Variance-minimal hedging under model risk - A discrete-time approach

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1. Introduction

Hedging life insurance liabilities have always been a challenge, particularly under model risk/misspecification. Neglecting model risk, optimality criteria like risk-minimizing can be adopted to find optimal hedging strategies. For instance, risk-minimizing strategies are able to match the liabilities of an insurance cohort when the cohort is sufficiently large, i.e. the law of large numbers applies (c.f. Föllmer and Sondermann (1996), Möller (1998, 2001), Schweizer (2001)). However, the strategies fail to be (variance) effective if the assumed model and the true model do not coincide. A misspecification of the hedging model gives rise to an additional variance term which cannot be diversified.

Robust hedging strategies are well established in the financial market literature, i.e. in a setup abstract of insurance risk. Just to quote a few, we refer to the papers of Avellaneda et al. (1995), Lyons (1995), Bergman et al. (1996), Dudenhausen et al. (1998), El Karoui et al. (1998), Hobson (1998) and Mahayni (2003). Here, a strategy is called robust if it gives a superhedge in a whole class of models, c.f. also Branger

The topic of model risk shall be called to attention when the question of interest rate and/or mortality rate generating process are studied. When the true models do not exist or are unknown, a meaningful approach to dealing with model risk to find robust strategies (or robust models), i.e. strategies which satisfy certain features with respect to a large class of models or mortality scenarios. For this purpose, Chen and Mahayni (2008) investigate the effectiveness of risk management strategies for endowment assurance products under combined misspecification scenarios (associated with both interest rate and mortality risk). The analysis is carried out by comparing the effectiveness of different hedging strategies, particularly the variance difference of the hedging errors. It is shown that a subset of hedging instruments (zero coupon bonds) may result in a lower variance than the corresponding strategy in all bonds when model misspecification is taken into consideration. Therefore, they address the importance of the combined effects of model risk from both the interest rate and mortality side.

Compared to Chen and Mahayni (2008), the present paper examines the effectiveness of the combined models in a discrete–time setup, i.e. a three–period model setup. We consider (insurance–linked) products where the timing of the payoff depends on a (mortality) model which is independent of the interest rate model. It is worth mentioning that the contracts considered here are more general and they contain pure endowment, term insurance and endowment insurance. In the considered three–period mortality model, the insured dies in the first or the second period or survives the second period. Concerning the interest rate, we work with Ho–Lee (1986) models where the price processes of (all) zero coupon bonds are modeled in a unique binomial model. With respect to the combined model, there exists a whole class of risk–minimizing hedging strategies if the interest rate model is complete. This is explained by the fact that some of the bonds which are needed for hedging can be replaced by a synthesizing strategy in other bonds. However, under model risk (wrong model assumptions), the effectiveness of the (intentionally) risk–minimizing hedging strategies depends on the set of hedging instruments. Or in other words, a robust hedging strategy is based on a suitable choice of the combined (hedging) model. This result is in accordance with Chen and Mahayni (2008). Besides the analysis conducted in Chen and Mahayni (2008) who focus on the variance difference of the hedging errors caused by diverse strategies, the total variance is derived in the present paper.
The remaining of the paper is organized as follows. Section 2 introduces the model setup for both the mortality and interest rate. In Section 3, a set of risk-minimizing strategies are derived. Section 4 examines the effectiveness of the strategies when model misspecification is taken into consideration. The focus is on the comparison of the strategies and under what combinations robust hedging can be achieved. In the following Section 5, we derive the total variance of the hedging errors. Section 6 concludes the paper.

2. Model Setup

All effects of mortality misspecification and interest rate misspecification can be analyzed in a three-period model setup.

2.1. Mortality Model. In a three-period mortality model, only three events are considered: the insured dies in the first, second period and the insured survives the second period. The cash flows under consideration are therefore defined on a probability space \((\Omega^M, \mathcal{F}^M, P^M)\) where

\[ \Omega^M = \{\omega^M_1, \omega^M_2, \omega^M_3\}, \]

\(\{\omega^M_1\}\) describes a premature death at time \(t_1\), \(\{\omega^M_2\}\) a premature death at time \(t_2\) and \(\{\omega^M_3\}\) contains death occurring at time \(t_3\) and survival of the insured. Throughout the paper, we use the superscript \(M\) and \(F\) to denote the parameters under mortality risk and financial market risk respectively. The cash flow \(C(\omega) = (c_1, c_2, c_3)\) where \(c_i\) denotes the payoff at \(t_i\) \((i = 1, 2, 3)\) of an insurance-linked product can be described by

\[
C(\omega^M) = \begin{cases} 
(c_1, 0, 0) & \text{for } \omega^M = \omega^M_1 \\
(0, c_2, 0) & \text{for } \omega^M = \omega^M_2 \\
(0, 0, c_3) & \text{for } \omega^M = \omega^M_3 
\end{cases}
\]

Remark 2.1. The above payoff structure describes an endowment insurance which combines pure endowment and term insurance. This construction is very general because pure endowment and term insurance can result from slight revision of the nature of the states.

Please note the mortality risk enters the hedging problem by the timing of the payoff. In the following we introduce an interest rate model with three features: Firstly, the interest rate model is complete without the introduction of model risk (parameter misspecification). Secondly, model risk can be explained by one wrong model parameter, i.e. the parameter which can be misspecified is the volatility of the short rate. Thirdly, a misspecified volatility is (automatically) consistent with the current term structure of interest. A natural way to assure the above features is given within a Ho–Lee model.

2.2. Interest Rate Model. The interest rate model is given by a Ho–Lee model, c.f. figure 1, where the price processes \(D(., .)\) of (all) zero coupon bonds are modeled in a unique binomial model. \(D_t(j, k+1)\) denotes the price at \(t_j\) of a zero bond with maturity \(t_{j+k+1}\) if there are \(i\) ups of the periodic interest rate, i.e. at \(t_j\) the time to
\[ D_i(j + 1, k + 1) := \frac{D_i(j, k + 1)}{D_i(j, 1)} \cdot h(p, \delta, k) \quad \text{“up”} \]
\[ D_i(j + 1, k) := \frac{D_i(j, k + 1)}{D_i(j, 1)} \cdot h^*(p, \delta, k) \quad \text{“down”} \]

**Figure 1.** Binomial structure of the Ho–Lee model.

maturity is \( t_{k+1} \). Further, \( h \) and \( h^* \) are described by

\[ h(p, \delta, k) = \frac{1}{p + (1-p)\delta^k} \]
\[ h^*(p, \delta, k) = \frac{\delta^k}{p + (1-p)\delta^k} = \delta^k h(p, \delta, k). \]

\( p \) is a constant transition probability (\( p \in [0, 1] \)) and \( \delta \in [0, 1] \), i.e.

\[ \delta = \exp \left\{ \frac{-\sigma}{\sqrt{p(1-p)}} \right\} \]

where \( \sigma \) gives the volatility of the short rate.

