Dynamical and structural self-organization: a study of friction, liquid-crystal nucleus growth, and supramolecular polymers through simple models

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We show that, in the continuous 1D Burridge-Knopoff model of multicontact friction, motion occurs via stick-slip sliding on a finite length rather than in avalanches, excluding the occurrence of self-organized criticality. We present strong numerical evidence that a transition from collective to strictly solitary motion occurs at a critical value of the interblock interactions. The solitary motion corresponds to successive stick-slip motion of one block between immobile neighbors, repeated periodically in time. This state persists also with open boundary conditions and moderate temperature.*

* The body of this chapter has been published as B. A. H. Huisman, and A. Fasolino. Phys. Rev. E. 72, 016107 (2005) Appendices 3.A to 3.E are added here as supplementary material.

Transition to strictly solitary motion in the Burridge-Knopoff model of multicontact friction

3.1 Introduction

Solid on solid sliding friction is often modeled by one dimensional spring-block models, meant to represent very different situations. At the atomic scale, friction is well described by the Frenkel-Kontorova, or by the Frenkel-Kontorova-Tomlinson model [1], where the blocks represent individual atoms in interaction with a surface represented as a rigid periodic modulation. At much larger length scales, the Burridge-Knopoff (BK) model, illustrated in Fig. 3.1, is used to describe sliding tectonic plates. In the BK model, the interaction with the underlying surface is replaced by a phenomenological velocity dependent friction force with a static and a dynamic contribution. The dynamics of tectonic sliding is usually studied by assuming a dynamic friction force that weakens as a function of velocity [2].

In all these models where energy is slowly fed to the system by the moving plate, the dynamics is not uniform but dominated by fast dissipative events corresponding to stick-slip motion of the individual blocks. In their velocity weakening BK model, Carlson and Langer have shown [2] that avalanches of all sizes occur, with a power
law size distribution compatible with the empirical Gutenberg-Richter law. This lack of an intrinsic length scale puts this deterministic continuous model into the larger class of systems which are said to display self-organized criticality (SOC) [3; 4], a term introduced [3] to describe the behavior of discrete sandpile automata. Since the finding of Carlson and Langer, the BK model has been studied intensively in this context, particularly in the two-dimensional discretized version proposed by Olami, Feder, and Christensen [5] (OFC). However, several authors claim or suggest that the model does not display criticality [7; 8; 6]. It has even been conjectured that the asymptotic avalanche size distribution is dominated by avalanches of size one, the fraction of larger avalanches converging towards zero as the system size increases [9].

Here we study the multicontact friction variant of the BK model, proposed by Persson [10] to model macroscopic sliding systems in the boundary lubrication regime. The BK model of multicontact friction uses a viscous dynamic friction proportional to velocity, which, contrary to the velocity weakening earthquake models, effectively reduces the range of interactions of the blocks. This approach is justified by previous studies of the same author [12] showing that, at low velocity, a thin lubricant layer exhibits a distribution of pinned solid islands that liquefy and begin to slide when the applied force exceeds a threshold value and pin again as their velocity vanishes.

We find that, after an initial transient, the motion occurs as successive domino-like slipping events of limited size rather than in avalanches, thereby excluding the occurrence of SOC. At a critical value of the interblock interactions close to realistic values for sliding surfaces in the boundary lubrication regime [10], the system reaches a dynamic regime, that we call a solitary state, where the motion occurs via periodic step-like slipping events of single blocks. Surprisingly, the solitary state is not destroyed by open boundary conditions, contrary to the behavior of OFC-models [14]. Also the solitary state is robust against small thermal fluctuations.

3.2 The BK model

The BK model of Fig. 3.1 consists of $N$ blocks of mass $m$ connected, at fixed distances $D$, to a plate moving at constant velocity $v_s$ by springs of spring constant $k_1$, and to nearest neighbor blocks by springs of spring constant $k_2$ and rest lengths $D$. The plate coordinate is $x = v_st$, and $q_i$ is the position of block $i$ with respect to its initial
equilibrium position \( q_i(0) = 0 \). The force on a block at rest (i.e. \( \dot{q}_i = 0 \)) is

\[
F_i = k_1 (x - q_i) + k_2 (q_{i+1} + q_{i-1} - 2q_i).
\tag{3.1}
\]

This force is balanced up to a threshold value \( F_s \) by the static friction force, so that a block remains motionless until it experiences a force \( F_i \geq F_s \). Once in motion, a block is subjected to a viscous force \(-2m\gamma \dot{q}_i\). If the block velocity \( \dot{q}_i \) vanishes, the static friction force is reintroduced by setting the block velocity to zero if it changes sign. For this reason we always remain in the underdamped regime. The discontinuity of the friction force at \( \dot{q}_i = 0 \) makes the system extremely nonlinear.

We introduce an dimensionless quantity characterizing the dynamic state of block \( i \):

\[
h_i \equiv \begin{cases} 0 & \text{if } \dot{q}_i = 0 \quad \text{(stick)} \\ 1 & \text{otherwise } \text{(slip)} \end{cases}
\tag{3.2}
\]

We also introduce

\[
\mathcal{H}_i(t) \equiv h_i(t) \cdot [h_{i+1}(t) + h_{i-1}(t)].
\tag{3.3}
\]

as the number \((0, 1, 2)\) of neighbors slipping while block \( i \) is slipping. Note that \( \mathcal{H}_i = 0 \) either when block \( i \) is at rest \( (h_i = 0) \) or when block \( i \) is moving while both neighbors are at rest \( (h_{i \pm 1} = 0, h_i = 1) \). Since the fraction of time a block is in motion can be quite small, it is useful to average Eq. 3.3 over a time \( \tau \) around \( t \)

\[
\langle \mathcal{H}_i(t) \rangle_{\tau} \equiv \frac{1}{\tau} \int_{t-\tau}^{t} \mathcal{H}_i(t') dt'.
\tag{3.4}
\]

yielding a continuous function, ranging between 0 and 2. By defining \( \bar{h}(t) \) as the fraction of blocks moving at time \( t \), the average over all moving blocks

\[
\langle H(t) \rangle_{\tau} \equiv \frac{1}{N} \sum_i \langle \mathcal{H}_i(t) \rangle_{\tau}, \quad \bar{h}(t) = \sum_i \frac{h_i}{N} \neq 0 \Rightarrow \bar{h}(t) = 0
\tag{3.5}
\]

constitutes an order parameter denoting if a system is either in solitary motion, i.e. \( \langle H \rangle_{\tau} = 0 \), or in collective motion, \( 0 < \langle H \rangle_{\tau} < 2 \).

