SUPPLEMENTARY MATERIAL

A PROOFS

A.1 PROOF OF THEOREM 1

Proof. We need to show that if $\boldsymbol{\zeta}_{I_i}$ and $\boldsymbol{\zeta}'_{I_i}$ are such that $(\boldsymbol{\zeta}_{I_i})_{\mathtt{pa}(i)} = (\boldsymbol{\zeta}'_{I_i})_{\mathtt{pa}(i)} = \boldsymbol{\eta}_{\mathtt{pa}(i)}$, then $\eta_i = \eta'_i$. To see that this is the case, observe that the system of equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}_{I_i})}$ is given by:

$$\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}_{I_i})} : \begin{cases} X_j(t) = \zeta_j(t) & j \in \mathcal{I} \setminus (\mathtt{pa}(i) \cup \{i\}) \,, \\ X_j(t) = \eta_j(t) & j \in \mathtt{pa}(i) \,, \\ f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)})(t) = 0 & X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i, \; 0 \le k \le n_i - 1 \,. \end{cases}$$

The equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}'_{I_i})}$ are similar, except with $X_j(t) = \zeta'_j(t)$ for $j \in \mathcal{I} \setminus (\mathtt{pa}(i) \cup \{i\})$. In both cases, the equations for all variables except X_i are solved already. The equation for X_i in both cases reduces to the same quantity by substituting in the values of the parents, namely

$$f_i(X_i, \boldsymbol{\eta}_{\mathtt{pa}(i)})(t) = 0 \,.$$

The solution to this equation in $\mathtt{Dyn}_i$ must be unique and independent of initial conditions, else the dynamic stability of the intervened systems $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}_{I_i})}$ and $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}'_{I_i})}$ would not hold, contradicting the dynamic structural stability of $(\mathcal{D}, \mathtt{Dyn})$. It follows that $\eta_i = \eta'_i$. □

A.2 PROOF OF THEOREM 2

Proof. By construction of the SCM, $\boldsymbol{\eta} \in \mathtt{Dyn}_{\mathcal{I}}$ is a solution of $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$ if and only if the following two conditions hold:

- for $i \in \mathcal{I} \setminus I$, $X_i(t) = \eta_i(t)\ \forall t$ is a solution to the differential equation $f_i(X_i, \boldsymbol{\eta}_{\mathtt{pa}(i)})(t) = 0$;
- for $i \in I$, $\eta_i(t) = \zeta_i(t)$ for all t.

which is true if and only if $\mathbf{X} = \boldsymbol{\eta}$ is a solution to $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ in $\mathtt{Dyn}_{\mathcal{I}}$. Thus, by definition of dynamic stability, $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ is dynamically stable with asymptotic dynamics describable by $\boldsymbol{\eta} \in \mathtt{Dyn}$ if and only if $\mathbf{X} = \boldsymbol{\eta}$ uniquely solves $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$. □

A.3 PROOF OF THEOREM 3

Proof. We need to show that the structural equations of $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$ and $(\mathcal{M}_{\mathcal{D}})_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ are equal. Observe that the equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ are given by:

$$\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)} : \begin{cases} X_i = \zeta_i, & i \in I \,, \\ f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)}) = 0, X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i, \; 0 \le k \le n_i - 1, & i \in \mathcal{I} \setminus I \,. \end{cases}$$

Therefore, when we perform the procedure to derive the structural equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$, we see that:

- if $i \in I$, the ith structural equation will simply be $X_i = \zeta_i$ since intervening on I_i does not affect variable X_i.
- if $i \in \mathcal{I} \setminus I$, the ith structural equation will be the same as for $\mathcal{M}_{\mathcal{D}}$, since the dependence of X_i on the other variables is unchanged.