**Remark 2.2.** The dynamics are already given under the so–called risk neutral measure, i.e. \( p = p^F = p^*. \)

2.3. **The Combined Model.** Under the assumption that interest rate and mortality risk are independent, the combined model is defined on the product probability space \( \{\Omega, \mathcal{F}, P\} = \{\Omega^M \otimes \Omega^F, \mathcal{F}^M \vee \mathcal{F}^F, P\} \). \( P \) is the probability measure defined on \( \{\Omega^M \otimes \Omega^F, \mathcal{F}^M \otimes \mathcal{F}^F\} \) and for an event \( A \in \mathcal{F} \), it holds \( P(A) = P^M(A)P^F(A) \).

3. **Risk–minimizing Strategies**

3.1. **Definitions.**

**Definition 3.1** (Discrete–time trading strategy). Let \( \phi_k = (\phi_k^{(1)}, \ldots, \phi_k^{(n)}) \) denote the number of bonds which are constantly held in the interval \( [t_k, t_{k+1}] \), \( k = 1, \ldots, n-1 \). The associated value process \( (V_k(\phi))_{k=0,1,\ldots,n} \) is given by

\[ V_0(\phi) := \sum_{i=1}^n \phi_0^{(i)} D(0, i) \]

and

\[ V_k(\phi) := \sum_{i=k}^n \phi_k^{(i)} D(k, i - k) \text{ for } k = 1, \ldots, n \]

The strategy is defined in a \( n \)-period model and the last trading occurs at \( t_{n-1} \).

In reality, due to trading restrictions or high transaction costs, it might be impossible or inconvenient to use all the bonds. Later on we compare the scenario where all the bonds are used with the scenario where the hedger just trades with a subset of bonds. The comparison is carried out by studying the distribution of hedging costs associated with the strategies under model risk. When only a subset of the bonds are used for hedging purpose, it is necessary to duplicate or synthesize the unavailable bonds. We assume that the bond to be synthesized matures prior to the maturity of the bonds used for hedging. In a 3-period model, it means the \( t_1 \)-zero bond is
duplicated by the \( t_2 \)- and \( t_3 \)-zero bonds. The following lemma is dedicated to the duplication of \( t_1 \)-bond in a 3-period Ho-Lee framework.

**Lemma 3.2 (Synthesizing strategy).** Let \( \bar{\phi} = (\bar{\phi}^{(2)}, \bar{\phi}^{(3)}) \) denote a trading strategy where \( \bar{\phi}^{(i)} \) denotes the number of bonds with maturity \( t_i \) (\( i = 2, 3 \)) where

\[
\begin{align*}
\bar{\phi}_0^{(2)} &= \frac{1 + \delta}{\delta h(p^*, \delta, 1)} \frac{D_0(0, 1)}{D_0(0, 2)} \\
\bar{\phi}_0^{(3)} &= -\frac{1}{\delta h(p^*, \delta, 2)} \frac{D_0(0, 1)}{D_0(0, 3)}
\end{align*}
\]

then it holds that \( \bar{\phi} \) is a synthesizing strategy for the bond with maturity \( t_1 \), i.e. \( V_0(\bar{\phi}) = D_0(0, 1) \) and \( V_1(\bar{\phi}) = D(1, 0) = 1 \) a.s..

**Proof:** The proof is given in the appendix, c.f. Appendix A.2

3.2. **Risk-minimizing hedge.** Insurance risk implies that no perfect hedge (a self-financing and at the same time duplicating strategy) can be found to hedge the considered payoff structure. Any hedge leads to a non-vanishing cost process. Moreover, model risk might cause an additional hedging cost.

Insurance risk associated with the insurance risk can be interpreted as the risk that the number of the insured who die at or survive a specified time differs from that calculated for the risk management purposes. This might happen because of an insufficient number of homogenous insurance takers, i.e. the law of large numbers fail to apply to small cohorts. Further, it might result from a wrong choice of mortality law, either on purpose or due to model uncertainty. Please note that the payment structure is based on the realized time of death/survival of the insured. Thus, the effectiveness of a (fixed income) hedging strategy depends on the assumption about whether the realized structure coincides with the assumed one.

Another factor which influences the effectiveness of a hedging strategy is the availability of the hedging instruments. Also the set of admissible strategies depends on the available instruments. Hedging is trivial if the hedging instruments include the claim to be hedged. For instance, a deterministic payoff can be statically hedged if the corresponding zero bonds are traded. In fact, in the ideal case of a large insurance cohort combined with a known mortality law, the payoff structure under consideration can be hedged in a model-independent way if a zero bond is available for each payment date.

Unless there is a perfect hedge, hedging strategies can only be compared according to a meaningful optimality criterium. According to Chen and Mahayni (2008), mean-self-financing is not a sufficient condition to construct a meaning strategy because any strategy where the initial investment coincides with the price of the claim to be hedged is mean-self-financing. Therefore, it is necessary to use an additional optimality criterion. In the following, we consider a conventional hedging criterion used in the incomplete market, i.e., the considered hedging strategies are risk-minimizing if model risk is neglected. Since a strategy which is risk-minimizing with respect to the measure \( P \) is also \( P \)-mean-self-financing, risk-minimizing feature contains
mean–self–financing feature.

In the analysis of risk–minimizing hedging, we look for an admissible strategy which minimizes the variance of the future costs at any time \( t \in [0, T] \). With no model risk, the risk–minimizing hedging strategy for the considered payoff structure is not unique. This is due to the fact that the unavailable hedging instrument can be synthesized by some self–financing strategy (c.f. Lemma 3.2), which leads to no extra hedging costs (variance). Abstracting from model risk together with the simplifying assumption that the real world measure \( P \) and \( P^* \) coincide, the criteria are formulated with respect to \( P^* \). Combined with Lemma 3.2, the set of risk–minimizing hedging strategies are determined in the following proposition.

