Cold electroweak baryogenesis and quantum cosmological correlations
van der Meulen, M.P.

Citation for published version (APA):
In this appendix we calculate the one loop correction to the two point function as given in equation (7.33). We first consider the diagrams with one external dashed line (diagrams A and D), and then the ones with two external dashed lines (diagrams B and C). The complete result is given in equation (7.45).

E.1 Diagrams A and D

We start with the diagrams with one external $G^R$ two point function. First we calculate the amputated diagrams, and then attach the external lines. The amputated diagrams are:

$$\begin{align*}
\text{A} & \quad \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) circle (0.5cm);
\filldraw[black] (0,0) circle (2pt);
\draw[thick, ->] (-0.5,0) -- (-0.2,0);
\draw[thick, ->] (-0.5,0) -- (-0.3,0);
\draw[thick, ->] (-0.5,0) -- (-0.4,0);
\node at (-0.5,0) {$k$};
\node at (0,0) {$\tau_1$};
\node at (0,0) {$\tau_2$};
\draw[thick, ->] (0,0) -- (0.5,0);
\draw[thick, ->] (0,0) -- (0.2,0);
\draw[thick, ->] (0,0) -- (0.3,0);
\node at (0.5,0) {$p'$};
\end{tikzpicture} \\
\text{D} & \quad \begin{tikzpicture}[baseline=-0.5ex]
\draw[thick] (0,0) circle (0.5cm);
\filldraw[black] (0,0) circle (2pt);
\draw[thick, ->] (-0.5,0) -- (-0.2,0);
\draw[thick, ->] (-0.5,0) -- (-0.3,0);
\draw[thick, ->] (-0.5,0) -- (-0.4,0);
\node at (-0.5,0) {$k$};
\node at (0,0) {$\tau_1$};
\node at (0,0) {$\tau_2$};
\draw[thick, ->] (0,0) -- (0.5,0);
\draw[thick, ->] (0,0) -- (0.2,0);
\draw[thick, ->] (0,0) -- (0.3,0);
\node at (0.5,0) {$p'$};
\end{tikzpicture}
\end{align*}$$

The amputated version of diagram A is given by

$$
\Lambda_{\text{amp}}(k, \tau_1, \tau_2) = \frac{-i(-i\lambda)^2}{H^8\tau_1\tau_2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \delta^3(k - p - p')G^R(p', \tau_1, \tau_2)F(p, \tau_1, \tau_2) = \frac{i\lambda^2}{(2\pi)^2k H^8\tau_1\tau_2} \int_0^{\infty} dp \int_{|p-k|}^{p+k} dp' G^R(p', \tau_1, \tau_2)F(p, \tau_1, \tau_2),
$$

(E.2)
where we have used the identity
\[
\int d^3p \delta^3(\mathbf{k} + \mathbf{p} + \mathbf{p}') f(k, p, p') = \frac{2\pi}{k} \int_0^\infty dp \int_{|p-k|}^{p+k} dp' p' f(k, p, p').
\] (E.3)

We will evaluate this integral below. For diagram D we see from equations (7.30) and (7.3) that it is equal to
\[
D_{\text{amp}}(k, \tau_1, \tau_2) = -i a^4(\tau_1) \delta_m(\tau_1 - \tau_2) = -\frac{-i\lambda^2}{4(2\pi)^2H^4\tau_1^4} \ln\left(\frac{\Lambda}{\mu}\right) \delta(\tau_1 - \tau_2),
\] (E.4)

where \(\Lambda\) is the ultraviolet momentum cutoff and \(\mu\) is a renormalization scale. The counterterm \(\delta Z\) is finite and leads to terms proportional to positive powers of \(\tau\), and is therefore left out.

We calculate the integral (E.2) by splitting the \(p\) integral in a small momentum part
\[
\int_0^{M_{\text{cm}}} dp
\]
and a large momentum part
\[
\int_{M_{\text{cm}}}^{\Lambda} dp
\]
with \(|M_{\text{cm}} \tau| \ll 1\) and \(M_{\text{cm}} > k\).

### E.1.1 Amputated Diagram for Small Internal Momenta

The integral in equation (E.2) is infrared divergent for \(p \rightarrow 0\). We regulate this divergence by giving the field a small mass \(m \ll H\), such that \(\nu = 3/2 - \delta\) with \(\delta = m^2/3H^2\). The \(F\) and \(G^R\) two point functions are then, using equations (7.17), (7.18) and (B.5),
\[
F(k, \tau_1, \tau_2) = \frac{\pi \sqrt{\tau_1 \tau_2}}{4a(\tau_1)a(\tau_2)} \text{Re}\left(H^{(1)}_\nu(-k\tau_1)H^{(1)*}_\nu(-k\tau_2)\right),
\] (E.5)
\[
G^R(k, \tau_1, \tau_2) = -\frac{\pi \sqrt{\tau_1 \tau_2}}{2a(\tau_1)a(\tau_2)} \theta(\tau_1 - \tau_2) \text{Im}\left(H^{(1)}_\nu(-k\tau_1)H^{(1)*}_\nu(-k\tau_2)\right).
\] (E.6)

Using (see [186])
\[
H^{(1)}_\nu(-k\tau) = J_\nu(-k\tau) + i \left(\frac{\cos \nu\pi}{\sin \nu\pi} J_\nu(-k\tau) - \frac{1}{\sin \nu\pi} J_{-\nu}(-k\tau)\right),
\] (E.7)
\[
J_\nu(-k\tau) = \frac{1}{\Gamma(\nu + 1)} \left(-\frac{1}{2} k^\nu\right)^\delta \left(1 + O(k^2\tau^2)\right),
\] (E.8)

and the identity \(\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin \nu\pi\), we obtain
\[
F(k, \tau_1, \tau_2) = \frac{H^2}{2k^4}(k^2\tau_1 \tau_2)^\delta,
\] (E.9)
\[
G^R(k, \tau_1, \tau_2) = \theta(\tau_1 - \tau_2) \frac{H^2}{3} \left(\tau_1 \frac{\tau_2}{\tau_1} \delta - \left(\frac{\tau_1}{\tau_2}\right)^\delta\tau_2^2\right).
\] (E.10)