Hence the structural equations for $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$ are given by:

$$\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})} : \begin{cases} X_i = \zeta_i, & i \in I \,, \\ X_i = F_i(\mathbf{X}_{\mathtt{pa}(i)}), & i \in \mathcal{I} \setminus I \,. \end{cases}$$

and therefore $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})} = (\mathcal{M}_{\mathcal{D}})_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$. □

A.4 PROOF OF COROLLARY 1

Proof. Corollary 1 follows very simply from the observation that if $(\mathcal{D}, \mathtt{Dyn})$ is structurally dynamically stable then so is $(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}, \mathtt{Dyn}_{\mathcal{I}\setminus I})$. The result then follows by application of Theorem 3. $\square$

B DERIVING THE DSCM FOR THE MASS-SPRING SYSTEM

Consider the mass-spring system of Example 1, but with $D \geq 1$ an arbitrary integer. We repeat the setup:

We have D masses attached together on springs. The location of the ith mass at time t is $X_i(t)$, and its mass is m_i. For notational ease, we denote by $X_0 = 0$ and $X_{D+1} = L$ the locations of where the ends of the springs attached to the edge masses meet the walls to which they are affixed. X_0 and X_{D+1} are constant. The natural length and spring constant of the spring connecting masses i and $i+1$ are l_i and k_i respectively. The ith mass undergoes linear damping with coefficient b_i, where b_i is small to ensure that the system is underdamped. The equation of motion for the ith mass ($1 \leq i \leq D$) is given by:

$$m_i \ddot{X}_i(t) = k_i[X_{i+1}(t) - X_i(t) - l_i] - k_{i-1}[X_i(t) - X_{i-1}(t) - l_{i-1}] - b_i \dot{X}_i(t)$$

so, defining

$$f_i(X_i, X_{i-1}, X_{i+1})(t) = m_i \ddot{X}_i(t) - k_i[X_{i+1}(t) - X_i(t) - l_i] + k_{i-1}[X_i(t) - X_{i-1}(t) - l_{i-1}] + b_i \dot{X}_i(t)$$

we can write the system of equations $\mathcal{D}$ for our mass-spring system as

$$\mathcal{D} : \{ \ f_i(X_i, X_{i-1}, X_{i+1})(t) = 0 \qquad i \in \mathcal{I} \, .$$

In the rest of this section we will explicitly calculate the structural equations for the DSCM derived from $\mathcal{D}$ with two different sets of interventions. First, we will derive the structural equations for the case that Dyn consists of all constant trajectories, corresponding to constant interventions that fix variables to constant values for all time. This illustrates the correspondence between the theory in this paper and that of Mooij et al. (2013). Next, we will derive the structural equations for the case that Dyn consists of interventions corresponding to sums of periodic forcing terms.

B.1 MASS-SPRING WITH CONSTANT INTERVENTIONS

In order to derive the structural equations we only need to consider, for each variable, the influence of its parents on it. (Formally, this is because of Theorem 1). Consider variable i. If we intervene to fix its parents to have locations $X_{i-1}(t) = \eta_{i-1}$ and $X_{i+1}(t) = \eta_{i+1}$ for all t, then the equation of motion for variable i is given by

$$m_i \ddot{X}_i(t) + b_i \dot{X}_i(t) + (k_i + k_{i-1}) X_i(t) = k_i[\eta_{i+1} - l_i] + k_{i-1}[\eta_{i-1} + l_{i-1}] \, .$$

There may be some complicated transient dynamics that depend on the initial conditions $X_i(0)$ and $\dot{X}_i(0)$ but provided that $b_i > 0$, we know that the $X_i(t)$ will converge to a constant and therefore the asymptotic solution to this equation can be found by setting $\ddot{X}_i$ and $\dot{X}_i$ to zero. Note that in general, we could explicitly find the solution to this differential equation (and indeed, in the next example we will) but for now there is a shortcut to deriving the structural equations.[7] The asymptotic solution is:

$$X_i = \frac{k_i[\eta_{i+1} - l_i] + k_{i-1}[\eta_{i-1} + l_{i-1}]}{k_i + k_{i-1}} .$$

Therefore the ith structural equation is:

$$F_i(X_{i-1}, X_{i+1}) = \frac{k_i[X_{i+1} - l_i] + k_{i-1}[X_{i-1} + l_{i-1}]}{k_i + k_{i-1}} .$$

[7] This is analogous to the approach taken in Mooij et al. (2013) in which the authors first define the Labelled Equilibrium Equations and from these derive the SCM.