**Proposition 3.3. (Set of risk–minimizing strategies)** Under the assumed probability measure \( \tilde{P} \), the risk–minimizing strategies \( \Phi^* = \{ \phi^*_t(\alpha) : t \in \{t_0, t_1, t_2\} \text{ and } \alpha \in [0,1] \text{ Big} \} \) are not unique. The non–uniqueness of the risk–minimizing strategies result from the synthesizing of the \( t_1 \)-bond. More specifically, the initial hedge is established as follows

\[
\phi^{(i)}_{t_0}(\alpha) = \tilde{\phi}^{(i)}_{t_0} + \alpha \delta h \tilde{\phi}^{(1)}_{t_0} \tilde{\phi}^{(i)} \\
\tilde{\phi}^{(1)} = -1 \\
\tilde{\phi}^{(2)} = \frac{1 + \delta}{\delta h(\tilde{\omega}^*, \tilde{\omega}, 1)} D_0(0,1) \\
\tilde{\phi}^{(3)} = -\frac{1}{\delta h(\tilde{\omega}^*, \tilde{\omega}, 2)} D_0(0,1)
\]

Immediately after \( t_1 \), the portfolio is rearranged according to

\[
\phi^{(i)}_t(M) = \phi^{(i)}_{t_1}(M), \phi^{(i)}_{t_1}(M), \phi^{(i)}_{t_1}(M) = \begin{cases} (0, \tilde{P}(\omega^M_1, \omega^M_2, \omega^M_3)) = (c_1, 0, 0) & \text{ for } \omega^M = \omega^M_1 \\
(0, \tilde{P}(\omega^M_1, \omega^M_2, \omega^M_3)) = (c_2, 0, 0) & \text{ for } \omega^M = \omega^M_2 \\
(0, \tilde{P}(\omega^M_1, \omega^M_2, \omega^M_3)) = (c_3, 0, 0) & \text{ for } \omega^M = \omega^M_3 \\
\end{cases}
\]

The last rebalancing occurs at time \( t_2 \) given that no premature death happens at \( t_2 \):

\[
\phi^{(i)}_{t_2}(M) = \phi^{(i)}_{t_2}(M), \phi^{(i)}_{t_2}(M), \phi^{(i)}_{t_2}(M) = \begin{cases} (0, 0, 0) & \text{ for } \omega^M = \omega^M_2 \\
(0, c_2, 0) & \text{ for } \omega^M = \omega^M_2 \\
(0, 0, c_3) & \text{ for } \omega^M = \omega^M_3 \\
\end{cases}
\]

or more precisely,

\[
\phi^{(i)}_{t_2} = \tilde{P}(\omega^M_i | \mathcal{F}^M_{t_2}) c_i \quad i = 2, 3
\]

**Proof:** Since there is no asset risk, the resulting risk-minimizing strategy is quite trivial. According to Møller (2001), the optimal number of \( t_i \)-bond \((i = 1, 2, 3)\) at \( t_j \) \((j = 0, 1, 2)\) corresponds to the product of the (assumed) conditional probabilities of the mortality events and the due payout at that time. The fact that \( t_1 \)-bond can be synthesized by \( t_2 \) and \( t_3 \) bonds leads to the non-uniqueness of the risk-minimizing strategies.
First, please note that for the pricing and hedging purposes, the insurer indeed uses assumed models/parameters. The derivation of the strategies is given with respect to the (assumed) measure \( \hat{P} \). All the hedging strategies are composed according to an assumed model where the assumed parameters are denoted by tilde. Concerning the mortality model, the strategy is therefore based on the assumed probabilities \( \hat{P}(\omega_i^M) \) instead of the “true” ones \( P(\omega_i^M) \).

Further, there are three possible trading dates \( t_0, t_1, t_2 \). Usually, a premature death occurs \( (\omega_1^M or \omega_2^M \text{ arise}) \), hedging stops at the death–occurring time. In order to simplify the calculation of the variance of the hedging cost in Section 5, we assume that immediately after the premature death–occurring time, the hedger buys exactly \( c_i \) (if \( t_i \) is the premature death time) in order to pay back the corresponding death benefits. By doing this, we do not have duplication costs at time \( t_1 \) and \( t_2 \), but just rebalancing costs. The strategy at time \( t_1 \) and \( t_2 \) are based on conditional consideration, i.e. conditional on the realizations of \( \omega_i^M \) at time \( t_1 \) and \( t_2 \). Particularly, provided that \( \omega_1^M \) has not occurred at time \( t_1 \) (or both \( \omega_1^M \) and \( \omega_2^M \) not yet occurred at time \( t_2 \)), i.e. the strategy is based on the survival probability at time \( t_1 \) (or \( t_2 \)). That is why \( (1 - \hat{P}(\omega_1^M)) \) stands in the denominator for the strategy at \( t_1 \). Contingent on the survival of \( t_2 \), the event of \( \omega_3^M \) occurs with certainty.

The strategies in the set differ only by the initial hedge because from time \( t_1 \) on, no synthesizing is necessary. The set of the initial hedge shows very flexible pattern. For \( \alpha = 0 \), \( \phi^*_0(\alpha) \) becomes \( \hat{\phi}_{t_0} = (\hat{\phi}^{(1)}_{t_0}, \hat{\phi}^{(2)}_{t_0}, \hat{\phi}^{(3)}_{t_0}) \), i.e. \( \hat{\phi} \) is the risk–minimizing strategy where all the three bonds are used for the hedging purpose and no synthesizing is used. The determination of \( \hat{\phi} \) can be straightforward conducted by the \( t_i \)-value \( (i = 0, 1, 2) \) of the payoff structure \( C(\omega^M) \). Whereas \( \alpha = 1 \) indicates the other extreme case, i.e. the risk–minimizing hedge is composed without the use of the zero bond with maturity \( t_1 \). Finally \( \alpha \in (0, 1) \), part of \( t_1 \)-bond is duplicated. Therefore, we call \( \alpha \) the replacement rate. The construction of \( \phi^*(\alpha) \) is derived from \( \hat{\phi} \) by replacing the unavailable part \( (\alpha \cdot \hat{\phi}_{t_0}^{(1)} \ t_1 \text{-bond}) \) by the \( t_2 \)– and \( t_3 \)–bond.