\(^1\)In [175] a similar split of integrals is used to calculate a similar integral. Note however that the integral there differs from the integral here, because the self-energy kernel of [175] is not the same as the amputated diagram A.
The integral is
\[
\frac{i\lambda^2}{(2\pi)^2 k^{1/4} M_{1/2}^{1/2}} \left( \frac{\theta(|\tau_1| - |\tau_2|) \frac{\tau_1^3}{\delta} - \theta(|\tau_1| + |\tau_2|) \frac{\tau_2^3}{\delta}}{2(\pi)^2 H^4(\tau_1 \tau_2)^3} \right) \int_0^{\tau_1 \tau_2} \frac{dp}{p^2} \int_{|p-k|}^{|p+k|} \frac{dp'}{p'^2} =
\]
\[
\frac{i\lambda^2 \theta(\tau_1 - \tau_2) \frac{\tau_1^3}{\delta}}{6(2\pi)^2 H^4(\tau_1 \tau_2)^3} \left( \frac{\tau_1^3 - \tau_2^3}{\delta} + 2\tau_1^3 \ln |M_{cm}| - 2\tau_2^3 \ln |M_{cm}| + O(\delta) \right).
\]
(E.11)

E.1.2 Amputated diagrams for large internal momenta

For large momenta we approximate the field to be massless and we use the two point functions of equations (7.24) and (7.25), which we write as
\[
F(k, \tau_1, \tau_2) = \frac{H^2}{2} \sum_{i=1}^3 F_i(k, \tau_1, \tau_2),
\]
(E.12)
\[
F_1(k, \tau_1, \tau_2) = \frac{1}{k^3} \cos k\Delta\tau,
\]
\[
F_2(k, \tau_1, \tau_2) = \frac{1}{k^2} \Delta\tau \sin k\Delta\tau,
\]
\[
F_3(k, \tau_1, \tau_2) = \frac{1}{k} \tau_1 \tau_2 \cos k\Delta\tau,
\]
with \(\Delta\tau = \tau_1 - \tau_2\), and similarly
\[
G_R(k, \tau_1, \tau_2) = \theta(\tau_1 - \tau_2) H^2 \sum_{i=1}^3 G_{R,i}(k, \tau_1, \tau_2),
\]
(E.13)
\[
G_{R,1}(k, \tau_1, \tau_2) = \frac{1}{k^3} \sin k\Delta\tau,
\]
\[
G_{R,2}(k, \tau_1, \tau_2) = -\frac{1}{k^2} \Delta\tau \cos k\Delta\tau,
\]
\[
G_{R,3}(k, \tau_1, \tau_2) = \frac{1}{k} \tau_1 \tau_2 \sin k\Delta\tau.
\]

In the following calculations we use the definitions
\[
\text{Si}(x) = \int_0^x dx' \frac{\sin x'}{x'}, \quad \text{Ci}(x) = -\int_x^\infty dx' \frac{\cos x'}{x'},
\]
(E.14)
which behave for small respectively large arguments as
\[
\text{Si}(x) = x + O(x^3), \quad \text{Si}(x) = \frac{\pi}{2} - \frac{\cos x}{x} - \frac{\sin x}{x^2} + O(x^{-3}),
\]
(E.15)
\[
\text{Ci}(x) = \gamma + \ln x - \frac{x^2}{4} + O(x^4), \quad \text{Ci}(x) = \frac{\sin x}{x} - \frac{\cos x}{x^2} + O(x^{-2}),
\]
(E.16)
and the identities

\[
\int_{p-k}^{p+k} dp' \frac{\sin p' \Delta \tau}{p'^2} = -\frac{\sin(p + k) \Delta \tau}{p + k} + \frac{\sin(p - k) \Delta \tau}{p - k} + \Delta \tau \left( \operatorname{Ci}((p + k) \Delta \tau) - \operatorname{Ci}((p - k) \Delta \tau) \right), \tag{E.17}
\]

\[
\int_{p-k}^{p+k} dp' \frac{\cos p' \Delta \tau}{p'^2} = -\frac{\cos(p + k) \Delta \tau}{p + k} + \frac{\cos(p - k) \Delta \tau}{p - k} - \Delta \tau \left( \operatorname{Si}((p + k) \Delta \tau) - \operatorname{Si}((p - k) \Delta \tau) \right), \tag{E.18}
\]

\[
\int_{p-k}^{p+k} dp' \sin p' \Delta \tau = -\frac{1}{\Delta \tau} (\cos(p + k) \Delta \tau - \cos(p - k) \Delta \tau) = \frac{2}{\Delta \tau} \sin k \Delta \tau \sin p \Delta \tau, \tag{E.19}
\]

\[
\int_{p-k}^{p+k} dp' \cos p' \Delta \tau = \frac{1}{\Delta \tau} (\sin(p + k) \Delta \tau - \sin(p - k) \Delta \tau) = \frac{2}{\Delta \tau} \sin k \Delta \tau \cos p \Delta \tau. \tag{E.20}
\]

Next we calculate the contributions

\[
\int_{M_m}^{\Lambda_{\alpha}(\tau_2)} dp \int_{p-k}^{p+k} dp' \left( \int_{p-k}^{p+k} d\tau F_i(p, \tau_1, \tau_2) \int_{p-k}^{p+k} d\tau' G_R^R(p', \tau_1, \tau_2) \right), \tag{E.21}
\]

for \(i\) and \(j\) from 1 to 3:

