F_k^a(\omega; y) - y)| + |x - y| \leq 1 + |x - y|,
\]

so that

\[
\lim_{k \to \infty} \left( \frac{F_k^a(x; \omega) - x}{k} - \frac{F_k^a(y; \omega) - y}{k} \right) = 0.
\]

It follows that the limit \( \lim_{k \to \infty} \frac{F_k^a(\omega; x) - x}{k} \), if it exists, is independent of \( x \).

Write

\[
F_a(\omega; x) = x + \delta_a(\omega; x),
\]

where the function \( \delta_a(\omega; x) \) is periodic with period one in the variable \( x \). We can consider \( \delta \) as a function defined on \( \mathbb{R}/\mathbb{Z} \). A simple induction argument gives for each \( k \in \mathbb{N} \),

\[
f_k^a(\omega; x) = x + \sum_{i=0}^{k-1} \delta \circ S^i(\omega; x)
\]

(6.5)

where \( S \) is the skew product system (see equation (1.1)) on \( \mathbb{R}/\mathbb{Z} \times \Omega \). Recall that \( \mu_a \times \nu^\infty \) is an \( S \)-invariant measure.

Rotation numbers for random circle maps are treated by Ruffino (98); this work includes a proof of the existence of rotation numbers, also part of the following proposition.

**Proposition 6.1.2.**

\[
\rho_a = \int_{\mathbb{R}/\mathbb{Z}} \mathbb{E}(\delta(x; \omega))d\mu_a(s) \quad \mathbb{P} - a.s.
\]

(6.6)

where \( \mathbb{E} \) is the expectation operator. The right hand side of (6.6) is independent of \( x \) and is a smooth and nondecreasing function of \( a \).

**Proof.** Consider the Birkhoff sum in equation (6.5)

\[
\frac{f_k^a(x; \omega) - x}{k} = \frac{1}{k} \sum_{i=0}^{k-1} \delta_a \circ S^i(s; \omega).
\]
6.1 Random circle diffeomorphisms

Figure 6.3: The function $a \mapsto \rho_a$. On the left the devil's staircase; the rotation number of the deterministic standard family. On the right the rotation number of the random standard family.

By Birkhoff’s ergodic theorem,

$$
\lim_{k \to \infty} \frac{f^k_a(x; \omega) - x}{k} = \int_{\mathbb{R}/\mathbb{Z}} \int_{\Omega} \beta_a(s; \omega) \mu_a(ds) \, d\mu(\omega) \quad \mu_a \times \mathbb{P} - \text{a.s.}
$$

The fact that the rotation number when it exist is independent of the initial point, see (6.1.1), implies that this equality holds for all $x$ and $\mathbb{P} - \text{a.s.}$.

Write $a \mapsto h(a)$ for the right hand side of (6.6). Smoothness of $h$ follows from smoothness of the stationary density $\phi_a$ and (6.7). Write the standard family as $R_a \circ f(x; \omega)$ where $f(x; \omega)$ is the random map $f(x; \omega) = x + \frac{\epsilon}{2\pi} \sin(2\pi x) + \sigma \xi(\omega)$ and $R_a$ is the translation with coefficient $a$. Then for $a_1 < a_2$ and $k \geq 1$, $(R_{a_1} \circ f(\cdot; \omega))^k < (R_{a_2} \circ f(\cdot; \omega))^k$ and thus,

$$
\rho_{a_1} = \lim_{k \to \infty} \frac{(R_{a_1} \circ f(\cdot; \omega))^k - Id}{k} \leq \lim_{k \to \infty} \frac{(R_{a_2} \circ f(\cdot; \omega))^k - Id}{k} = \rho_{a_2}.
$$

6.1.2 The skew product system

The random circle diffeomorphisms in the previous sections induces a skew product system given by equation (1.1).

$$
S(\omega; x) = \left( \sigma \omega; x + \frac{\epsilon}{2\pi} \sin(2\pi x) + a + \epsilon \omega_0 \right)
$$

Similar to the unforced standard circle map there exist so-called Arnold tongues-regions in the parameter space on which the rotation number stays constant, see
Figure (6.3). The reason of this is usually the existence of an attracting invariant graph inside the tongue. On the boundaries of the tongue this attracting graph collides with an repelling (unstable) invariant graph in a saddle node bifurcation as defined before. For our purpose it is convenient to study only those bifurcations which take place on the boundary of the tongue with rotation number zero, that is when $|a| < \frac{\pi}{\omega} - \varepsilon$. In order to do so we fix $\kappa \in [0,1]$ and $\varepsilon > 0$, thus obtaining a one parameter family depending on $a$. As long as $\varepsilon$ is not too large, there exist an attracting and a repelling invariant t curve at $a = 0$. Increasing or decreasing $a$ leads to the disappearance of the two curves after collision in a saddle-node bifurcation.