Except the initial time, the strategies in the set take the same values at the rest of time points. Therefore, if the goal is oriented towards a comparison among the set of risk–minimizing strategies under model risk, it is sufficient to compare the initial hedge of the strategies. For the analysis in the following section, we drop the subscript \( t_0 \) and use the simplified notations \( \hat{\phi}, \phi^*, \phi^{*, i} \) to denote \( \hat{\phi}_{t_0}, \phi^*_{t_0}, \phi^{*, i}_{t_0} \).

4. Expected Costs and Variance under Model Misspecification

Different from pricing and hedging, the efficiency of a strategy is usually examined under the true measure \( P^* \). In the following, we focus on examining the effectiveness of the risk–minimizing strategies stated in Proposition 3.3 under model risk by studying the distributions of the hedging errors associated with the strategies.

Concerning the mortality model, the strategy is therefore based on the assumed probabilities \( \hat{P}(\omega_i^M) \) instead of the “true” ones \( P(\omega_i^M) \), c.f. Proposition 3.3. Regarding the model risk related to the interest rate model, the hedging strategies are composed according to the parameter \( \tilde{\sigma} \) instead of the true parameter \( \sigma \) (\( \tilde{\delta} \) instead of \( \delta \)). Particularly, we shall mention that the synthesizing strategy given in Lemma
3.2 is constructed with the assumed $\tilde{\delta}$ (or $\tilde{\sigma}$).

**Lemma 4.1.** The discounted hedging cost associated with $\tilde{\phi}$ is determined by

$$L^*_1(\tilde{\phi}) = -G^*_1(\tilde{\phi}),$$

where $G^*_1$ denotes the discounted gain associated with $\tilde{\phi}$. In particular, it holds

$$E \left[ L^*_1(\tilde{\phi}) \right] = 0$$
$$\text{Var} \left[ L^*_1(\tilde{\phi}) \right] = E \left[ (L^*_1(\tilde{\phi}))^2 \right] = E \left[ (G^*_1(\tilde{\phi}))^2 \right].$$

**Proof:** The discounted hedging cost at time $t$ is defined as the difference between the discounted portfolio value and the discounted gain at that time minus the initial value of the strategy, i.e.,

$$L^*_1(\tilde{\phi}) = V^*_1(\tilde{\phi}) - G^*_1(\tilde{\phi}) - V_0(\tilde{\phi})$$
$$= V_1(\tilde{\phi})D(0, 1) - G^*_1(\tilde{\phi}) - D(0, 1)$$
$$= -G^*_1(\tilde{\phi}) = \sum_{i=2}^{3} \tilde{\phi}^{(i)}_0 (D^*(1, i - 1) - D^*(0, i))$$

From line 2 to 3 we use $V_1(\tilde{\phi}) = 1$. Further, due to $E[D^*(1, i - 1)] = D(0, i)$, we obtain immediately $E \left[ L^*_1(\tilde{\phi}) \right] = 0$ and $\text{Var} \left[ L^*_1(\tilde{\phi}) \right] = E \left[ (G^*_1(\tilde{\phi}))^2 \right].$

The effectiveness of the risk-minimizing hedging strategies can be investigated by looking at the distribution of the hedging errors generated by the strategies under the true measure $P^*$ (or $P$). In the remaining of this section, we first carry out a comparative analysis, i.e. we compare strategies with a non–zero replacement rate $\alpha$ with the basic strategy $\hat{\phi} = \phi^*(\alpha = 0)$. In Section 5, we calculate the total variance of the strategies.

**Definition 4.2 (Variance difference).** The variance difference $AV(\alpha)$ is defined by the difference of the variance of the hedging errors generated by $\phi^*(\alpha)$ ($\alpha > 0$) and of $\hat{\phi}$, i.e.

$$AV(\alpha) = \text{Var} \left[ L^*(\phi^*(\alpha)) \right] - \text{Var} \left[ L^*(\hat{\phi}) \right]$$

Obviously, $AV(\alpha) = 0$ when there exists no model misspecification.

**Lemma 4.3.** Let

$$f \left( p^*, \delta, \tilde{\delta} \right) := (1 + \tilde{\delta})(1 - \delta) \frac{h(1)}{h(1)} - (1 - \delta^2) \frac{h(2)}{h(2)}$$

then it holds

$$f \left( p^*, \delta, \tilde{\delta} \right) = (1 - \delta) h(1) h(2) \left[ p^* \left( \tilde{\delta} - \delta \right) + (1 - p^*) \left( \tilde{\delta}^2 - \delta^2 \right) \right],$$

with $h(i) := h(p^*, \delta, i)$.

**Proof:** The proof is given in the appendix, c.f. Appendix A.3.
**Proposition 4.4 (Additional Variance).**

\[
AV = \left(\alpha \hat{\phi}_0^{(1)}\right)^2 V + 2\alpha \hat{\phi}_0^{(1)} CV
\]

where

\[
V = E \left(\left(G^*_1(\hat{\phi})\right)^2\right)
\]

\[
CV = E \left[G^*_1(\hat{\phi})G^*_1(\bar{\phi}) - \sum_{i=2}^{3} P(\omega^M_i) c_i E[D^*(1, i-1)G^*_1(\bar{\phi})]\right]
\]

In particular, it holds

\[
V = p^*(1-p^*) (D(0, 1))^2 \frac{f^2(p^*, \delta, \bar{\delta})}{\bar{\delta}^2}
\]

\[
CV = p^*(1-p^*) D(0, 1) \frac{f(p^*, \delta, \bar{\delta})}{\bar{\delta}} \times \sum_{i=2}^{3} c_i \left(P(\omega^M_i) - P(\omega^M_i)\right) D(0, i) h(i-1)(1 - \delta_i^{-1})
\]

**Proof:** The proof is given in the appendix, c.f. Appendix A.4.

**Corollary 4.5 (Effects of model misspecification).** For \(p^* = 0.5\) it holds

(a) In the case that there is no model misspecification of the interest rate process \((\bar{\delta} = \bar{\sigma} = \sigma)\), it holds \(AV = 0\).

(b) In the case that there is no model misspecification of the mortality law \((\bar{P} = P)\), the additional variance is simply given by the variance of \(\alpha \hat{\phi}_0^{(1)}\) times the discounted hedging costs of the bond synthesizing strategy, i.e.

\[
AV = \left(\alpha \hat{\phi}_0^{(1)}\right)^2 E \left[\left(G^*_1(\bar{\phi})\right)^2\right] = Var[\alpha \hat{\phi}_0^{(1)} G^*_1(\bar{\phi})] > 0
\]

(c) Concerning the interesting case, i.e. model misspecification exists for both the interest rate process and the mortality law, the “additional variance” (variance difference) can be positive as well as negative. In particular, \(AV > 0\) if \(\bar{P}(\omega^M_i) > P(\omega^M_i)\) for \(i = 2, 3\) in combination with \(\bar{\delta} > \delta\).