**#1: \(F_1(p, \tau_1, \tau_2)G_R^R(p', \tau_1, \tau_2)\)**

\[
\int_{M_m}^{\Lambda_{\alpha}(\tau_2)} dp \int_{p-k}^{p+k} dp' \sin p' \Delta \tau = \left. \int_{M_m}^{\Lambda_{\alpha}(\tau_2)} dp \right\} - \cos p \Delta \tau \left( -\frac{\sin(p + k) \Delta \tau}{p + k} + \frac{\sin(p - k) \Delta \tau}{p - k} + \Delta \tau \left( \operatorname{Ci}((p + k) \Delta \tau) - \operatorname{Ci}((p - k) \Delta \tau) \right) \right), \tag{E.22}
\]
Appendix E - One loop correction to two point function

#2: \( F_1(p, \tau_1, \tau_2) G^R_2(p', \tau_1, \tau_2) \)

\[
- \Delta \tau \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \frac{\cos p \Delta \tau}{p^2} \int_{p-k}^{p+k} dp' \frac{\cos p' \Delta \tau}{p'} = \\
\Delta \tau \left[ \frac{\cos p \Delta \tau}{p} \left( \text{Ci}((p + k) \Delta \tau) - \text{Ci}((p - k) \Delta \tau) \right) \right]_{M_{cm}}^{\Lambda \alpha(\tau_2)} + \\
\Delta \tau \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \left\{ \Delta \tau \frac{\sin p \Delta \tau}{p} \left( \text{Ci}((p + k) \Delta \tau) - \text{Ci}((p - k) \Delta \tau) \right) \right\} + \\
- \cos p \Delta \tau \left( \frac{\cos (p + k) \Delta \tau}{p + k} - \frac{\cos (p - k) \Delta \tau}{p - k} \right),
\]

(E.23)

#3: \( F_1(p, \tau_1, \tau_2) G^R_3(p', \tau_1, \tau_2) \)

\[
\tau_1 \tau_2 \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \frac{\cos p \Delta \tau}{p^2} \int_{p-k}^{p+k} dp' \frac{\sin p' \Delta \tau}{p'} = \\
\frac{\tau_1 \tau_2 \sin k \Delta \tau}{\Delta \tau} \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \sin 2p \Delta \tau = \\
\frac{\tau_1 \tau_2 \sin k \Delta \tau}{\Delta \tau} \left[ - \frac{\sin 2p \Delta \tau}{p} + 2 \Delta \tau \text{Ci}(2p \Delta \tau) \right]_{M_{cm}}^{\Lambda \alpha(\tau_2)},
\]

(E.24)

#4: \( F_2(p, \tau_1, \tau_2) G^R_1(p', \tau_1, \tau_2) \)

\[
\Delta \tau \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \frac{\sin p \Delta \tau}{p} \int_{p-k}^{p+k} dp' \frac{\sin p' \Delta \tau}{p^2} = \\
\int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \Delta \tau \frac{\sin p \Delta \tau}{p} \left( - \frac{\sin (p + k) \Delta \tau}{p + k} + \frac{\sin (p - k) \Delta \tau}{p - k} + \frac{\Delta \tau \left( \text{Ci}((p + k) \Delta \tau) - \text{Ci}((p - k) \Delta \tau) \right)}{p} \right),
\]

(E.25)

#5: \( F_2(p, \tau_1, \tau_2) G^R_2(p', \tau_1, \tau_2) \)

\[
- \Delta \tau^2 \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \frac{\sin p \Delta \tau}{p} \int_{p-k}^{p+k} dp' \frac{\cos p' \Delta \tau}{p'} = \\
- \Delta \tau^2 \int_{M_{cm}}^{\Lambda \alpha(\tau_2)} dp \frac{\sin p \Delta \tau}{p} \left( \text{Ci}((p + k) \Delta \tau) - \text{Ci}((p - k) \Delta \tau) \right), \quad \text{(E.26)}
\]
Appendix E - One loop correction to two point function

#6: \( F_2(p, \tau_1, \tau_2) G_3^R(p', \tau_1, \tau_2) \)
\[
\Delta \tau \tau_1 \tau_2 \int_{M_m}^{\Lambda a(\tau_2)} dp \frac{\sin p \Delta \tau}{p} \int_{\tau - k}^{p + k} dp' \sin p' \Delta \tau = \\
\tau_1 \tau_2 \sin k \Delta \tau \int_{M_m}^{\Lambda a(\tau_2)} dp \frac{1 - \cos 2p \Delta \tau}{p} = \\
\tau_1 \tau_2 \sin k \Delta \tau \left[ \ln p - \text{Ci}(2p \Delta \tau) \right]_{M_m}^{\Lambda a(\tau_2)}, \quad (E.27)
\]

#7: \( F_3(p, \tau_1, \tau_2) G_1^R(p', \tau_1, \tau_2) \)
\[
\tau_1 \tau_2 \int_{M_m}^{\Lambda a(\tau_2)} dp \cos p \Delta \tau \int_{\tau - k}^{p + k} dp' \frac{\sin p' \Delta \tau}{p'^2} = \\
\tau_1 \tau_2 \int_{M_m}^{\Lambda a(\tau_2)} dp \cos p \Delta \tau \left( - \frac{\sin(p + k) \Delta \tau}{p + k} + \frac{\sin(p - k) \Delta \tau}{p - k} + \Delta \tau \left( \text{Ci}(p + k \Delta \tau) - \text{Ci}(p - k \Delta \tau) \right) \right) = \\
- \frac{\tau_1 \tau_2}{2} \left[ \sin k \Delta \tau \left( \ln(p^2 - k^2) - \text{Ci}(2(p + k) \Delta \tau) - \text{Ci}(2(p - k) \Delta \tau) \right) + \cos k \Delta \tau \left( \text{Si}(2(p + k) \Delta \tau) - \text{Si}(2(p - k) \Delta \tau) \right) \right]_{M_m}^{\Lambda a(\tau_2)} + \\
\tau_1 \tau_2 \Delta \tau \int_{M_m}^{\Lambda a(\tau_2)} dp \cos p \Delta \tau \left( \text{Ci}(p + k \Delta \tau) - \text{Ci}(p - k \Delta \tau) \right), \quad (E.28)
\]