**Proposition 6.1.3.** Suppose that $T = \mathbb{Z}$, i.e. the shift operator in equation (6.8) is bijective. For $|a| < \frac{\pi}{\omega} - \varepsilon$, the graph $G$ of the random point attractor $x^* : \Omega \rightarrow V_a$ is $S$-invariant, attracting and continuous for all $(\omega, x) \in \Omega \times V_a$.

**Proof.** Let $V_a = \text{supp}(m)$ the support for the unique stationary measure $(V_a$ is connected because $|a| < \frac{\pi}{\omega} - \varepsilon$). Recall from proposition 6.1.1 that $V_a = \bigcup_{\omega \in \Delta} x_{\omega} = [x_{-1}, x_i]$. On this interval, the random circle map stratifies the following contraction inequality

$$\left| \frac{\partial f^n(\omega; x)}{\partial x} \right| < c\delta^n$$

(6.9)

for all $\omega \in \Omega$ and $x \in V_a$, and for some $0 < \delta < 1, c > 0$. This because the $V_a = \bigcup_{\omega \in \Delta} x_{\omega}$ and $x_\omega$ is an hyperbolic attracting fixed point for $f_\omega$ for each $\omega$. Then we reproduce a classical result of Hirsh and Pugh (1970); Hirsh, Pugh and Shub (1977) using the graph transform approach.

Let $C^0$ the space of continuous functions $x : \Omega \rightarrow V_a$. We define the metric of uniform convergence $d_u$, on $C^0$ by

$$d(x, x') = \sup_{\omega \in \Omega} d_u(x(\omega), x'(\omega))$$

making $C^0$ a complete metric space. The graph transform is a function $\Phi : C^0 \rightarrow C^0$ and is defined by

$$\Phi(x)(\omega) = S(\sigma^{-1}(\omega); x(\sigma^{-1}(\omega)))$$

By induction

$$\Phi^n(x)(\omega) = S^n(\sigma^{-n}(\omega); x(\sigma^{-n}(\omega)))$$

Now by equation (6.9),

$$d(\Phi^n(x)(\omega), \Phi^n(x')(\omega)) = d(S^n(\sigma^{-n}(\omega); x(\sigma^{-n}(\omega))), S^n(\sigma^{-n}(\omega), x'(\sigma^{-n}(\omega))))$$

$$\leq c\delta^n d(x(\sigma^{-n}(\omega)), x'(\sigma^{-n}(\omega)))$$

$$\leq c\delta^n d(x, x')$$

for any $x, x' \in C^0$. This means that $\Phi$ is contracting, and therefore has a unique attracting fixed point $x^*$ by the contraction mapping principle. Since the graph transform takes functions in $C^0$ as its arguments, this fixed point is a continuous function. Now by definition

$$x^*(\sigma(\omega)) = \Phi(x^*)(\sigma(\omega)) = S((\sigma^{-1}\sigma(\omega)); x^*(\sigma^{-1}\sigma(\omega))) = S(\omega; x^*(\omega))$$
6.2 Random unimodal maps

and so the graph of $x^*$ is invariant under $S$. Finally, by induction, $x^*(\sigma^n(\omega)) = S^n(\omega; x^*)$ and hence

$$d(x^*(\sigma^n(\omega)), S^n(\omega; x)) \leq c\delta^n d(x^*, x)$$

and so the graph of $x^*$ is attracting under $S$.

Remark 6.1.3.

1. We have already shown, in more than one occasion, the invariance and the attracting property of the random fixe point. Only the continuity is a new result in the above proposition.

2. If the stationary measure has a full support ($\text{supp}(m) = S^1$), the graph cannot be continuous. The graph is not defined for all $\omega \in \Omega$, see also Lemma 4.4.1.

According to the bifurcation scenario described in the beginning of this section, we expect the existence of a continuous invariant attracting graph inside the Arnold tongues. This graph has support $\Omega \times \text{supp}(m)$. At bifurcation the support of the graph explode to fill densely the space $\Omega \times S^1$. This happen exactly when the support of the stationary measure explodes to become all of the circle.

To illustrate this situation numerically, we replace the infinite dimensional space $\Omega$ with the torus and the shift with the Arnold cat map. This choice is purely for numerical purposes and shouldn’t be seen as a legitime substitution, although there exist some similarities between the two skew products systems. All two dynamical systems are systems which are forced by a chaotic map in the base. Skew-products with chaotic forcing in the base have drawn more attention last years, see (97; 112; 21).