The equations of motion are

\[
m\ddot{q}_i = h_i \left[ -2m\gamma \dot{q}_i + k_1 (x - q_i) + k_2 (q_{i+1} + q_{i-1} - 2q_i) \right],
\]

where

\[
h_i(t + dt) = \begin{cases} 0, & \dot{q}_i(t)\dot{q}_i(t + dt) < 0 \\ 1, & F_i(t + dt) \geq F_s \\ h_i(t), & \text{otherwise} \end{cases}
\tag{3.6}
\]

with \( dt \) the time step of numerical integration. The equations of motion are made dimensionless by scaling time by \( \sqrt{m/k_1} \), positions by \( F_s/k_1 \) and forces by \( F_s \):

\[
\ddot{q}_i = h_i \left[ -2\gamma \ddot{q}_i - \omega_0^2 q_i + \ddot{h}_2 (q_{i+1} + q_{i-1}) + x \right] \equiv h_i \sigma_i
\tag{3.7}
\]
Figure 3.2: Time dependence of (a) the average force $\bar{\sigma}$ (b) the fraction $\bar{h}$ of blocks moving and (c) the measure of collective behavior $\langle H \rangle_\tau$ for $N = 10000$, $\gamma = 0.5$, $k_2 = 1$, $\tau = 0.5$. Panel (a) and (b) reproduce Fig. 4 of Ref. [10] extended to larger time. Notice in (c) the transition around $t \sim 20000$ to solitary motion, causing $\bar{\sigma}$ and $\bar{h}$ (see insets of panels (a) and (b)) to become periodic in time. Note that $\Delta t \gg \tau$ in this figure.

with $\omega_0 = \sqrt{1+2k_2}$ and $\sigma_i$ denoting the total force on block $i$ irrespective of its dynamic state $h_i$. Note that $\bar{k}_2$ and $\bar{\gamma}$ are in units of $k_1$ and $\sqrt{k_1/m}$ respectively, and that $\bar{F}_s = 1$. We will only consider dimensionless quantities, and will omit the tilde from now on.

The Eqs. of motion (3.7) are integrated by a fourth order Runge-Kutta algorithm with time step $dt = 0.005$. The initial positions $q_i(0)$ are chosen from a uniform random distribution $q = [-0.005, 0.005]$; furthermore $x(0) = 0$ and $\dot{q}_i(0) = 0$. We use periodic boundary conditions, unless specified otherwise. The width of the random distribution determines the duration of the transient collective stick-slip behavior. We consider a driving velocity $v_s = 0.005$, which is low enough to be in the limit $v_s \ll \max(\dot{q}_i)$ characterizing typical tribological experiments.
3.3 Solitary versus collective motion

In Figs. 3.2(a) and 3.2(b) we show the average force $\bar{\sigma}$ and the fraction $\bar{h}$ of moving blocks as in Ref. [10] on a much longer timescale. The initial collective stick-slip behavior is due to the very narrow distribution of forces below $F_s$ at $t = 0$. At the first such collective event almost all blocks slip at the same time ($\bar{h} \approx 1$). As time progresses the distribution of forces $P(\sigma)$ widens and the number of blocks slipping at the same time decreases. After $t \sim 1000$, at any time a number of blocks is moving and, at $t \sim 1800$, the system is said to be in a steady state in Ref. [10].

In the steady state however, the fraction $\bar{h}$ of moving blocks keeps decreasing, indicating that the system is still equilibrating towards a more favorable state. Finally, at $t = 20,000$, we find that $\bar{\sigma}$ and $\bar{h}$ become periodic in time. It is shown in Fig. 3.2(c) that $(H)_r \approx 0$ when the system becomes periodic. This indicates that blocks slip in a step-like fashion between immobile nearest neighbors ($h_{i\pm 1} = 0$), whence the name of solitary motion. Once this is the case for the motion of all blocks for longer than the interval between successive slips of the same block, the system is trapped in this solitary state and becomes periodic.

Analytical results give a rationale for this behavior. For solitary motion ($h_{i\pm 1} = 0$ when $h_i = 1$) the equations of motion (3.7) become decoupled, and the motion of a single block is that of a discontinuously driven, damped harmonic oscillator. For initial conditions $q_i(0) = \dot{q}_i(0) = 0$ and $F_i(0) = F_s = 1$ (i.e. $k_2(q_{i+1} + q_{i-1}) + x = F_s$), and by assuming $v_s \ll \max(\dot{q}_i)$:

$$\dot{q}_i + 2\gamma \dot{q}_i + \omega_0^2 q_i = F_s. \tag{3.8}$$

The solution of Eq. (3.8) for the underdamped case ($\gamma < \omega_0$)

$$q_i(t) = \frac{F_s}{\omega_0^2} \left[ 1 - \exp(-\gamma t) \left( \frac{2}{\omega} \sin(\omega t) + \cos(\omega t) \right) \right] \tag{3.9}$$

reaches zero velocity after a time

$$\delta t = \frac{\pi}{\omega}, \text{ with } \omega = \sqrt{\omega_0^2 - \gamma^2}. \tag{3.10}$$

In a time $\delta t$ the block travels a distance [15]

$$\Delta q = \frac{F_s}{\omega_0^2} \left[ 1 + \exp \left( -\gamma \pi / \omega \right) \right]. \tag{3.11}$$

The interval $\Delta t$ between consecutive slip events of the same block is given by

$$\Delta t = \Delta q / v_s, \tag{3.12}$$

because, although most of the time a block is not moving, its average velocity has to be equal to the plate velocity $v_s$. The fraction of time a block is moving, is simply the ratio of the duration of a slip event and the interval between them: $h = \delta t / \Delta t$. 

Figure 3.3: (a) Forces $\sigma_i$ for part of a system in a solitary state, with $k_2 = 1$, $\gamma = 0.5$, $N = 10,000$ and (b) the distribution of forces measured over $N = 10,000$ blocks over a period of 3000 time steps. Dashed lines from top to bottom indicate $\sigma = 1 - \Delta q$, $\sigma = 1 - (1 + k_2)\Delta q$ and $\sigma = 1 - (1 + 2k_2)\Delta q$ respectively. Arrows indicate which blocks will be in the high force range next. The peak in the distribution at $\sigma < 1 - (1 + 2k_2)\Delta q$ is caused by moving blocks, and vanishes for $v_s = 0$.

In the interval $\Delta t$ between slip events, the force $F_i$ acting on block $i$ is slowly increased by the movement of the plate by an amount $\Delta q$ ($k_1\Delta q$ in dimensional units), and by the sudden movement of both neighbors by an amount $2k_2\Delta q$. Therefore, the force directly after the slip event is $F_{\text{min}} = 1 - (1 + 2k_2)\Delta q$ (since $F_s = 1$). We can identify three ranges of the forces acting on a block:

$$
egin{align*}
1 - (1 + 2k_2)\Delta q & \leq F_i \leq 1 - 2k_2\Delta q & \text{low} \\
1 - (1 + k_2)\Delta q & \leq F_i \leq 1 - k_2\Delta q & \text{medium} \\
1 - \Delta q & \leq F_i \leq 1 & \text{high}
\end{align*}
$$

A block is in the low force range after it has slipped, moves to the medium range when one neighbor has slipped, and to the high range when both neighbors have slipped. Movement within each range is caused by the slow motion of the plate.