#8: \( F_3(p, \tau_1, \tau_2) G_2^R(p', \tau_1, \tau_2) \)
\[
- \tau_1 \tau_2 \Delta \tau \int_{M_m}^{\Lambda a(\tau_2)} dp \cos p \Delta \tau \int_{\tau - k}^{p + k} dp' \frac{\cos p' \Delta \tau}{p'} = \\
- \tau_1 \tau_2 \Delta \tau \int_{M_m}^{\Lambda a(\tau_2)} dp \cos p \Delta \tau \left( \text{Ci}(p + k \Delta \tau) - \text{Ci}(p - k \Delta \tau) \right), \quad (E.29)
\]

#9: \( F_3(p, \tau_1, \tau_2) G_3^R(p', \tau_1, \tau_2) \)
\[
\tau_1^2 \tau_2^2 \int_{M_m}^{\Lambda a(\tau_2)} dp \cos p \Delta \tau \int_{\tau - k}^{p + k} dp' \sin p' \Delta \tau = \frac{\tau_1^2 \tau_2^2}{\Delta \tau^2} \sin k \Delta \tau \left[ \sin^2 p \Delta \tau \right]_{M_m}^{\Lambda a(\tau_2)}. \quad (E.30)
\]
Appendix E - One loop correction to two point function

Together this becomes

\[
\cos \frac{p \Delta \tau}{p} \left( \frac{\sin(p+k) \Delta \tau}{p+k} - \frac{\sin(p-k) \Delta \tau}{p-k} \right) + \frac{\tau_1 \tau_2 \sin \frac{k \Delta \tau}{2}}{2} (2 \text{Ci}(2p \Delta \tau) + \text{Ci}(2(p+k) \Delta \tau) + \text{Ci}(2(p-k) \Delta \tau) + \\
\ln \frac{p^2 - k^2}{2} - 2 \frac{\sin \frac{2p \Delta \tau}{p}}{p \Delta \tau} - \frac{\tau_1 \tau_2}{2} \cos k \Delta \tau \left( \text{Si}(2(p+k) \Delta \tau) - \text{Si}(2(p-k) \Delta \tau) \right) + \\
\int_{M_{cm}}^{\Lambda a(\tau_2)} dp \left( \frac{\sin(p+k) \Delta \tau}{(p+k)^2} - \frac{\sin(p-k) \Delta \tau}{(p-k)^2} + \frac{\tau_1 \tau_2}{\Delta \tau^2} \sin k \Delta \tau \sin^2 p \Delta \tau \right)
\]

(E.31)

For the upper limit the boundary term vanishes as \(1/\Lambda^2\), except the last term which we will discuss below. The lower limit of the boundary term gives (where we use that \(|M_{cm} \tau_i| \ll 1\))

\[
\frac{2}{3} k \Delta \tau^3 - 2 k \tau_1 \tau_2 \Delta \tau \left( -2 + \gamma + \ln 2 M_{cm} \Delta \tau \right) + O(\tau_i^4).
\]

(E.32)

Using Mathematica, the integral in (E.31) becomes for \(|M_{cm} \tau_i| \ll 1\)

\[
\frac{k \Delta \tau^3}{9} \left( 8 - 6 \gamma - 6 \ln 2 M_{cm} \Delta \tau \right) + O(\tau_i^4).
\]

(E.33)

Together equation (E.31) becomes

\[
\frac{2k}{3} \left( \tau_1^3 - \tau_2^3 \right) \left( \frac{7}{3} - \gamma - \ln 2 M_{cm} (\tau_1 - \tau_2) \right) - \frac{2}{3} k \tau_1 \tau_2 (\tau_1 - \tau_2) + \\
\frac{\tau_1 \tau_2}{\Delta \tau^2} \sin k \Delta \tau \sin^2 \Lambda a(\tau_2) \Delta \tau + O(\tau_i^4).
\]

(E.34)

The term that contains the \(\sin^2 \Lambda\) is logarithmically divergent for \(\Lambda \to \infty\). This can be seen as follows. Consider the integral

\[
\int_{-\infty}^{\infty} d\Delta \tau \theta(\Delta \tau) f(\Delta \tau) \frac{\sin^2 \Lambda a(\tau_2) \Delta \tau}{\Delta \tau} = \frac{1}{2} \int_{0}^{\infty} d\Delta \tau f(\Delta \tau) \frac{1 - \cos \left( \frac{-2 \Lambda}{\tau_1 - \Delta \tau} \right)}{\Delta \tau},
\]

where \(f(\Delta \tau)\) is a test function. The integral can be split into two integrals

\[
\int_{0}^{\infty} = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\eta} + \int_{\eta}^{\infty},
\]