Hence the SCM for $(\mathcal{D}, \texttt{Dyn}_c)$ is:

$$\mathcal{M}_\mathcal{D} : \left\{ \; X_i = \frac{k_i[X_{i+1} - l_i] + k_{i-1}[X_{i-1} + l_{i-1}]}{k_i + k_{i-1}} \qquad i \in \mathcal{I}\,. \right.$$

We can thus use this model to reason about the effect of constant interventions on the asymptotic equilibrium states of the system.

B.2 SUMS OF PERIODIC INTERVENTIONS

Suppose now we want to be able to make interventions of the form:

$$\texttt{do}\big(X_i(t) = A\cos(\omega t + \phi)\big)\,. \tag{4}$$

Such interventions cannot be described by the DSCM derived in Section B.1. In this section we will explicitly derive a DSCM capable of reasoning about the effects of such interventions. It will also illustrate why we need dynamic structural stability.

By Theorem 1, to derive the structural equation for each variable we only need to consider the effect on the child of intervening on the parents according to interventions of the form (4). Consider the following linear differential equation:

$$m\ddot{X}(t) + b\dot{X}(t) + kX(t) = g(t)\,. \tag{5}$$

In general, the solution to this equation will consist of two parts—the *homogeneous* solution and the *particular* solution. The homogeneous solution is one of a family of solutions to the equation

$$m\ddot{X}(t) + b\dot{X}(t) + kX(t) = 0 \tag{6}$$

and this family of solutions is parametrised by the initial conditions. If $b > 0$ then all of the homogeneous solutions decay to zero as $t \longrightarrow \infty$. The particular solution is any solution to the original equation with arbitrary initial conditions. The particular solution captures the asymptotic dynamics due to the forcing term g. Equation 5 is a linear differential equation. This means that if $X = X_1$ is a particular solution for $g = g_1$ and $X = X_2$ is a particular solution for $g = g_2$, then $X = X_1 + X_2$ is a particular solution for $g = g_1 + g_2$.

In order to derive the structural equations, the final ingredient we need is an explicit representation for a particular solution to (5) in the case that $g(t) = A\cos(\omega t + \phi)$. We state the solution for the case that the system is underdamped—this is a standard result and can be verified by checking that the following satisfies (5):

$$X(t) = A'\cos(\omega t + \phi')$$

where

$$A' = \frac{A}{\sqrt{[k - m\omega^2]^2 + bm\omega^2}}\,, \qquad\qquad \phi' = \phi - \arctan\left[\frac{b\omega}{k - m\omega^2}\right]\,. \tag{7}$$

Therefore if we go back to our original equation of motion for variable X_i

$$m_i\ddot{X}_i(t) + b_i\dot{X}_i(t) + (k_i + k_{i-1})X_i(t) = k_i[X_{i+1}(t) - l_i] + k_{i-1}[X_{i-1}(t) + l_{i-1}]$$

and perform the intervention

$$\texttt{do}(X_{i-1}(t) = A_{i-1}\cos(\omega_{i-1}t + \phi_{i-1}), X_{i+1}(t) = A_{i+1}\cos(\omega_{i+1}t + \phi_{i+1}))$$

we see that we can write the RHS of the above equation as the sum of the three terms

$$\begin{aligned} g_1(t) &= k_{i-1}l_{i-1} - k_il_i\,, \\ g_2(t) &= k_{i-1}A_{i-1}\cos(\omega_{i-1}t + \phi_{i-1})\,, \\ g_3(t) &= k_iA_{i+1}\cos(\omega_{i+1}t + \phi_{i+1})\,. \end{aligned}$$

Using the fact that linear differential equation have superposable solutions and (7), we can write down the resulting asymptotic dynamics of X_i:

$$\begin{aligned} X_i(t) = & \frac{k_{i-1}l_{i-1} - k_i l_i}{k_i + k_{i-1}} \\ & + \frac{k_{i-1}A_{i-1}}{\sqrt{[k_i + k_{i-1} - m_i\omega_{i-1}^2]^2 + b_i m_i \omega_{i-1}^2}} \cos\left(\omega_{i-1}t + \phi_{i-1} - \arctan\left[\frac{b_i\omega_{i-1}}{k_i + k_{i-1} - m_i\omega_{i-1}^2}\right]\right) \\ & + \frac{k_i A_{i+1}}{\sqrt{[k_i + k_{i-1} - m_i\omega_{i+1}^2]^2 + b_i m_i \omega_{i+1}^2}} \cos\left(\omega_{i+1}t + \phi_{i+1} - \arctan\left[\frac{b_i\omega_{i+1}}{k_i + k_{i-1} - m_i\omega_{i+1}^2}\right]\right) . \end{aligned}$$

However, note that if we were using `Dyn` consisting of interventions of the form of equation (4), then we have just shown that the mass-spring system would not be structurally dynamically stable with respect to this `Dyn`, since we need two periodic terms and a constant term to describe the motion of a child under legal interventions of the parents.