Figure 3.3(a) shows a snapshot of the forces on part of the chain in the solitary regime. Peaks of only one block are present in the lower and higher force range, separated by slanted lines in the medium force range where most of the blocks reside. In Fig. 3.3(b) we show the distribution of forces $P(\sigma)$ around the time of the snapshot of Fig. 3.3(a). $P(\sigma)$ is peaked at $\sigma = 1$, and $\sigma = 1 - (1 + 2k_2)\Delta q$ due to the predominance of lines with a small slope.

The distribution of forces $P(\sigma)$ in the solitary state shown in Fig. 3.3(b) is highly
3.3 Solitary versus collective motion

symmetric, hence its mean $\bar{\sigma}$ can be approximated by the center of the distribution:

$$\bar{\sigma} \approx 1 - \frac{(1 - (1 + 2k_2)\Delta q)}{2} = \frac{1}{2} (1 - \exp(-\gamma \pi/\omega)), \quad (3.13)$$

where we have made use of Eq. 3.11 for $\Delta q$. The friction force measured in experiments is the lateral force acting on the support,

$$f = \sum_{i=1}^{N} (q_i - x) \approx -N\bar{\sigma} \quad (3.14)$$

where we have assumed in Eq. 3.7 that $\sum_i (q_{i+1} - q_i) \approx 0$ and $\sum_i \dot{q}_i \approx Nv_s \approx 0$. Since the kinetic friction force Eq. 3.14 is normalized by the static friction force $F_s$, this result implies that the ratio of the kinetic to the static friction force in the solitary state can be used to extract, from experiments, the ratio $\gamma/\omega$ characterizing the sliding system.

The analysis of the behavior of the forces $\sigma_i$ in the solitary state, leads us to define a typical length scale in the system. We find that solitary motion requires two consecutive blocks in the high force range to be separated by an arbitrary number of blocks in the medium force range, and by exactly one block in the low force range. The blocks in the medium force range are arranged in monotonically increasing or decreasing slanted lines that will reach the high energy region one after the other. However, since in the time $\delta t$ it takes block $i$ to slip, the upper surface travels a distance $v_s \delta t$, the absolute slope of the lines is constrained by

$$\left| \frac{d\sigma}{dt} \right| \geq v_s \delta t. \quad (3.15)$$

Since, in the strictly solitary state, each slanted line must start and end in the medium force zone $\Delta \sigma = \Delta q$ wide, the minimum slope also limits the number of blocks along the line to

$$N_{\text{line}} = \left| \frac{di}{d\sigma} \right| \Delta \sigma \leq \frac{\Delta q}{\delta tv_s} = \frac{\Delta t}{\delta t}. \quad (3.16)$$

The finite duration $\delta t$ of a slip event introduces a typical length scale, contrary to systems displaying SOC. Strictly speaking, in a continuous model, the size of an avalanche is given by the number of blocks performing simultaneous motion. By this definition, in a system in solitary motion, all avalanches are of size one. However, sequences up to $N_{\text{line}}$ of size one avalanches can and do occur.

Next, we show in Fig. 3.4 the time evolution of $\langle H \rangle_s$ for three values of $k_2$. We find that a transition occurs at a critical relative value of the spring constant $k_2^c \sim 1.5$. Below $k_2^c$, $\langle H \rangle_s$ smoothly decreases to zero, signaling the occurrence of the solitary state, whereas, above $k_2^c$, $\langle H \rangle_s$ reaches a constant finite value. An estimate of parameters for realistic sliding lubricated surfaces [10] gives $k_2 \sim 1$. For values of $k_2$
just below $k_2^c$ there is an initial, relatively smooth decrease of $\langle H \rangle_\tau$, but the solitary state is reached only after many attempts. This process is shown in the left panel of Fig. 3.5 where we show a gray scale map of the order parameter $\langle H \rangle_\tau$ for $k_2$ below and above $k_2^c$. The initial uniform band corresponds to collective stick-slip motion (see Fig. 3.2). This behavior is followed by a very short period of almost uniform motion with velocity $v_s$, appearing as black regions in the figure. Notice that uniform motion has been shown to be unstable for models with a velocity weakening friction force. Afterward, for $k_2 < k_2^c$, domains of solitary motion of different sizes grow and shrink, until finally the complete system is in a solitary state. For $k_2 > k_2^c$, the system remains in a collective state, characterized by the fact that neighboring blocks move simultaneously for part of their movement, much like domino toppling. However large patches of solitary motion that expand and disappear are also present. We expect the probability that, for $k_2 > k_2^c$, a patch of solitary motion extends to the whole system to vanish for $N \to \infty$.

There are several indications that the transition to solitary motion is of first order. For a given system size, increasing $k_2$ towards $k_2^c$ increases the time needed to reach the solitary state, but does not change qualitatively the shape of the curve shown in Fig. 3.4 for $k_2 = 1.25$. Moreover, once the solitary state is reached, if $k_2$ is increased in small steps above $k_2^c$ the system readjusts to remain in a now metastable solitary state, in analogy to overheating a system above $T_c$. We wish to underline however, that the evidence for a sharp transition is only numerical.

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1 Black regions in Fig. 3.5 at $t \sim 2000$ represent groups of blocks moving with almost constant velocity $\dot{q}_i \sim v_s$.
3.4 Stability of the solitary state

Next we study the stability of the solitary state for small perturbations caused by thermal fluctuations. Due to these fluctuations a block may temporarily obtain enough energy to slip, even though it is experiencing a force smaller than the static friction force. Following Persson [10] we can define an energy barrier for block $i$ as

$$\Delta E_i = U(F_s, q_{i+1}, x) - U_i(F_i, q_{i+1}, x),$$

(3.17)

where $U_i(F_i, q_{i+1}, x)$ is the potential energy of block $i$, and $U(F_s, q_{i+1}, x)$ is the potential energy of the same block, moved to where it would experience the static friction force $F_s = 1$, while keeping the position of the neighboring blocks and of the plate fixed. The potential energy $U_i$ is given by

$$U_i = \frac{1}{2} (x - q_i)^2 + \frac{k_2}{2} (q_{i+1} - q_i)^2 + \frac{k_2}{2} (q_{i-1} - q_i)^2$$

$$= g(q_{i\pm 1}, x) + \frac{F_i^2}{2\omega_0^2}.$$  

(3.18)

Since $g(q_{i\pm 1}, x)$ does not depend on $F_i$, Eqs. 3.17 and 3.18 give

$$\Delta E_i = \Delta E_{\text{max}} (1 - F_i^2), \quad \text{with} \quad \Delta E_{\text{max}} = 1/2\omega_0^2,$$

(3.19)

The probability that block $i$ slips (i.e. overcomes the energy barrier) within a time $dt$ is assumed to be

$$P_i(dt) = \nu \exp(-\Delta E_i/k_BT)dt,$$

(3.20)
where $\nu$ is an attempt frequency, $T$ the temperature and $k_B$ the Boltzmann constant. In practice, finite temperature is simulated by drawing a random number $r_i = [0,1]$ at each integration step for each block, and if $r_i < P_i(dt)$, where $dt$ is the integration time step size, the static friction force is decreased to zero by setting $h_i = 1$.