(E.36)
where $\eta$ is used as a regulator time, which we take to zero in the end, after taking the limit $\Lambda \to \infty$. In the first integral we can approximate

$$\frac{\Delta \tau}{\tau - \Delta \tau} \approx \frac{\Delta \tau}{\tau_1}, \quad f(\Delta \tau) \approx f(0), \quad (E.37)$$

so that it becomes

$$\lim_{\epsilon \to 0} \int_\eta^{\epsilon} d\Delta \tau \ f(0) \ \frac{1 - \cos \left( \frac{-2\Lambda \Delta \tau}{\eta} \right)}{\Delta \tau} =$$

$$\lim_{\epsilon \to 0} f(0) \left( \ln \frac{\eta}{\epsilon} - \text{Ci}(\frac{-2\Lambda \eta}{H\tau_1}) + \text{Ci}(\frac{-2\Lambda \epsilon}{H\tau_1}) \right) = f(0) \left( \gamma + \ln \frac{-2\Lambda \eta}{H\tau_1} \right), \quad (E.38)$$

where we have taken $\text{Ci}(\frac{-2\Lambda \eta}{H\tau_1}) \to 0$, and $\text{Ci}(\frac{-2\Lambda \epsilon}{H\tau_1}) \to \gamma + \ln(\frac{-2\Lambda \epsilon}{H\tau_1})$. The remaining integral is

$$\lim_{\Lambda \to \infty} \int_{\eta}^{\infty} d\Delta \tau \ f(\Delta \tau) \ \frac{1 - \cos \left( \frac{-2\Lambda \Delta \tau}{\tau - \Delta \tau} \right)}{\Delta \tau} = \int_{\eta}^{\infty} d\Delta \tau \ f(\Delta \tau) \Delta \tau, \quad (E.39)$$

where the term with the cosine vanishes, provided that the test function $f(\Delta \tau)$ vanishes sufficiently fast as $\Delta \tau \to \infty$. Together we obtain for $\Lambda \to \infty$

$$\int_{-\infty}^{\infty} d\Delta \tau \ \theta(\Delta \tau) f(\Delta \tau) \ \frac{\sin^2 \Lambda a(\tau_2) \Delta \tau}{\Delta \tau} =$$

$$\int_{-\infty}^{\infty} d\Delta \tau \ f(\Delta \tau) \ \frac{1}{2} \left[ \theta(-\eta + \Delta \tau) \ + \delta(\Delta \tau) \left( \gamma + \ln \frac{-2\Lambda \eta}{H\tau_1} \right) \right], \quad (E.40)$$

which is in the language of distributions

$$\theta(\Delta \tau) \ \frac{\sin^2 \Lambda a(\tau_2) \Delta \tau}{\Delta \tau} = \frac{1}{2} \left[ \theta(-\eta + \Delta \tau) \ + \delta(\Delta \tau) \left( \gamma + \ln \frac{-2\Lambda \eta}{H\tau_1} \right) \right]. \quad (E.41)$$

Using this result in equation (E.34), gathering the right prefactors and adding the contribution from the counterterm (E.4), we obtain for the large momentum contribution

$$\frac{i\lambda^2 \theta(\tau_1 - \tau_2)}{2(2\pi)^2 H^4 (\tau_1 \tau_2)^2} \left[ \frac{2}{3} \tau_1^3 - \frac{2}{3} \tau_2^3 \right] \left( \frac{7}{3} - \gamma - \ln 2M_{cm} \Delta \tau \right) + \frac{2}{3} \tau_1 \tau_2 (\tau_1 - \tau_2) +$$

$$\frac{1}{2} \tau_1 \tau_2 \left[ \theta(-\eta + \Delta \tau) \ + \delta(\Delta \tau) \left( \gamma + \ln \frac{-2\Lambda \eta}{H\tau_1} \right) \right]. \quad (E.42)$$
E.1.3 Attaching the external lines

Adding the small and large momenta contributions, we obtain for the amputated diagrams A and D:

\[
\Lambda_{\text{amp}}(k, \tau_1, \tau_2) + D_{\text{amp}}(k, \tau_1, \tau_2) = \frac{i\lambda^2 \theta(\tau_1 - \tau_2)}{6(2\pi)^2 H^4(\tau_1 \tau_2)^4} \left( \frac{\tau_1^3 - \tau_2^3}{\tau_2} \right) \left( \frac{1}{\delta} + \frac{14}{3} - 2\gamma \right) + \\
- 2\tau_1 \tau_2 (\tau_1 - \tau_2) + 2\tau_2^3 \ln \left| \frac{\tau_2}{2(\tau_1 - \tau_2)} \right| - 2\tau_1^3 \ln \left| \frac{\tau_1}{2(\tau_1 - \tau_2)} \right| + \\
\frac{3}{2} \left( \frac{\tau_1 \tau_2}{\tau_1 - \tau_2} \right)^2 \left[ \theta(-\eta + \tau_1 - \tau_2) + \delta(\tau_1 - \tau_2) \left( \gamma + \ln \frac{-2\mu\eta}{H\tau_1} \right) \right] + \mathcal{O}(\tau_1^4) + \mathcal{O}(\delta),
\]

where the dependence on \( M_{\text{cm}} \) has dropped out. The full correlation function is obtained by

\[
-i \int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 G^R(k, \tau, \tau_1) F(k, \tau, \tau_2) \left( \Lambda_{\text{amp}}(k, \tau_1, \tau_2) + D_{\text{amp}}(k, \tau_1, \tau_2) \right).
\]

Because the external momentum \( k \) is small, i.e. \( |k\tau_i| \ll 1 \), we can use the expanded versions of the two point functions (7.37), (7.38) (or the ones of (E.9), (E.10), but this gives only corrections of order \( \mathcal{O}(\delta) \)). Using the integrals

\[
\int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 \left( \tau^3 - \tau_1^3 \right) \left( \tau_1^3 - \tau_2^3 \right) \left( \tau_1 \tau_2 \right)^4 = \frac{1}{3} \left( 1 + 2 \ln \frac{\tau}{\tau_H} + \frac{3}{2} \ln^2 \frac{\tau}{\tau_H} \right) + \mathcal{O}\left( \frac{\tau}{\tau_H} \right),
\]

\[
\int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 \left( \tau^3 - \tau_1^3 \right) (\tau_1 - \tau_2) = -\frac{1}{12} \left( 11 + 6 \ln \frac{\tau}{\tau_H} \right) + \mathcal{O}\left( \frac{\tau}{\tau_H} \right),
\]