**Proof:** The result follows immediately from the observation that

\[
f(0.5, \delta, \bar{\delta}) = \begin{cases} 
> 0 & \text{for } \bar{\delta} > \delta \\
= 0 & \text{for } \bar{\delta} = \delta \\
< 0 & \text{for } \bar{\delta} < \delta.
\end{cases}
\]

Combining this observation with Proposition 4.4 the result follows immediately.

Figures 2 and 3 illustrate the effects stated in Corollary 4.5. Figure 2 demonstrates how the variance difference depends on the assumed volatility of the spot rate when the assumed and true mortality law coincide. The variance difference is then reduced to the product of \(\left(\alpha \hat{\phi}_0^{(1)}\right)^2\) and the variance of the synthesizing strategy. In this case, the variance difference is always positive and reaches highest value when \(\alpha = 1\) or \(-1\). The variance difference depends on the absolute value of \(\alpha\) only. That is why the variance difference curve for \(\alpha = 0.5\) overlap with that for \(\alpha = -0.5\). Figure 3 gives
an illustration of the variance difference when model misspecification exists for both the interest rate or mortality model. As an example, the assumed \( \bar{\sigma} = 1 \), \( \delta = \frac{1}{2} \), \( \alpha = 0.5 \), \( \tilde{\sigma} = 1 \) gives the general result. Hence, as \( \bar{\sigma} \) moves from the area \( \bar{\sigma} < \sigma \) to \( \bar{\sigma} = \sigma \) and further to \( \bar{\sigma} > \sigma \), the variance difference \( \text{AV} \) moves from positive to zero and further to negative values.

5. Total Variance under Model Risk

In the last section we have calculated the additional variance which arises in the case \( \alpha \neq 0 \) instead of \( \alpha = 0 \). We are now interested in the calculation of the total variance of the hedging error under model risk. To achieve the result, it is enough to consider the variance in the case \( \alpha = 0 \). Adding the additional variance \( \text{AV}(\alpha) \) then gives the general result.

First, let us recall the strategy \( \phi^* \) when \( \alpha = 0 \). The number held in \( t_j \)-bond (for \( j = 1, 2, 3 \)) at each time \( t_i, i = 0, 1, 2 \) is given by (c.f. Proposition (3.3)):

\[
\phi_{t_0}^{(s_j)} = \tilde{P}(\omega_j^M)c_j; \quad \phi_{t_1}^{(s_j)} = \tilde{P}(\omega_j^M|\mathcal{F}_{t_1}^M)c_j; \quad \phi_{t_2}^{(s_j)} = \tilde{P}(\omega_j^M|\mathcal{F}_{t_2}^M)c_j.
\]

The discounted cost associated with the strategy \( \phi^* \) can be expressed either as the difference of the value and the gain process or as follows:

\[
L^*(\phi^*) = \sum_{i=1}^{3} (\phi_{t_2}^{(s_i)} - \phi_{t_1}^{(s_i)})D^*(1, i - 1) + \sum_{i=1}^{3} (\phi_{t_1}^{(s_i)} - \phi_{t_0}^{(s_i)})D^*(0, i)
\]

\[
= \sum_{i=1}^{3} (\tilde{P}(\omega_i^M|\mathcal{F}_{t_2}^M)c_i - \tilde{P}(\omega_i^M|\mathcal{F}_{t_1}^M)c_i)D^*(1, i - 1)
\]

\[
+ \sum_{i=1}^{3} (\tilde{P}(\omega_i^M|\mathcal{F}_{t_1}^M)c_i - \tilde{P}(\omega_i^M)c_i)D^*(0, i)
\]
Based on the realizations of the mortality risk, the discounted cost can be rewritten as

\[ L^*(\phi^*) = 1_{\{\omega_1^M\}} \sum_{i=1}^{3} \tilde{P}(\omega_i^M) D^*(0, i) + 1_{\{\omega_2^M\}} (c_2 D^*(1, 1) - A) + 1_{\{\omega_3^M\}} (c_3 - A) \]

with \( A := -\sum_{i=2}^{3} \frac{\tilde{P}(\omega_i^M)}{1 - \tilde{P}(\omega_i^M)} c_i D^*(1, i - 1) + \sum_{i=2}^{3} \frac{\tilde{P}(\omega_i^M)}{1 - \tilde{P}(\omega_i^M)} c_i D^*(0, i). \)

It is noted that \( E[A] = 0 \) because it holds \( E[D^*(1, i - 1)] = D(0, i) \). The variance of \( L^*(\phi^*) \) is derived as follows

\[
\text{Var}[L^*] = \text{Var} \left[ 1_{\{\omega_1^M\}} \sum_{i=1}^{3} \tilde{P}(\omega_i^M) D^*(0, i) + 1_{\{\omega_2^M\}} (c_2 D^*(1, 1) - A) + 1_{\{\omega_3^M\}} (c_3 - A) \right]
\]

\[
= \left( \sum_{i=1}^{3} \tilde{P}(\omega_i^M) D^*(0, i) \right)^2 P(\omega_i^M)(1 - P(\omega_i^M))
\]

\[
+ P(\omega_2^M) E \left[ (c_2 D^*(1, 1) - A)^2 \right] - (P(\omega_2^M))^2 (E[c_2 D^*(1, 1) - A])^2
\]

\[
+ P(\omega_3^M) E \left[ (c_3 - A)^2 \right] - (P(\omega_3^M))^2 (E[c_3 - A])^2
\]

\[
+ 2P(\omega_1^M) \left( \sum_{i=1}^{3} \tilde{P}(\omega_i^M) D^*(0, i) \right) \left\{ P(\omega_2^M) E[c_2 D^*(1, 1) - A]
\]

\[
+ P(\omega_3^M) E[c_3 - A] \right\} + 2P(\omega_2^M) P(\omega_3^M) E[c_2 D^*(1, 1) - A] E[c_3 - A]
\]