In Fig. 3.6 the time dependence of the order parameter for collective motion $\langle H\rangle_\tau$ is shown at different temperatures for $k_2 = 1$ where at $T = 0$ the solitary state is stable. For low temperatures the order parameter goes to zero in a way similar to the zero temperature case, although small fluctuations do occur. These fluctuations grow with increasing temperature, until the system cannot maintain the solitary state\(^2\). The solitary state is therefore stable at low temperatures. In Fig. 3.7 we show a gray scale map of the order parameter per block $i$, as a function of time. At low temperature the system evolves towards the solitary state in much the same way as in the zero temperature case (compare with Fig 3.5). Collective motion occurs only very locally, very weakly, and only often involves the direct neighbors of the blocks that were thermally excited. At higher temperatures, larger patches of collective motion appear, without ever extending to the whole system.

We recall that once every block in the system is moving in a solitary fashion for longer than the time between two slips $\Delta t$, the complete system is trapped in the solitary state. Since each block slips the same distance $\Delta q$, the system becomes

\[ E(t) = E_{\text{max}}/50 \]

\[ E(t) = E_{\text{max}}/30 \]

\[ E(t) = E_{\text{max}}/20 \]

\[ E(t) = E_{\text{max}}/15 \]

\[ t(\times 10^4) \]

![Figure 3.6](image.png)

**Figure 3.6:** Order parameter for collective motion $\langle H\rangle_\tau$ as a function of time for different temperatures, and for $\tau = \Delta t$, $k_2 = 1$, $\gamma = 0.5$, $N = 10,000$. The solitary state is stable at least up to $k_B T = \Delta E_{\text{max}}/30$.

\(^2\) At low temperatures the system can escape the solitary state only very locally (left panel of Fig. 3.7), for a brief period of time. At higher temperatures large groups of blocks that are not in the solitary state arise (right panel of Fig. 3.7), and only at very high temperatures (not shown) these domains start dominating the dynamics.
periodic with a period \( \Delta t \). A finite temperature gives rise to a finite probability for a block to slip at a force \( F_s < F_s \), and since the distance a block slips is proportional to the force acting on the block at the moment it slips (Eq. 3.11), a thermally induced slip event breaks the perfect periodicity of the solitary state. However, as clearly shown in Fig. 3.7 the nature of the motion is not drastically different from strictly solitary motion.

Lastly in Fig. 3.8 we show that the solitary state is not destroyed by open boundary conditions. One can recognize the region of collective motion at the edges, because the interval between successive slip events of the same block is larger than in the solitary state. These regions of collective, nonperiodic motion appear, expand and shrink at the boundary, but do not extend to the interior of the sample. This is remarkable because open boundary conditions are expected to destroy simple periodic states [14].

### 3.5 Summary and conclusions

In summary we have shown that the motion in the continuous BK model with viscous friction at low driving velocities occurs in domains of finite size presenting an intrinsic length scale, thereby excluding the occurrence of SOC. Below a critical value of the interblock interaction the system evolves to a strictly solitary, periodic state with successive slipping of individual blocks among immobile neighbors. The solitary state is
Figure 3.8: Force $\sigma_i(t)$ in a system of $N = 512$ blocks with $\gamma = 0.5$ and $k_2 = 1$, with open boundary conditions. Black is $\sigma_i(t) = 1$ and white is $\sigma_i(t) = 1 - (1 + 2k_2)\Delta q$. Note that the boundary conditions do not change the solitary state of the bulk ($60 < i < 470$) even after $t = 90000$ time steps ($\sim 10^5\Delta t$). Also note the difference in interval $\Delta t$ between solitary (bulk) and collective (edges) slip events.

stable against small thermal fluctuations and open boundary conditions. In the range of parameters estimated to describe actual sliding systems [10], this model predicts strictly solitary motion, for which Eq. 3.14 can be used to measure the damping and stiffness of the sliding system.


Supplementary material

Appendix 3.A Derivation of parameters

In this section we will describe how the problem of boundary lubrication can be mapped onto the Burridge and Knopoff model, as illustrated in Fig 2.4. Furthermore, we derive the values of the spring constants $k_2$ and $k_1$ and the size of the viscous damping constant $\gamma$. Throughout this chapter we have given $k_2$ in units of $k_1$. Only in this appendix will we briefly depart from this convention, as we derive the size of $k_2$ and $k_1$.

The material of the sliding surface around a solidified part of the lubrication layer is described by blocks of equal mass $m = \rho D^3$, with $D$ the typical length of a solid island and $\rho$ the density of the bulk material (See Fig. 3.1). The transverse elasticity of the material is represented by a spring, connecting the upper surface to the block, of constant $k_1$. The lateral elasticity is represented by a spring of strength $k_2$. The force acting on block $i$ is denoted by $F_i = \sigma_i \delta A$, with $\delta A = D^2$ and $\sigma_i$ the stress at the position of block $i$. The pinning (freezing) of the lubrication layer, and the slipping (melting) is modeled by pinning the block to the lower surface and allowing it to slip when the stress $\sigma_i$ on block $i$ is larger than some critical value $\sigma_s$, the static friction force of the block per unit area. (Note that only in macroscopic, dry friction the static friction is independent of the area, it does not apply at the atomic level). The block will stick again when its velocity vanishes, modeling the solidification of the lubrication layer in the absence of a shear force. If all blocks have the same size $\delta A$, which in our version of the BK model they do, then the critical force corresponding to $\sigma_s$ is $F_s = \sigma_s \delta A$. When an island melts, the elastic energy stored in the contact is released and partially turned into heat. This dissipation of energy is represented by a viscous (dynamic) friction force, acting on the moving block.