\[
\int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 \left( \tau_1^3 - \tau_2^3 \right) \left( \tau_1 \tau_2 \right)^4 \left( \tau_1^3 \ln \left| \frac{\tau_2}{2(\tau_1 - \tau_2)} \right| - \tau_2^3 \ln \left| \frac{\tau_1}{2(\tau_1 - \tau_2)} \right| \right) = \\
\frac{1}{18} \left( \frac{97}{6} - 18 \zeta(3) - 2\pi^2 - 6 \ln 2 + (13 - 3\pi^2 - 12 \ln 2) \ln \frac{\tau}{\tau_H} + \\
3 \ln 2 - 3 \ln^2 \frac{\tau}{\tau_H} \right) + \mathcal{O}\left( \frac{\tau}{\tau_H} \right),
\]

\[
\int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 \left( \tau^3 - \tau_1^3 \right) \left[ \theta(-\eta + \tau_1 - \tau_2) + \delta(\tau_1 - \tau_2) \left( \gamma + \ln \frac{-2\mu\eta}{H\tau_1} \right) \right] = \\
\frac{1}{6} \left( 8 - 2\gamma - \pi^2 - 2 \ln \frac{2\mu}{H} + 6 \left( 1 - \gamma - \ln \frac{2\mu}{H} \right) \ln \frac{\tau}{\tau_H} \right) + \mathcal{O}\left( \frac{\tau}{\tau_H} \right).
\]
Appendix E - One loop correction to two point function

(recall that \( \eta \) is sent to zero), this becomes

\[
\frac{\lambda^2}{36(2\pi)^2k^3}\left\{ \frac{1}{3\delta} + \frac{194}{27} - \frac{7}{6}\gamma - \frac{17}{36}\pi^2 - \frac{2}{3}\ln 2 - 2\zeta(3) - \frac{1}{2}\ln \frac{2\mu}{H} + \right.
\]
\[
\left. \left( \frac{2}{3\delta} + \frac{127}{18} - \frac{17}{6}\gamma - \frac{3}{2}\pi^2 - \frac{4}{3}\ln 2 - 3\frac{\ln 2}{2\mu H} \right) \ln \frac{\tau}{\tau_H} + \right.
\]
\[
\left. \left( \frac{1}{2\delta} + \frac{8}{3} - \gamma - \ln 2 \right) \ln^2 \frac{\tau}{\tau_H} + \frac{1}{3}\ln^3 \frac{\tau}{\tau_H} + {\cal O}\left( \frac{\tau}{\tau_H} \right) + {\cal O}(\delta) \right\}. \tag{E.49}
\]

There is an equal contribution from the diagram with \( \tau_1 \) and \( \tau_2 \) interchanged. Note that there is no dependence on \( \ln k/\mu \) for \( |k\tau| \ll 1 \).

\section*{E.2 Diagrams B and C}

The amputated versions of the diagrams with two external \( G^R \) propagators are

\[ \begin{array}{c}
\textbf{B} \quad \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram.png}} \\
\tau_1 \\
\tau_2
\end{array}
\end{array} \quad \begin{array}{c}
\textbf{C} \quad \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram.png}} \\
\tau_1 \\
\tau_2
\end{array}
\end{array} \tag{E.50}
\]

They translate to

\[
B_{\text{amp}}(k, \tau_1, \tau_2) = \frac{(-i\lambda)^2}{2H^8(\tau_1\tau_2)^4} \int \frac{d^3pd^3p'}{(2\pi)^3} \delta^3(k - p - p')F(p', \tau_1, \tau_2)F(p, \tau_1, \tau_2)
\]
\[
= -\lambda^2 2(2\pi)^2kH^8(\tau_1\tau_2)^4 \int_0^{\infty} dp' \int_{|p-k|}^{p+k} dp' F(p', \tau_1, \tau_2)F(p, \tau_1, \tau_2), \tag{E.51}
\]

\[
C_{\text{amp}}(k, \tau_1, \tau_2) = \frac{(-i)^2(-i\lambda)^2}{8H^8(\tau_1\tau_2)^4} \int \frac{d^3pd^3p'}{(2\pi)^3} \delta^3(k - p - p')G^R(p', \tau_1, \tau_2)G^R(p, \tau_1, \tau_2)
\]
\[
= \lambda^2 8(2\pi)^2kH^8(\tau_1\tau_2)^4 \int_0^{\infty} dp' \int_{|p-k|}^{p+k} dp' G^R(p', \tau_1, \tau_2)G^R(p, \tau_1, \tau_2), \tag{E.52}
\]

where both diagrams have a factor 1/2 for symmetry. Diagram C has an additional factor 1/4 from the vertex with three dashed lines (7.29). We split the \( p \) integral again into a small momentum part and a large momentum part.

\subsection*{E.2.1 Amputated Diagrams for Small Internal Momenta}

For small internal momenta we use the expanded propagators (E.9) and (E.10).
Diagram B. The integral is

\[-\frac{\lambda^2}{2(2\pi)^2 k H^2 (\tau_1 \tau_2)^4} \int_{0}^{M_{\text{cm}}} dp \int_{|p-k|}^{p+k} dp' \frac{\delta}{(pp')^2} = \]

\[-\frac{\lambda^2}{8(2\pi)^2 k H^2 (\tau_1 \tau_2)^4 (2\delta - 1)} \left[ \int_{0}^{k} dp p^{-2+2\delta} (p + k)^{-1+2\delta} - (k - p)^{-1+2\delta} \right] + \int_{k}^{M_{\text{cm}}} dp p^{-2+2\delta} ((p + k)^{-1+2\delta} - (p - k)^{-1+2\delta}) \right]. \tag{E.53}