\[
= \left( \sum_{i=1}^{3} \tilde{P}(\omega_i^M) D^*(0, i) \right)^2 P(\omega_i^M)(1 - P(\omega_i^M))
\]

\[
+ P(\omega_2^M) E \left[ (c_2 D^*(1, 1) - A)^2 \right] - (P(\omega_2^M))^2 (c_2 D^*(0, 2))^2
\]

\[
+ P(\omega_3^M) E \left[ (c_3 - A)^2 \right] - (P(\omega_3^M))^2 (c_3)^2
\]

\[
+ 2P(\omega_1^M) \left( \sum_{i=1}^{3} \tilde{P}(\omega_i^M) D^*(0, i) \right) \left\{ P(\omega_2^M) c_2 D^*(0, 2)
\]

\[
+ P(\omega_3^M) c_3 \right\} + 2P(\omega_2^M) P(\omega_3^M) (c_2 D^*(0, 2)) c_3
\]

with

\[
E \left[ (c_2 D^*(1, 1) - A)^2 \right] = c_2^2 E[(D^*(1, 1))^2] - 2c_2 E[AD^*(1, 1)] + E[A^2]
\]

\[
= c_2^2 D(0, 2)^2 h(1)(p^* + \delta^2(1 - p^*))^2 + 2c_2 \sum_{i=2}^{3} \frac{\tilde{P}(\omega_i^M)}{1 - \tilde{P}(\omega_i^M)} c_i D(0, i) D(0, 2)
\]

\[
[h(i - 1)h(1)(p^* + \delta^{-1} \delta^1(1 - p^*)) - 1] + \sum_{i=2}^{3} \left( \frac{\tilde{P}(\omega_i^M)}{1 - \tilde{P}(\omega_i^M)} \right) c_i^2 (D(0, 1))^2
\]

\[
p^*(1 - p^*) (D_0(1, i - 1) - D_1(1, i - 1))^2
\]

\[
+ \frac{2\tilde{P}(\omega_2^M) \tilde{P}(\omega_3^M)}{(1 - \tilde{P}(\omega_1^M))^2} D(0, 2) D(0, 3) [h(1)h(2)(p^* + \delta^3(1 - p^*)) - 1]
\]
\[
E \left[ (c_3 - A)^2 \right] = c_3^2 - 2c_3 E[A] + E[A^2] = c_3^2 + \text{Var}[A]
\]

\[
= c_3^2 + \sum_{i=2}^{3} \left( \frac{\tilde{P}(\omega_i^M)}{1 - \tilde{P}(\omega_i^M)} \right) c_2^2 (D(0,1))^2 p^*(1 - p^*)(D_0(1, i - 1) - D_1(1, i - 1))^2
\]

\[
+ 2 \frac{\tilde{P}(\omega_2^M)\tilde{P}(\omega_3^M)}{(1 - \tilde{P}(\omega_1^M))^2} D(0,2)D(0,3)[h(1)h(2)(p^* + \delta^3(1 - p^*)) - 1].
\]

For the above derivations we have used

\[
\text{Var}[D^*(1, i)] = \text{Var}[D(0, 1)D(1, i)] = (D(0,1))^2 p^*(1 - p^*)(D_0(1, i - 1) - D_1(i, i - 1))^2
\]

\[
\text{Cov}[D^*(1, 1)D^*(1, 2)] = D(0, 2)D(0, 3)[h(1)h(2)(p^* + \delta^3(1 - p^*)) - 1].
\]

Table 5 illustrates the total variance (for \(\alpha = 0\)) and the variance difference \(AV_T\) for several mortality scenarios. The total variance depends on the real volatility of the bond and both the assumed and true mortality model (but not on the assumed term structure model). The more the survival probability (occurrence of \(\omega_3^M\)) is underestimated, the lower the resulting total variance.

Although the variance difference \(AV_T\) is quite small compared to the total variance. However, due to its non-monotonic sign, this term is non-negligible, particularly when there exists mortality misspecification. According to Proposition 4.4, the variance difference \(AV_T\) can be decomposed into two parts, a variance part \(V\) and two times a covariance part \(CV\), where \(V\) is the variance of synthesizing costs. When there exists no mortality misspecification, \(AV_T\) is reduced to the variance of synthesizing costs. This is why always positive \(AV_T\) is observed in this case, expect that \(AV_T = 0\) when there exists no model risk with the term structure of the interest rate (c.f. figure 2). The larger the extent of misspecification, i.e. the higher the distance \(|\tilde{\sigma} - \sigma|\), the higher is the variance of the synthesizing costs. However, the magnitude of this variance is pretty small, in particular in relation to the total variance difference. Therefore, the variance of synthesizing costs does not play a crucial role for \(AV_T\).

Second, when mortality model is misspecified, \(AV_T\) could demonstrate both negative or positive sign due to the covariance component. Observe that \(CV\) plays a deciding role in the magnitude of \(AV_T\). The sign is decided through the combined model (c.f. Corollary 4.5). For instance, an overestimation of the volatility of the bond (\(\tilde{\sigma} > \sigma\)) combined with an overestimation of death probabilities (the occurrence of \(\omega_1^M\) and \(\omega_2^M\)) will lead to a negative variance difference, i.e. by using less bonds, the hedger is indeed better off. He would be even better off when there is an underestimation of the volatility of the bond (\(\tilde{\sigma} < \sigma\)) and at the same time an overestimation of the survival probability (occurrence of \(\omega_3^M\)).

6. Conclusion

The paper examines the effectiveness of the combined term structure and mortality model in a three–period model setup. With respect to the combined model, a class of risk–minimizing hedging strategies can be found if no model misspecification
is available. However, under model risk (wrong model assumptions/parameter misspecifications), the effectiveness of the originally risk-minimizing hedging strategies cannot always be retained and it highly depends on the choice of the hedging instruments and the combined (hedging) model. This result is in accordance with Chen and Mahayni (2008). Further, this paper conducts not only a comparative analysis of the variance difference of the hedging errors (associated with different strategies), but calculates the total variance of the hedging errors.