In Ref. [10] the value of the spring constants are estimated by use of the elastic continuum theory. The displacement of a patch of area $\delta A = D \times D$, due to a stress $\sigma$ applied to that patch only, and the ratio between this displacement and the displacement of a patch a distance $D$ away, is related to the mechanical model. The displacement $d$ of the patch due to a force $F = \sigma D^2$ is [11]:

\[
    d \approx \frac{F}{\rho c^2 D},
\]

where $c$ is the velocity of sound. The displacement a distance $D$ away from where $F$ is applied, is approximately $d/3$. This can be related to the displacement of block $q_0$ and $q_{\pm 1}$, due to a force $F$ applied to block 0 only:

\[
    k_1 q_i + k_2 (2q_i - q_{i-1} - q_{i+1}) = F \delta_{i0},
\]

(3.22)
The left hand side has the form of a discrete second order differential equation, so that in the continuum limit Eq. 3.22 resembles the steady state Helmholtz equation:

\[ aq(x) - b \frac{d^2 q(x)}{dx^2} = c \delta(x), \]

which is known to have solutions of the form: \( q(x) = A \exp(-\alpha|x|) \). We will therefore try the ansatz \( q_i = A \exp(-\alpha|i|) \). After substitution of the ansatz for \( i = 0 \), Eq. 3.22 becomes:

\[
\begin{align*}
  k_1 A + k_2 (2A - A \exp(-\alpha|1|) - A \exp(-\alpha|1|)) &= F \\
  A(k_1 + 2k_2 (1 - \exp(-\alpha))) &= F 
\end{align*}
\]

whence,

\[ q_0 = A = \frac{F}{k_1 + 2k_2 (1 - \exp(-\alpha))}. \tag{3.24} \]

Substitution of the ansatz for \( i \neq 0 \) gives:

\[
\begin{align*}
  k_1 \exp(-\alpha|i|) + k_2 (2 \exp(-\alpha|i|) - \exp(-\alpha|i - 1|) - \exp(-\alpha|i + 1|)) &= 0, \\
  k_1 + k_2 (2 - \exp(+\alpha) - \exp(-\alpha)) &= 0 
\end{align*}
\]

where both sides where divided by \( A \), and where the equations are equal to zero because the force is only exerted on block \( i = 0 \). To solve the \( i \neq 0 \) case for \( u = \exp(-\alpha) \), we multiply both sides by \( \exp(-\alpha) \):

\[
\begin{align*}
  k_1 \exp(-\alpha) + k_2 (2 \exp(-\alpha) - 1 - \exp(-2\alpha)) &= 0 \\
  k_1 u + k_2 (2u - 1 - u^2) &= u^2 - \frac{1}{k_2} u + 1 = 0 \\
  u &= \frac{k_1 + 2k_2}{2k_2} \pm \frac{1}{2} \sqrt{\left( \frac{k_1 + 2k_2}{k_2} \right)^2 - 4} \\
  \exp(-\alpha) &= \frac{1}{2} \left( \frac{k_1}{k_2} + 2 \pm \sqrt{\left( \frac{k_1}{k_2} \right)^2 + 4 \frac{k_1}{k_2}} \right). \tag{3.25} 
\end{align*}
\]

We need to relate \( q_{\pm 1} = A \exp(-\alpha) \), and \( q_0 = A \) to the continuum limit, \( q_0 = d \), and \( q_{\pm 1}/q_0 = 1/3 \):

\[
\begin{align*}
  \frac{F}{k_1 + (4/3)k_2} &= \frac{F}{pc^2D} \\
  \frac{1}{2} \left( \frac{k_1}{k_2} + 2 \pm \sqrt{\left( \frac{k_1}{k_2} \right)^2 + 4 \frac{k_1}{k_2}} \right) &= 1/3
\end{align*}
\]
3. A Derivation of parameters

First we solve the latter equation

\[ \frac{k_1}{k_2} \pm \sqrt{\left( \frac{k_1}{k_2} \right)^2 + 4 \frac{k_1}{k_2}} = 2/3 - 2 \]

where only the solution with a minus sign will lead to a real valued result

\[ \sqrt{\left( \frac{k_1}{k_2} \right)^2 + 4 \frac{k_1}{k_2}} = 4/3 + \frac{k_1}{k_2} \]
\[ \frac{k_1}{k_2} = \frac{4}{3} \]  

(3.26)

The values of the spring constants can now be derived:

\[ \frac{1}{k_1 + (4/3)k_2} = \frac{1}{\rho c^2 D} \]
\[ k_1 + (4/3)(3/4)k_1 = 2k_1 = \rho c^2 D \]
\[ k_1 = \frac{\rho c^2 D}{2} \]  

(3.27)
\[ k_2 = \frac{3\rho c^2 D}{8} \]  

(3.28)

These results (most notably Eq. 3.26) differ from those derived in [10], where \( k_2 = 3/2k_1 \) was found, versus our result \( k_1 = 3/4k_2 \). Note that in our result \( k_2 < k_1 \), contrary to Persson’s result \( k_2 > k_1 \). With these results, it is stated in Ref. [10] that: \( k_1 \approx k_2 \approx \rho c^2 D \). As is explained in section 3.3, the solitary state spontaneously arises for \( k_2 \lesssim 1.5k_1 \), so for both estimates the solitary state description is appropriate. We have followed Persson, and chose \( k_1 = k_2 \).

After displacing the patch, the patch returns to its equilibrium position, and dissipates the energy by sending out an elastic wave pulse into the medium. In Ref. [10] a damping force \(-m\zeta(\dot{q}_t - \dot{x})\) is introduced into the equation of motion, and the damping due to this dissipation is estimated in Ref. [13]

\[ 2\gamma = \zeta \approx \frac{1}{8\pi} \frac{m \omega_0^5}{\rho c^3} \xi, \text{ with } \omega_0 = \sqrt{\frac{k_1 + 2k_2}{m}} \]  

(3.29)

where \( \xi \approx 3 \) is the phonon contribution to the transverse damping force, and where we have introduced \( \gamma = \zeta/2 \) for convenience. Using Eqs. 3.27 and 3.28 and the definition of the mass of a block, we get

\[ \zeta = \frac{25c^2\xi}{128D\pi} \approx \frac{1}{4} \sqrt{\frac{k_1}{m}} \]  

(3.30)

Although this is a factor 4 smaller than what Persson estimated, we have followed Persson, and chose \( \gamma = 0.5 \sqrt{\frac{k_1}{m}} \).
Appendix 3.B  Integrating the equations of motion

In this Appendix we explain in more detail how we integrate the equations of motion of the blocks and how we let a block stick or slip. The equations of motion are integrated by means of a fourth order Runge-Kutta algorithm. To integrate using this algorithm we must rewrite equation 3.7, for a system of $N$ blocks, as a system of $2N$ first order differential equations. As the position and velocity of the upper surface should also be integrated, it is convenient to define $q_0 = x$ and $\dot{q}_0 = v_s$, so that we have to integrate a system of $2N + 2$ first order differential equations

\[
\frac{d}{dt} \begin{pmatrix}
q_0 \\
q_1 \\
q_2 \\
\vdots \\
q_{N-1} \\
q_N \\
\dot{q}_1 \\
\dot{q}_2 \\
\vdots \\
\dot{q}_{N-1} \\
\dot{q}_N
\end{pmatrix} = \begin{pmatrix}
v_s \\
\dot{q}_1 \\
\dot{q}_2 \\
\vdots \\
\dot{q}_{N-1} \\
\dot{q}_N
\end{pmatrix} = \begin{pmatrix}
-2\gamma\dot{q}_1 + (q_0 - q_1) + k_2(q_2 + q_N - 2q_1) \cdot h_1 \\
-2\gamma\dot{q}_2 + (q_0 - q_2) + k_2(q_3 + q_1 - 2q_2) \cdot h_2 \\
\vdots \\
-2\gamma\dot{q}_{N-1} + (q_0 - q_{N-1}) + k_2(q_N + q_{N-2} - 2q_{N-1}) \cdot h_{N-1} \\
-2\gamma\dot{q}_N + (q_0 - q_N) + k_2(q_1 + q_{N-1} - 2q_N) \cdot h_N
\end{pmatrix} \quad (3.31)
\]