The integral on the middle line of (E.53) is finite, but the individual parts are infrared divergent. Therefore we calculate the individual parts for \( \delta > 1/2 \), and in the end use analytic continuation to \( \delta \ll 1 \). The integrals are (using \( p = k \varepsilon \))

\[\int_{0}^{1} dx x^{-2+2\delta} (1 + x)^{-1+2\delta} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1 + n - 2\delta)}{n! \Gamma(1 - 2\delta)} \int_{0}^{1} dx x^{-2+n+2\delta} \]

\[= -\frac{1}{2\delta} + \ln 2 + O(\delta), \tag{E.54}\]

\[\int_{0}^{1} dx x^{-2+2\delta} (1 - x)^{-1+2\delta} = B(-1 + 2\delta, 2\delta) = \frac{1}{\delta} - 2 + O(\delta), \tag{E.55}\]

\[\int_{1}^{M_{\text{cm}}/k} dx x^{-2+2\delta} (1 + x)^{-1+2\delta} = \int_{k/M_{\text{cm}}}^{1} dy \frac{y^{1-4\delta}}{(1 + y)^{1-2\delta}} \]

\[= 1 - \frac{k}{M_{\text{cm}}} - \ln 2 + \ln \left(1 + \frac{k}{M_{\text{cm}}}\right) + O(\delta), \tag{E.56}\]

\[\int_{1}^{M_{\text{cm}}/k} dx x^{-2+2\delta} (x - 1)^{-1+2\delta} = \int_{k/M_{\text{cm}}}^{1} dy \frac{y^{1-4\delta}}{(1 - y)^{1-2\delta}} \]

\[= \int_{0}^{1-k/M_{cm}} dz z^{-1+2\delta} (1 - z) + O(\delta) \]

\[= \frac{1}{2\delta} - 1 + \frac{k}{M_{\text{cm}}} + \ln \left(1 - \frac{k}{M_{\text{cm}}}\right) + O(\delta), \tag{E.57}\]

where we have used analytic continuation in the first two integrals and \( y = 1/x \) and \( z = 1 - y \) in the latter two. The right hand side of equation (E.53) becomes

\[-\frac{\lambda^2}{8(2\pi)^2 k^3 H^4 (\tau_1 \tau_2)^4 (2\delta - 1)} \left( -\frac{2}{\delta} + 4 - 2 \frac{k}{M_{\text{cm}}} + \ln \left(1 + \frac{k}{M_{\text{cm}}}\right) + \frac{1}{2} \ln \left(\frac{M_{\text{cm}} + k}{M_{\text{cm}} - k}\right) + 2 \ln (k^2 \tau_1 \tau_2) + O(\delta) \right) =

\[-\frac{\lambda^2}{4(2\pi)^2 k^3 H^4 (\tau_1 \tau_2)^4} \left( \frac{1}{\delta} + \frac{k}{M_{\text{cm}}} - \frac{1}{2} \ln \left(\frac{M_{\text{cm}} + k}{M_{\text{cm}} - k}\right) + 2 \ln (k^2 \tau_1 \tau_2) + O(\delta) \right). \tag{E.58}\]
Appendix E - One loop correction to two point function

Diagram C. From equations (E.52) and (E.10) we see directly that diagram C does not give late time contributions and also does not have an infrared divergence.

E.2.2 Amputated diagrams for large internal momenta

The contributions from large internal momenta can be calculated in a similar way as is used for diagram A in section E.1.2.

Diagram B. The sum of integrals

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \int_{M_{cm}}^{\lambda a(\tau_3)} dp \, p F_i(p, \tau_1, \tau_2) \int_{p-k}^{p+k} dp' \, p' F_j(p', \tau_1, \tau_2),$$  \hspace{1cm} (E.59)

(where $\tau_3 = \tau_1, \tau_2$, depending on which time is earlier), is equal to

$$\left[ \cos \frac{p \Delta \tau}{p} \left( \frac{\cos(p+k)\Delta \tau}{p+k} - \frac{\cos(p-k)\Delta \tau}{p-k} \right) + \right.$$ \nonumber

$$\left. - \tau_1 \tau_2 \sin k \Delta \tau \left( \text{Si}(2p\Delta \tau) + 2 \frac{\cos^2 p \Delta \tau}{p \Delta \tau} \right) + \right.$$ \nonumber

$$\left. \frac{\tau_1 \tau_2}{\Delta \tau} \sin k \Delta \tau \left( p + \frac{\sin 2 p \Delta \tau}{2 \Delta \tau} \right) \right]_{M_{cm}}^{\lambda a(\tau_3)} +$$ \nonumber

$$\int_{M_{cm}}^{\lambda a(\tau_3)} dp \, \cos \frac{p \Delta \tau}{p} \left\{ \cos \frac{(p+k)\Delta \tau}{(p+k)^2} - \frac{\cos(p-k)\Delta \tau}{(p-k)^2} \right.$$ \nonumber

$$\left. + \Delta \tau \left( \frac{\sin(p+k)\Delta \tau}{p+k} - \frac{\sin(p-k)\Delta \tau}{p-k} \right) + \right.$$ \nonumber

$$\left. - pt_1 \tau_2 \left( \cos(p+k)\Delta \tau \frac{p+k}{p+k} - \cos(p-k)\Delta \tau \frac{p+k}{p} \right) \right\}. \hspace{1cm} (E.60)$$

The only ultraviolet term comes from the last term of the boundary term and is, including the correct prefactor:

$$-\lambda^2 \sin k \Delta \tau \left[ \frac{\text{Si}(2p\Delta \tau)}{2 \Delta \tau} \right]_{M_{cm}}^{\lambda a(\tau_3)}. \hspace{1cm} (E.61)$$