<table>
<thead>
<tr>
<th>Mortality combination</th>
<th>Total variance ($10^8$)</th>
<th>$\sigma = 0.01$</th>
<th>$\sigma = 0.02$</th>
<th>$\sigma = 0.03$</th>
<th>$\sigma = 0.01$</th>
<th>$\sigma = 0.02$</th>
<th>$\sigma = 0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = (0.2, 0.2, 0.6)$</td>
<td></td>
<td>4.06</td>
<td>4.03</td>
<td>4.01</td>
<td>0.0780072</td>
<td>0</td>
<td>0.70074</td>
</tr>
<tr>
<td>$\tilde{P} = (0.2, 0.2, 0.6)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P = (0.1, 0.1, 0.8)$</td>
<td></td>
<td>2.65</td>
<td>2.64</td>
<td>2.63</td>
<td>-51.50</td>
<td>0</td>
<td>479.63</td>
</tr>
<tr>
<td>$\tilde{P} = (0.2, 0.2, 0.6)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P = (0.3, 0.3, 0.4)$</td>
<td></td>
<td>4.22</td>
<td>4.18</td>
<td>4.15</td>
<td>51.66</td>
<td>0</td>
<td>-478.23</td>
</tr>
<tr>
<td>$\tilde{P} = (0.2, 0.2, 0.6)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Table 1. Total variance and $AV_T$ for different mortality scenarios with parameters: $p = 0.2; r = 0.03; c = \{10000, 20000, 30000\}, \tilde{\sigma} = 0.02; \alpha = 1.$
A.1. Useful results.

**Lemma A.1.**

\[
E[D^*(1, i - 1)D^*(1, j - 1)] = D_0(0, i)D_0(0, j)h(i - 1)h(j - 1)[p^* + \delta^{i-1}\delta^{j-1}(1 - p^*)],
\]

\[
E[D^*(1, i - 2)D^*(2, j - 2)] = E[D^*(1, i - 2)D^*(1, j - 1)]
\]

\[
= D_0(0, i - 1)D_0(0, j)h(i - 2)h(j - 1)[p^* + \delta^{i-2}\delta^{j-1}(1 - p^*)],
\]

with \(h(i) := h(p^*, \delta, i)\). In particular

\[
E[D^*(1, i - 1)D^*(1, i - 1)] = (D_0(0, i))^2 \left(p^* h^2(i - 1) + (1 - p^*)h^2(i - 1)\delta^2(i-1)\right).
\]

**Proof:**

\[
E[D^*(1, i - 1)D^*(1, j - 1)] = E[D_0(0, 1)D(1, i - 1)D_0(0, 1)D(1, j - 1)]
\]

\[
= (D_0(0, 1))^2 [p^*D_1(1, i - 1)D_1(1, j - 1) + (1 - p^*)D_0(1, i - 1)D_0(1, j - 1)].
\]

The result follows immediately with

\[
D_1(1, i - 1) = \frac{D_0(0, i)}{D_0(0, 1)}h(i - 1)
\]

\[
D_0(1, i - 1) = \frac{D_0(0, i)}{D_0(0, 1)}h^*(i - 1) = \frac{D_0(0, i)}{D_0(0, 1)}\delta^{i-1}h(i - 1)
\]

Similarly we use the simplified notations \(h(i)\) and \(h^*(i)\) to denote \(h(p^*, \delta, i)\) and \(h^*(p^*, \delta, i)\).

A.2. **Proof of Lemma 3.2.** Notice that the condition \(V_1(\tilde{\phi}) = D(1, 0) = 1\) imply

\[
\tilde{\phi}_0^{(2)} D_1(1, 1) + \tilde{\phi}_0^{(3)} D_1(1, 2) = 1
\]

\[
\tilde{\phi}_0^{(2)} D_0(1, 1) + \tilde{\phi}_0^{(3)} D_0(1, 2) = 1
\]

i.e.

\[
\tilde{\phi}_0^{(2)} D_0(0, 2) D_0(0, 1) h(p^*, \delta, 1) + \tilde{\phi}_0^{(3)} D_0(0, 3) D_0(0, 1) h(p^*, \delta, 2) = 1
\]

\[
\tilde{\phi}_0^{(2)} \delta D_0(0, 2) D_0(0, 1) \delta h(p^*, \delta, 1) + \tilde{\phi}_0^{(3)} D_0(0, 3) D_0(0, 1) \delta^2 h(p^*, \delta, 2) = 1
\]

which gives immediately the above representation. Finally, notice that

\[
V_0(\tilde{\phi}) := \tilde{\phi}_0^{(2)} D_0(0, 2) + \tilde{\phi}_0^{(3)} D_0(0, 3)
\]

\[
= \left(\frac{1 + \delta}{\delta h(p^*, \delta, 1) - \delta h(p^*, \delta, 2)}\right) D_0(0, 1)
\]

\[
= \left[\frac{1 + \delta}{\delta} (p^* + \delta(1 - p^*)) - \frac{1}{\delta} (p^* + \delta^2(1 - p^*))\right] D_0(0, 1) = D_0(0, 1)
\]
A.3. **Proof of Lemma 4.3.** Notice that

\[(1 + \tilde{\delta})(1 - \delta)\frac{h(1)}{h(1)} - (1 - \delta^2)\frac{h(2)}{h(2)}\]

\[= (1 - \delta)\left[ (1 + \tilde{\delta})\frac{h(1)}{h(1)} - (1 + \delta)\frac{h(2)}{h(2)} \right] \]

\[= (1 - \delta)\left[ (1 + \tilde{\delta})\frac{p + (1 - p)\tilde{\delta}}{p + (1 - p)\delta} - (1 + \delta)\frac{p + (1 - p)\tilde{\delta}^2}{p + (1 - p)\delta^2} \right] \]

\[= (1 - \delta)h(1)h(2)\left[ (1 + \tilde{\delta})(p + (1 - p)\tilde{\delta})(p + (1 - p)\delta^2) \right. \]

\[\left. - (1 + \delta)(p + (1 - p)\delta)(p + (1 - p)\tilde{\delta}^2) \right] \]

\[= (1 - \delta)h(1)h(2)\left[ (p + \tilde{\delta} + (1 - p)\tilde{\delta}^2)(p + (1 - p)\delta^2) \right. \]

\[\left. - (p + \delta + (1 - p)\delta^2)(p + (1 - p)\tilde{\delta}^2) \right] \]

\[= (1 - \delta)h(1)h(2)\left[ p\left( \tilde{\delta} - \delta \right) + (1 - p)\left( \tilde{\delta}^2 - \delta^2 \right) \right] \]

A.4. **Proof of Proposition 4.4.** For the following derivation, \(L^* := L^*(\phi^*(\alpha))\)