The system of Eq. 3.31 is integrated with fixed step size $t_{\text{step}}$. After each Runge-Kutta integration step, $h_i$ is determined as follows. First it is determined whether the velocity has changed sign. If this is the case the block sticks:

\[
h_i(t) = \begin{cases} 
h_i(t - t_{\text{step}}), & \dot{q}_i(t - t_{\text{step}}) \cdot \dot{q}_i(t) \geq 0 \\
0, & \dot{q}_i(t - t_{\text{step}}) \cdot \dot{q}_i(t) < 0.
\end{cases} \quad (3.32)
\]

Then it is determined if the force on the block has become higher than the critical force $\sigma_a$, in which case the block starts slipping:

\[
h_i(t) = \begin{cases} 
h_i(t - t_{\text{step}}), & \sigma_i < \sigma_a \\
1, & \sigma_i \geq \sigma_a.
\end{cases} \quad (3.33)
\]

The temperature is incorporated into the model by calculating the probability $w_i$ that a block is excited over the barrier according to Eq. 3.20, and generating a uniform random number $0 < r_i < 1$. If $r_i \leq w_i$ then the block starts slipping:

\[
h_i(t) = \begin{cases} 
h_i(t - t_{\text{step}}), & r_i > w_i t_{\text{step}} \\
1, & r_i \leq w_i t_{\text{step}}.
\end{cases} \quad (3.34)
\]

The Runge-Kutta integration algorithm is fourth order accurate in time, and therefore we do not need many integration steps per slip-event of a single particle. However, the accuracy of the solution is not only determined by the order of the integration
algorithm, but also by how accurately we resolve the strong discontinuities in the motion of a block, which switches from a zero to a nonzero velocity and vice versa in one time step. We have chosen a time step of $t_{\text{step}} = 0.005$. This way we describe the motion of a block in the solitary state by about 380 integration steps. With this time-step a moving block can cause a neighboring block to slip at a force of at most 0.5% above the static friction force $\sigma_s$, and the motion of the upper surface can cause a block to slip with a force at most 0.0025% above $\sigma_s$. The maximum absolute discontinuity in velocity when we stop a block by applying the rule of Eq. 3.32 is of the order $v \sim 0.005$, and for a block in the solitary state it is maximally $v = 0.002$.

Appendix 3.C Features of the solitary state

3.C.1 Motion of the blocks in the solitary state

In Section 3.3 we have shown that the solitary state spans the complete system only at low $k_2$. In this appendix we study the motion of a block, i.e. the position and velocity of a block, and the force acting on it, during a solitary slip event, as a function of $k_2$. This gives insight into why the solitary state is harder to reach at higher $k_2$, and explains the small peak in the distribution of Fig. 3.3b at $\sigma < 1 - (1 + 2k_2)\Delta q$.

In section 3.3 we have calculated the distance a block slips $\Delta q$ (Eq.3.11) in the solitary state. In Fig. 3.9 we show this distance as a function of the spring constant $k_2$. As $k_2$ increases, $\Delta q$ decreases and so does the interval $\Delta t$ between subsequent slip events (Eq. 3.12). Because the duration of a slip event $\delta t$ (Eq. 3.10) is finite, the length $N_{\text{line}}$ of Eq. 3.16 also decreases. This may be one of the reasons why the solitary state becomes less prominent at high $k_2$. The velocity $\dot{q}$
Figure 3.10: (a) The maximum negative force $\sigma_{\text{max}}$ during one slip event, and the force right after one slip event $\sigma_i(\delta t)$, with $\delta t = \pi/\omega$, as a function of $k_2$. (b) The maximum velocity $\dot{q}_{\text{max}}$, during a slip event, as a function of $k_2$. In both panels $\gamma = 0.5$.

$$\dot{q}_i(t) = \frac{F_s}{\omega} \exp(-\gamma t) \sin(\omega t)$$

(3.35)

of a block also decreases with $k_2$, and the negative force slowing a moving block down during a slip-event is stronger

$$\ddot{q}_i(t) = \sigma_i(t) = F_s \exp(-\gamma t) \left[ \cos(\omega t) - \frac{\gamma}{\omega} \sin(\omega t) \right].$$

(3.36)

The block reaches a maximum velocity at time $t_{v,\text{max}}$

$$\frac{d\dot{q}_i}{dt} = \sigma_i = 0 \Rightarrow t_{v,\text{max}} = \omega^{-1} \arctan \left( \frac{\omega}{\gamma} \right)$$

(3.37)

and a maximum (negative) force at $t_{\sigma,\text{max}}$

$$\frac{d\sigma}{dt} = 0 \Rightarrow t_{\sigma,\text{max}} = -\frac{1}{\omega} \arctan \left( \frac{2\omega \gamma}{\omega^2 - \gamma^2} \right) + n \frac{\pi}{\omega}, \quad n = \ldots, -1, 0, 1, \ldots$$

(3.39)

$$\sigma_{\text{max}} = \begin{cases} 
- F_s \exp \left[ -\frac{\gamma}{\omega} \arctan \left( \frac{2\omega \gamma}{\omega^2 - \gamma^2} \right) + \frac{\pi}{\omega} \right] & \text{for } \gamma < \frac{\omega}{\sqrt{2}} \\
F_s \exp \left[ -\frac{2\omega \gamma}{\gamma^2 - \omega^2} \right] & \text{for } \frac{\omega}{\sqrt{2}} < \gamma < \omega_0
\end{cases}$$

(3.40)

This force gives rise to the small peak in Fig.3.3 at $\sigma < 1 - (1 + 2k_2)\Delta q$. In Figure 3.10 $\sigma_{\text{max}}$, $\sigma_i(\delta t)$ and $\dot{q}_{\text{max}}$ are shown as a function of $k_2$. Finally, in Fig. 3.11, we
show \(q_i, \dot{q}_i\) and \(\sigma_i\) for \(k_2 = 1\) and \(\gamma = 0.5\), which are the parameters we have used the most in this chapter.