The skew-product in equation (6.8) become

$$S : T^2 \times \mathbb{R}/\mathbb{Z} \rightarrow T^2 \times \mathbb{R}/\mathbb{Z}$$

$$S(t; x) = \left( At; x + \frac{\kappa}{2\pi} \sin(2\pi x) + a + \epsilon \sin(2\pi(t_1 + t_2)) \right)$$

Where $A$ is the automorphism of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

The following figures show the attracting invariant graphs for different values of the bifurcations parameter $a$. The figures show a remarkable difference in the geometry of these graphs according to proposition 6.9 and Lemma 4.4.1. Note also that the dynamics of $S$ is chaotic on the attracting curve in all cases. We have plotted the attracting curve for $t_2 = C$ for some constant $C$.

6.2 Random unimodal maps

This section is devoted to the investigation of the randomized version of the logistic family

$$f_a(x; \omega) = (a + \sigma \omega)x(1-x).$$

(6.10)
Figure 6.4: Pictures obtained from equation (6.8) with \( \kappa = 0.99 \) and \( \varepsilon = 0.05 \). (a) Upper left figure shows attracting invariant curve for \( a = 0 \), i.e. \( |a| < \frac{\kappa}{\pi} - \varepsilon \) and we are inside the Arnold tongue with rotation number equals to zero. The invariant graph is clearly continuous. (b) Upper right figure shows attracting invariant curve for \( a = 0.15 \), i.e. \( |a| > \frac{\kappa}{\pi} - \varepsilon \) and the attracting graph is dense. (c) Lower left figure shows attracting invariant curve for \( a = 0.5 \), i.e parameter values corresponding to the Arnold tongue with rotation number \( \frac{1}{2} \). (d). Lower right figure shows attracting invariant curve for \( a = 0.54 \), i.e we are outside the Arnold tongue and the invariant curve is dense in the state space.

on \([0,1]\). The random parameter \( \omega \) be chosen from a uniform distribution on \( \Delta = [-1,1] \). Note that \( f_a \) is not a random diffeomorphism and also that \( x = 0,1 \) are fixed points for all values of \( \omega \). Throughout this section we will assume that

\[
a + \sigma \omega \in (1, 4),
\]

for all \( \omega \in \Delta \). As a consequence, the interval \([0,1]\) is mapped into itself by each map \( x \mapsto f_a(x; \omega) \) and the fixed point at the origin is repelling. Any stationary measure will therefore have support contained in \((0,1)\). We will demonstrate that there is only one stationary measure.

**Proposition 6.2.1.** The random logistic map \( f_a \) has a unique stationary measure.

**Proof.** We collect some facts from unimodal dynamics needed in the sequel of the proof. The following facts hold for unimodal maps with negative Schwarzian derivative such as the logistic map \( x \mapsto ax(1-x) \). By Guckenheimer’s theorem, see (29, Theorem III.4.1), \( x \mapsto ax(1-x) \) possesses a unique attractor \( \Lambda_a \). The attractor \( \Lambda_a \) is either a periodic attractor, a solenoidal attractor, or a finite union of intervals on which the map acts transitively. In all cases, the omega-limit set of the critical point \( c \) (with \( c = \frac{1}{2} \) for the logistic map) is contained in \( \Lambda_a \). In fact, if \( \Lambda_a \) is not a periodic
attractor, then \( c \) is contained in \( \Lambda_a \). It follows from a result of Misiurewicz, see (29, Theorem III.3.2), that the basin of attraction of \( \Lambda_a \) is an open and dense subset of \((0,1)\).

We will distinguish the following two cases.

**Case (i):** There exists \( \omega \in \Delta \), so that \( c \in \Lambda_{a+\sigma_\omega} \).

**Case (ii):** otherwise.

The two cases are treated separately.

**Case (i):** Write

\[
W = \bigcap_{n \geq 0} \bigcup_{i \geq n} f_a^n(c;\Delta^N)
\]

for the omega-limit set of \( c \) under all possible random iterations. Observe that \( W \) is an invariant set. From the properties of the noise, \( W \) consists of a finite union of intervals. Note that \( c \in W \), so that \( W \) equals the closure of the positive orbit \( \bigcup_{n \geq 0} f_a^n(c;\Delta^N) \) of \( c \) under all possible random iterations. We will prove that for each \( x \in (0,1) \), \( y \in W \) and \( \varepsilon > 0 \), there exist \( n > 0 \) and \( \bar{\omega} \in \Delta^N \) with the property that

\[
|f_a^n(x;\bar{\omega}) - y| < \varepsilon.
\]

This implies the \( W \) is the unique minimal invariant set, which in turn implies the theorem in the first case.