This illustrates the fact that we may sometimes be only interested in a particular set of interventions that may not itself satisfy structural dynamic stability, and that in this case we must consider a larger set of interventions that *does*. In this case, we can consider the modular set of trajectories generated by trajectories of the following form for each variable:

$$X_i(t) = \sum_{j=1}^{\infty} A_i^j \cos(\omega_i^j t + \phi_i^j)$$

where for each i it holds that $\sum_{j=1}^{\infty} |A_i^j| < \infty$ (so that the series is absolutely convergent and thus does not depend on the ordering of the terms in the sum). Call this set $\texttt{Dyn}_{qp}$ ("quasi-periodic"). By equation (7), we can write down the structural equations

$$\begin{aligned} & F_i\left(\sum_{j=1}^{\infty} A_{i-1}^j \cos(\omega_{i-1}^j t + \phi_{i-1}^j), \sum_{j=1}^{\infty} A_{i+1}^j \cos(\omega_{i+1}^j t + \phi_{i+1}^j)\right) \\ = & \frac{k_{i-1}l_{i-1} - k_i l_i}{k_i + k_{i-1}} \\ & + \sum_{j=1}^{\infty} \frac{k_{i-1}A_{i-1}^j}{\sqrt{[k_i + k_{i-1} - m_i(\omega_{i-1}^j)^2]^2 + b_i m_i (\omega_{i-1}^j)^2}} \cos\left(\omega_{i-1}^j t + \phi_{i-1}^j - \arctan\left[\frac{b_i\omega_{i-1}^j}{k_i + k_{i-1} - m_i(\omega_{i-1}^j)^2}\right]\right) \\ & + \sum_{j=1}^{\infty} \frac{k_i A_{i+1}^j}{\sqrt{[k_i + k_{i+1} - m_i(\omega_{i+1}^j)^2]^2 + b_i m_i (\omega_{i+1}^j)^2}} \cos\left(\omega_{i+1}^j t + \phi_{i+1}^j - \arctan\left[\frac{b_i\omega_{i+1}^j}{k_i + k_{i+1} - m_i(\omega_{i+1}^j)^2}\right]\right) . \end{aligned}$$

Since this is also a member of $\texttt{Dyn}_{qp}$, the mass-spring system is dynamically structurally stable with respect to $\texttt{Dyn}_{qp}$ and so the equations F_i define the Dynamic Structural Causal Model for asymptotic dynamics.

C DYNAMIC BAYESIAN NETWORK REPRESENTATION

By using Euler's method, we can obtain a (deterministic) Dynamic Bayesian Network representation of the mass-spring system. For $D = 2$, this yields

$$DBN : \begin{cases} X_1^{(t+1)\Delta} = X_1(t\Delta) + \Delta \dot{X}_1(t\Delta) \\ \dot{X}_1^{(t+1)\Delta} = \dot{X}_1(t\Delta) + \frac{\Delta}{m_1}\Big[k_1 X_2(t\Delta) - b_1\dot{X}_1(t\Delta) - (k_0 + k_1)X_1(t\Delta) + k_0 l_0 - k_1 l_1\Big] \\ \\ X_2^{(t+1)\Delta} = X_2(t\Delta) + \Delta \dot{X}_2(t\Delta) \\ \dot{X}_2^{(t+1)\Delta} = \dot{X}_2(t\Delta) + \frac{\Delta}{m_2}\Big[k_1 X_1(t\Delta) - b_2\dot{X}_2(t\Delta) - (k_1 + k_2)X_2(t\Delta) + k_1 l_1 - k_2 l_2 + k_2 L\Big] \\ \\ X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i \quad k \in \{0, 1\},\ i \in \{1, 2\} . \end{cases} \tag{8}$$