### 3.C.2 Friction force as a function of driving velocity in the solitary state

In Eq. 3.13 the average force \(\bar{\sigma}\) is estimated as the center of the distribution \(P(\sigma)\), i.e. as \(\bar{\sigma} = \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2}\). However, this only holds for \(v_s = 0\). For a finite driving velocity \(v_s\), a fraction of the blocks \(\bar{h}\)

\[
\bar{h} = \frac{\Delta t}{\Delta t} = \frac{\pi v_s}{\omega \Delta q}
\]

is in motion. As for a moving block

\[
\frac{1}{t_{\text{end}} - t_{\begin{smallmatrix}\text{begin} \end{smallmatrix}}} \int_{t_{\begin{smallmatrix}\text{begin} \end{smallmatrix}}}^{t_{\text{end}}} \sigma dt = \dot{q}(t_{\text{end}}) - \dot{q}(t_{\begin{smallmatrix}\text{begin} \end{smallmatrix}}) = 0
\]

on average, moving blocks do not exert any force on the upper surface and the friction force becomes

\[
\bar{\sigma} = \frac{1}{2} \left(1 - \frac{\bar{h}}{}\right) \left(\omega_0^2 \Delta q\right)
\]

\[
= \frac{1}{2} \left(1 - v_s \frac{\pi}{\omega \Delta q}\right) \left(\omega_0^2 \Delta q\right)
\]
which decreases linearly with the driving velocity \( v = v_s / v_0 \). This is in contrast to what is stated in Ref. [10], where creep is described as having a constant dynamical friction force as a function of velocity. Figures 3.12(c) and (d) show the average friction force and the average fraction of moving blocks as calculated from Eqs. 3.44 and 3.41 respectively. For comparison Figures 3.12(a) and (b) are simulation results taken from Ref. [10]. Up to \( \ln(v_s / v_0) \approx -3 \) (where \( v_0 = F_0 \sqrt{m / k_t} \)) the solitary state approximation reproduces the simulations, at higher driving velocities however, the motion in the BK model does not resemble that of the solitary state. This is because some blocks never stick after they slip, and are dragged at \( \dot{q} = v_s \). These blocks exert a force \(-2\gamma v_s\) on the upper surface.

To summarize, the BK model of multicontact friction exhibits velocity weakening friction at low velocities and viscous friction at high velocities. At low velocities the friction force can be explained by the stick-and-slip motion of the solitarily moving
blocks; the friction force calculated for a system in the solitary state reproduces the velocity weakening friction force measured in the BK model very well. At higher driving velocities less and less blocks move solitarily, and the friction enters the viscous regime in which the blocks are following the upper surface at the same velocity.

3.C.3 Characterizing the structure of the solitary state

In the solitary state, the position of the blocks must be structured in a very specific way. In this appendix we will study the structure of a solitary state in more detail. In the solitary state each block moves a well defined distance during a slip event, and the motion of each block is periodic in time. All blocks must therefore be equivalent, and thus periodic boundary conditions are required. Although we have shown that large parts of a system can move solitarily, even when open boundary conditions are used, or small temperature fluctuations are allowed, here we will only focus on a purely solitary state, for a system with periodic boundary conditions at $T = 0$.

Although the position of a block is the most natural variable when describing block motion, to understand the solitary state it is more convenient to study the stress $\sigma_i$ on each block. After all, the condition for two blocks to move solitarily is that at no time two neighboring blocks should have a force in the same range $\sigma_i > 1 - k_2 \Delta q$. The force $F$ on a block can be discretized into three regimes $F_i'$,

$$
1 - (1 + 2k_2) \Delta q \leq F_i \leq 1 - 2k_2 \Delta q \quad \text{low, } F_i' = -1;
$$

$$
1 - (1 + k_2) \Delta q \leq F_i \leq 1 - k_2 \Delta q \quad \text{medium, } F_i' = 0;
$$

$$
1 - \Delta q \leq F_i \leq 1 \quad \text{high, } F_i' = 1;
$$

They correspond to a block that has just slipped, a block for which one neighbor has slipped, and a block for which both neighbors have slipped and which itself is about to slip, respectively. In the high force regime, the position of the upper surface determines, via the spring $k_1$, when the block will slip. Here we will consider only one sequence of slip events, and remember which blocks have slipped and which have not.

In this discretization the rules for the slip of blocks are as follows

$$
\sigma_{i-1} \rightarrow \sigma_{i-1} + 1
$$

$$
\sigma_i \rightarrow -1
$$

$$
\sigma_{i+1} \rightarrow \sigma_{i+1} + 1 \quad \text{(3.45)}
$$

In Section 3.3 we mentioned that two blocks in the high force regime should be separated by exactly one block in the low force regime and an arbitrary number of blocks in the medium force regime. By using Eq. 3.45 on a few example configurations, shown in Tables 3.1-3.3, we will explain why this is necessarily the case. We will always start with block $i = 3$ in the high force range, and place either zero, one or more blocks in the low force position between block $i = 3$ and its periodic image. Blocks that have already slipped are marked in gray in Tables 3.1-3.3, and will not be allowed to slip again.
In Table 3.1 we start with a configuration where we place no blocks in the low force regime. The high force block is only separated from its periodic image by blocks in the medium force range. The high force regime “propagates” in both directions from \( i = 3 \), until both meet at \( i = 6 \). In this configuration blocks 5 and 6, or 6 and 1 would have moved at the same time, breaking the requirement for solitary motion. In Table 3.2 two blocks (\( i = 5 \) and \( i = 1 \)) are placed in the low force regime between two blocks in the high force regime (in this case again block \( i = 3 \) and its periodic images). After three slip events none of the blocks are in the high force range, and three neighboring blocks are in the medium regime. The motion of the upper surface would cause these three blocks to reach the high force regime at some point, which would again break the requirement for solitary motion. Adding even more blocks in the low force regime gives the same results. Therefore, to be in the solitary state, two blocks in the high force regime have to be separated by exactly one block in the low force regime, as illustrated in Table 3.3.

For a chain of \( N \) blocks there are \( 3^N \) ways of distributing the blocks into the three force regimes. In the following we will calculate how many solitary configurations there are for an infinitely slow moving upper surface. There must be an even number \( n \) of blocks alternatingly having a force \( F'_i = 1 \) and \( F'_i = -1 \). There are \( 2^{(N)} \) possible

<table>
<thead>
<tr>
<th>( t )</th>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( t_2 )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( t_3 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( t_4 )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Configuration starting with one block (\( i = 3 \)) in the high force regime and the rest in the medium regime. At \( t_4 \) the force on block \( i = 6 \) is twice the static friction force, and so either blocks 5 and 6, or blocks 6 and 1 will move at the same time.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( t_2 )</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( t_3 )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: Configuration starting with one block (\( i = 3 \)) in the high force regime, two blocks (\( i = 1 \) and \( i = 5 \)) in the low force regime and the rest in the medium regime. At \( t_3 \) there are no blocks in the high force regime and neighboring blocks 1, 5 and 6 are in the medium regime. These three will move at the same time.
3.C Features of the solitary state

ways to distribute \( n \) of these alternating blocks, and so there are

\[
2 \sum_{n=\text{even} \neq 0}^{N} \binom{N}{n}
\]

ways of creating a solitary configuration. The binomial theorem states that:

\[
(x + y)^N = \sum_{i=0}^{N} \binom{N}{n} x^{n-k} y^k.
\]