Fix \( x \in (0,1) \), \( y \in W \), \( \varepsilon > 0 \). From the construction of \( W \), there exist \( \omega_1 \in \Delta^N \), \( i > 0 \), so that \( |f_a^i(c;\omega_1) - y| < \varepsilon \). By continuity of \( x \mapsto f_a^i(x;\omega_1) \), the same holds with \( c \) replaced by a point from a \( \delta \) neighborhood of \( c \) for some \( \delta > 0 \). We need to establish the existence of \( \bar{\omega} \in \Delta^N \) and \( j > 0 \) so that \( |f_a^j(x;\bar{\omega}) - c| < \delta \). Let \( \omega_2 \in \Delta \) be such that \( c \in \Lambda_{a+\sigma_\omega} \). Since the basin of attraction of \( \Lambda_{a+\sigma_\omega} \) is open and dense, there exists \( \omega_3 \in \Delta \) with \( x_1 = f_a(x;\omega_3) \) contained in the basin of attraction of \( \Lambda_{a+\sigma_\omega} \). For \( i \) large, \( x_i = f_a^{i-1}(x_1;\omega_2,\omega_2,\cdots) \) is as close as desired to \( \Lambda_{a+\sigma_\omega} \). If \( \Lambda_{a+\sigma_\omega} \) is a finite union of intervals, we get that \( x_i \) is contained in \( \Lambda_{a+\sigma_\omega} \) for large enough \( i \). As inverse images of \( c \) for \( x \mapsto f_a(x;\omega_2) \) are dense in \( \Lambda_{a+\sigma_\omega} \), one deduces that there exist \( \omega_4 \in \Delta^N \) and \( k > 0 \) so that \( f_a^k(x_{i+1};\omega_4) \) lies in a \( \delta \) neighborhood of \( c \). Indeed, if \( \Lambda_{a+\sigma_\omega} \) is a finite union of intervals, then we find \( \omega_5 \) with \( x_{i+1} = f(x_1;\omega_5) \) equal to an inverse image of \( c \). If \( \Lambda_{a+\sigma_\omega} \) is a solenoidal attractor, then \( x \mapsto f_a(x;\omega_2) \) is infinitely renormalizable. In this case one can use \( \omega_4 = (\omega_2,\omega_2,\cdots) \). Also for a periodic attractor \( \Lambda_{a+\sigma_\omega} \) containing \( c \) one uses \( \omega_4 = (\omega_2,\omega_2,\cdots) \).

**Case (ii):** By Guckenheimer’s theorem, \( x \mapsto f_a(x;\omega) \) possesses a unique periodic attractor for each \( \omega \in \Delta \). Write \( V \) for the union of \( \Lambda_{a+\sigma_\omega} \) over \( \omega \in \Delta \). Define

\[
W = \bigcup_{i \geq 0} f_a^i(V;\Delta^N).
\]

This is clearly an invariant set. Note that we do not claim that \( c \) is outside of \( W \). Arguments as before prove (6.12) with this definition of \( W \): for suitable noise one finds an orbit starting at \( x \) that approaches a point in \( V \) and then with further iterates approaches \( y \in W \).
Remark 6.2.1.
The above proof applies to show that a random unimodal map $g(x; \omega)$ with negative Schwarzian derivative for each $\omega$ (the invoked theorem by Guckenheimer is true for these maps) has a unique stationary measure.

Figure 6.5: Numerically computed Birkhoff averages $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f_\omega^i(x))$ of $\phi(x) = 1_{[0.4, 0.6]}$ for the logistic family (left picture) and the random logistic family with $\sigma = 0.005$ (right picture), for parameters $a$ ranging from 3.8 to 3.9. The flat part in the left picture, where the average equals 1/3, runs from a saddle node bifurcation to a homoclinic bifurcation. The numerical computations show that for the random logistic family these are replaced by their random versions; for parameters from the flat part the stationary measure is supported on three disjoint intervals cycled by the random map.

Perturbing away from the deterministic logistic family one sees that both random saddle node bifurcations and random homoclinic bifurcations occur in the random logistic family $\{f_a\}$ for small noise levels. Typically one can expect the following scenario. We start by recalling some facts concerning the dynamics of the deterministic map $f_a(\cdot; 0)$. The map $f_a(\cdot; 0)$ is called renormalizable if there exists an interval $I$ and a positive integer $q$, so that $f_a^q(\cdot; 0) \subset I$. Let $[a_-, a_+]$ be a maximal interval so that $f_a(\cdot; 0)$ is renormalizable for $a \in [a_-, a_+]$ with $q$ constant. Then $f_a$ undergoes a saddle node bifurcation at $a = a_-$ involving a periodic orbit of period $q$. At $a = a_+$, $f_a$ undergoes a homoclinic bifurcation, where an iterate of $f_a$ maps the critical point onto a periodic orbit of period $q$. For small noise levels (i.e. $\sigma$ small) one expects a random saddle node bifurcation near $a = a_-$ and a random homoclinic bifurcation near $a = a_+$. Figure 6.5 illustrates this by computing Birkhoff averages for the logistic family and the random logistic family. See (59) for explanations of the computations for the logistic family.