First, notice that

\[L^*(\phi^*) = \sum_{i=1}^{3} 1_{\{\omega_i^M\}}(c_i^* - (V_0(\phi^*) + G_i^*(\phi^*)))\]

such that

\[E[L^*(\phi^*)] = \sum_{i=1}^{3} P(\omega_i^M)c_i D(0, i) - V_0(\phi^*)\]

\[E[(L^*(\phi^*))^2] = \sum_{i=1}^{3} P(\omega_i^M)E_F\left[ (c_i^* - (V_0(\phi^*) + G_i^*(\phi^*))^2 \right] \]

In particular, we have

\[E[(L^*(\phi^*))^2] = \sum_{i=1}^{3} P(\omega_i^M)E_F\left[ (c_i^* - (V_0(\phi^*) + G_i^*(\phi^*(\omega_i^M))))^2 \right] \]

Using

\[V_0(\phi^*) = V_0(\hat{\phi})\]

\[G_i^*(\phi^*(\omega_i^M)) = G_i^*(\hat{\phi}(\omega_i^M)) + \alpha_0(\omega_i^M)G_i^*(\hat{\phi})\]

First here \(G_i, i\) denotes both time and state of nature. For time 1 and 2, the strategy \(\phi^*\) and \(\hat{\phi}\) indeed have the same value, only the initial hedge differs. That means only when the payoff of the contract occurs at time 1, i.e. \(\omega_1\) occurs, the gain and consequently the cost differs from these two strategies \(\phi^*\) and \(\hat{\phi}\) gives
\[ E[(L^*(\phi^*))^2] = \sum_{i=1}^{3} P(\omega_i^M) E_F \left[ (c_i^* - V_0(\phi) - G_i^*(\phi(\omega_i^M)) - \alpha \phi_0^{(1)} G_1^*(\phi))^2 \right] \]

\[ = \sum_{i=1}^{3} P(\omega_i^M) E_F \left[ (c_i^* - V_0(\phi) - G_i^*(\phi(\omega_i^M)))^2 \right] \]

\[ + \sum_{i=1}^{3} P(\omega_i^M) \left( \alpha \phi_0^{(1)} \right)^2 E_F \left[ (G_1^*(\phi))^2 \right] \]

\[ -2 \alpha \phi_0^{(1)} \sum_{i=1}^{3} P(\omega_i^M) E_F \left[ G_i^*(\phi)(c_i^* - V_0(\phi) - G_i^*(\phi(\omega_i^M))) \right] \]

\[ = \ E[(L^*(\phi))^2] + \left( \alpha \phi_0^{(1)} \right)^2 E_F \left[ (G_1^*(\phi))^2 \right] \]

\[ -2 \alpha \phi_0^{(1)} \sum_{i=1}^{3} P(\omega_i^M) E_F \left[ G_i^*(\phi)c_i^* \right] + 2 \alpha \phi_0^{(1)} \sum_{i=1}^{3} P(\omega_i^M) E_F \left[ G_1^*(\phi)G_i^*(\phi) \right] \]

\[ = \ E[(L^*(\phi))^2] + \left( \alpha \phi_0^{(1)} \right)^2 E_F \left[ (G_1^*(\phi))^2 \right] \]

\[ + 2 \alpha \phi_0^{(1)} E_F \left[ G_1^*(\phi)G_i^*(\phi) \right] - 2 \alpha \phi_0^{(1)} \sum_{i=2}^{3} P(\omega_i^M) E_F \left[ G_1^*(\phi)c_i^* \right] \]

where the last equality is due to the observation that

\[ E_F \left[ G_i^*(\phi)c_i^* \right] = c_i D(0, 1) E_F \left[ G_i^*(\phi) \right] = 0. \]

With respect to the calculation of \( E \left[ (G_1^*(\phi))^2 \right] \) it is convenient to recall that

\[ E \left[ (G_1^*(\phi))^2 \right] = V_{ar} \left[ G_1^*(\phi) \right] \]

and, because of \( V_1^*(\phi) = 0 \), that

\[ G_1^*(\phi) = \sum_{i=1}^{3} \phi_0^{(1)} D^*(1, i - 1). \]

Observing that there are only two possible realizations of \( G_1^*(\phi) \), we immediately have

\[ V_{ar} \left[ G_1^*(\phi) \right] = p^*(1 - p^*) \left( \sum_{i=1}^{3} \phi_0^{(i)} (D_1^*(1, i - 1) - D_0^*(1, i - 1)) \right)^2 \]

\[ = p^*(1 - p^*) \left( \sum_{i=1}^{3} \phi_0^{(i)} D_0(0, i) h(i - 1)(1 - \delta^{i-1}) \right)^2 \]

Inserting \( \bar{\phi}_0 = (\bar{1}, \frac{1 + \delta}{\delta h(1)} \frac{D_0(0, 1)}{D(0, 2)}, -\frac{1}{\delta h(2)} \frac{D_0(0, 1)}{D(0, 3)}) \) immediately gives

\[ E \left[ (G_1^*(\phi))^2 \right] = p^*(1 - p^*) (D(0, 1))^2 \frac{1}{\delta^2} \left[ (1 + \delta)(1 - \delta) \frac{h(1)}{h(1)} - (1 - \delta^2) \frac{h(2)}{h(2)} \right]^2 \]
Analogous arguments give

\[ E \left[ G_1^*(\hat{\phi})G_1^*(\bar{\phi}) \right] = p^*(1 - p^*)D(0, 1) \frac{1}{\delta} \left[ (1 + \bar{\delta})(1 - \hat{\delta}) \frac{h(1)}{h(1)} - (1 - \hat{\delta}^2) \frac{h(2)}{h(2)} \right] \]

\[ \times \sum_{i=2}^{3} c_i \bar{P}(\omega_i^M)D(0, i)h(i - 1)(1 - \delta^{-i}) \]

and

\[ \sum_{i=2}^{3} P(\omega_i^M)c_i E[D^*(1, i - 1)G_1^*(\bar{\phi})] \]

\[ = p^*(1 - p^*)D(0, 1) \frac{1}{\delta} \left[ (1 + \bar{\delta})(1 - \hat{\delta}) \frac{h(1)}{h(1)} - (1 - \hat{\delta}^2) \frac{h(2)}{h(2)} \right] \]

\[ \times \sum_{i=2}^{3} c_i P(\omega_i^M)D(0, i)h(i - 1)(1 - \delta^{-i}) \]
REFERENCES