The only term that gives late time contributions is the first line in the integral. It is

$$-\lambda^2 \frac{1}{4(2\pi)^2 k^4 H^4(\tau_1 \tau_2)} \left( - \frac{k}{M_{cm}} + \frac{1}{2} \ln \frac{M_{cm}+k}{M_{cm}-k} + O(\tau_i^4) \right). \hspace{1cm} (E.62)$$

146
Appendix E - One loop correction to two point function

Diagram C. The sum of integrals

\[ \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{M_{em}}^{\Lambda a(\tau_3)} dp \ g^{R_i}(p, \tau_1, \tau_2) \int_{p-k}^{p+k} dp' \ g^{R_j}(p', \tau_1, \tau_2), \tag{E.63} \]

is equal to

\[ \left[ \frac{\sin p \Delta \tau}{p} \left( \frac{\sin(p + k) \Delta \tau}{p + k} - \frac{\sin(p - k) \Delta \tau}{p - k} \right) \right. \]
\[ + \tau_1 \tau_2 \sin k \Delta \tau \left( 3 \sin(2p \Delta \tau) - \frac{2 \sin^2(p \Delta \tau)}{p \Delta \tau} \right) \]
\[ \left. + \frac{(\tau_1 \tau_2)^2}{\Delta \tau} \sin k \Delta \tau \left( p - \frac{\sin 2p \Delta \tau}{2 \Delta \tau} \right) \right]_{M_{em}}^{\Lambda a(\tau_3)}. \tag{E.64} \]

Only the last term of the boundary term is ultraviolet divergent:

\[ \frac{\lambda^2 \theta(\tau_1 - \tau_2) \sin k \Delta \tau}{8(2\pi)^2 k \hbar^4 (\tau_1 \tau_2)^2 \Delta \tau} \left[ p - \frac{\sin 2p \Delta \tau}{2 \Delta \tau} \right]_{M_{em}}^{\Lambda a(\tau_3)}. \tag{E.65} \]

The diagram with the vertices interchanged gives the same result, except that \( \theta(\tau_1 - \tau_2) \) is replaced by \( \theta(\tau_2 - \tau_1) \). There are no further late time contributions.

**Ultraviolet divergences.** The ultraviolet divergent terms of diagrams B (E.61), C (E.65), and C with the vertices interchanged, add up to

\[ \left[ -\frac{\lambda^2 \sin k \Delta \tau}{8(2\pi)^2 k \hbar^4 (\tau_1 \tau_2)^2 \Delta \tau} \right]_{M_{em}}^{\Lambda a(\tau_3)}. \tag{E.66} \]

which is finite and does not give late time contributions.
E.2.3 ATTACHING THE EXTERNAL LINES

Adding the small and large momenta contributions, we obtain for the amputated diagrams B and C:

\[
\begin{align*}
B_{\text{amp}}(k, \tau_1, \tau_2) + C_{\text{amp}}(k, \tau_1, \tau_2) &= \frac{-\lambda^2}{4(2\pi)^2 k^3 H^4(\tau_1 \tau_2)^4} \left( \frac{1}{\delta} + 2 \ln(k^2 \tau_1 \tau_2) + \mathcal{O}(\tau_i) + \mathcal{O}(\delta) \right), \quad (E.67)
\end{align*}
\]

where the dependence on \( M_{cm} \) has dropped out. The full correlation function is obtained by

\[
- \int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 G^R(k, \tau, \tau_1)G^R(k, \tau, \tau_2) \left( B_{\text{amp}}(k, \tau_1, \tau_2) + C_{\text{amp}}(k, \tau_1, \tau_2) \right). \quad (E.68)
\]

Because the external momentum \( k \) is small, i.e. \( |k\tau| \ll 1 \), we can use the expanded version of the \( G^R \) propagator (7.38) (or the one of (E.10), but this gives only corrections of \( \mathcal{O}(\delta) \)). Using the integrals

\[
\begin{align*}
\int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 \frac{(\tau^3 - \tau_1^3)(\tau^3 - \tau_2^3)}{(\tau_1 \tau_2)^4} &= \frac{1}{9} + \frac{2}{3} \ln \frac{\tau}{\tau_H} + \ln^2 \frac{\tau}{\tau_H} + \mathcal{O}(\frac{\tau}{\tau_H}), \quad (E.69)
\end{align*}
\]

\[
\begin{align*}
\int_{\tau_H}^{\tau} d\tau_1 \int_{\tau_H}^{\tau} d\tau_2 \frac{(\tau^3 - \tau_1^3)(\tau^3 - \tau_2^3)}{(\tau_1 \tau_2)^4} \ln(k^2 \tau_1 \tau_2) &= \frac{1}{27} \left( 2 + 6 \ln(-k\tau_H) + 12 \left( 1 + 3 \ln(-k\tau_H) \right) \ln \frac{\tau}{\tau_H} + 27 \left( 1 + 2 \ln(-k\tau_H) \right) \ln^2 \frac{\tau}{\tau_H} + 27 \ln \ln \frac{\tau}{\tau_H} \right) + \mathcal{O}(\frac{\tau}{\tau_H}), \quad (E.70)
\end{align*}
\]

this becomes

\[
\frac{\lambda^2}{36(2\pi)^2 k^3} \left( \frac{1}{90} + \frac{4}{27} + \frac{4}{9} \ln(-k\tau_H) + \left( \frac{2}{3\delta} + \frac{8}{9} + \frac{8}{3} \ln(-k\tau_H) \right) \ln \frac{\tau}{\tau_H} + \left( \frac{1}{\delta} + 2 + 4 \ln(-k\tau_H) \right) \ln^2 \frac{\tau}{\tau_H} + 2 \ln \ln \frac{\tau}{\tau_H} + \mathcal{O}(\frac{\tau}{\tau_H}) + \mathcal{O}(\delta) \right). \quad (E.71)
\]