If we choose \( x = 1 \) and \( y = 1 \) then

\[
(1 + 1)^N = 2^N = \sum_{i=0}^{N} \binom{N}{n}
\]

and if we choose \( x = 1 \) and \( y = -1 \) then

\[
(1 - 1)^N = 0 = \sum_{i=0}^{N} \binom{N}{n} (-1)^n
\]

adding equations 3.48 and 3.49 gives

\[
2^N = \sum_{i=0}^{N} \binom{N}{n} (1^n + (-1)^n) = 2 \sum_{n=\text{even}}^{N} \binom{N}{n},
\]

and since we cannot have zero blocks with force \( F_1 \pm 1 \) there are

\[
2 \sum_{n=\text{even} \neq 0}^{N} \binom{N}{n} = 2^N - 2,
\]

<table>
<thead>
<tr>
<th>( t ) ( i )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( t_3 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( t_5 = t_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 3.3: Configuration starting with one block \( i = 3 \) in the high force regime, one block \( i = 6 \) in the low force regime and the rest in the medium regime. At \( t_5 \) the configuration of \( t_1 \) is retrieved and no blocks have moved in concert.
ways of creating a solitary configuration. For large $N$ the fraction of solitary configurations decreases exponentially with the chain length

$$\frac{2^N - 2}{3^N} \approx \exp(-0.4N)$$

(3.52)

which may explain why it takes much longer for larger chains to reach the solitary state. It is remarkable that the solitary state readily spans systems as large as $N = 32,000$, suggesting that the solitary state is not just a finite size effect. In practice, with a finite upper surface velocity $v_s$, the distance between two consecutive blocks in the high force regime is limited by Eq. 3.16, and this distance decreases with increasing $k_2$. This limits the amount of possible configurations for higher $k_2$. The last requirement that further decreases the amount of solitary states is that the forces on neighboring blocks in different force ranges should be at least $k_2 \Delta q + v_s \delta t$ apart, as blocks that slip increase the force on neighboring blocks by $k_2 \Delta q$, and the motion of the upper surface during this slip event increases the force by $v_s \delta t$. This is another reason why increasing the sliding velocity decreases the amount of solitary states.

To illustrate the motion in the solitary state, in Figure 3.14 we show the time evolution of part of a system in the solitary state for exactly one period. Blocks in the high force regime are shown in black, blocks in the medium force regime are shown in gray, and blocks in the low force regime are shown in white. For comparison the forces at all these times are shown in Fig. 3.15.

Appendix 3.D Creating a solitary state

Although we know how to calculate the forces on the blocks from their positions, and we know the requirements on these forces for the system to be in a solitary state, we do not yet know how to generate block positions that satisfy the solitary state. In this Appendix we show how to generate a system in the solitary state by first applying the rules on the block forces, and then calculating the block positions by inverting the equations of motion.

One can easily create a system of forces in the solitary state, by following a few simple rules. It is easiest to start with a block in the high force regime.

- After each block in the high force regime the next block must either be in the low force regime, or in the medium force regime with a force $\Delta F = k_2 \Delta q + v_s \delta t$ less then the one in the high force regime.

- After each block in the low force regime the next block must either be in the high force regime, or in the medium force regime with a force $\Delta F = k_2 \Delta q + v_s \delta t$ more then the one in the low force regime.

- In the medium force regime it depends on whether you came from either the low (a) or the high (b) force regime.
3.D Creating a solitary state

![Figure 3.13](image)

**Figure 3.13:** Algorithmically generated solitary state. (a) The forces $F_i$ of the solitary state, as described in Section 3.D. Dashed lines denote the boundaries between the low, medium and high force regime. (b) The positions calculated from the forces of (a), with Eq. 3.57. The dashed line at $q_i = 0$ is a guide to the eye.

Parameters: $\gamma = 0.5, k_2 = 1, N = 100, v_s = 0.005$.

- After each block, coming from a low force regime, the next block must either be in the high force regime, with a force $\Delta F = k_2 \Delta q$ more then this block, or in the medium force regime, with a force $\Delta F = v_s \delta t$ higher.

- After each block, coming from a high force regime, the next block must either be in the low force regime, with a force $\Delta F = k_2 \Delta q$ less then this block, or in the medium force regime, with a force $\Delta F = v_s \delta t$ lower.

After creating a system of $N$ forces, one must make sure that the solitary state rules also apply when taking periodic boundary conditions into consideration. In Fig. 3.13(a) we have used these rules, captured in an algorithm, to generate a random solitary state.

To use these rules to generate input for the numerical integration of the equations of motion (Eq. 3.7), one needs the positions, not the forces. How to determine block positions from their forces is discussed in the following section.

### 3.D.1 Determining block positions from the forces

In Eqs. 3.53 to 3.57 we show in detail how to derive the positions corresponding to a distribution of the forces along the chain that gives rise to solitary motion. For a
Figure 3.14: Snapshots of part of a system in solitary motion. Note that the time between two slip events of the same block is $\Delta q/v_s = 92.5$. The black dots on the surfaces show the relative position of the upper and lower surface. Blocks are colored in order of increasing force, with white blocks experiencing the maximum negative force, and black blocks experiencing the maximum positive force. A white cross denotes a block that is about to slip, a black cross denotes a block that has just slipped, a black circle denotes a moving block.

Parameters: $\gamma = 0.5$, $k_2 = 1$, $N = 1000$, $v_s = 0.005$. 

$t = 0.00$

$t = 6.61$

$t = 13.22$

$t = 19.83$

$t = 26.44$

$t = 33.05$

$t = 39.66$

$t = 46.27$

$t = 52.88$

$t = 59.49$

$t = 66.10$

$t = 72.71$

$t = 79.32$

$t = 85.93$

$t = 92.54$
Figure 3.15: 15 snapshots of the forces $\sigma_i$ in part of a system in solitary motion. The forces correspond to the positions shown in Fig. 3.14. Note that the time between two slip events of the same block is $\Delta q/v_s = 92.5$.

Parameters: $\gamma = 0.5$, $k_2 = 1$, $N = 1000$, $v_s = 0.005$. 
system in which all velocities \( \dot{q}_i \) are zero, all the forces are given by

\[
x - q_i + k_2(q_{i+1} + q_{i-1} - 2q_i) = f_n
\]

(3.53)

\[
k_2q_{i+1} + k_2q_{i-1} - (1 + 2k_2)q_i = f_n - x = f'_n
\]

(3.54)

This is the discrete version of the Helmholtz equation

\[
aq(x) - b \frac{d^2q(x)}{dx^2} = f(x),
\]

which has solutions of the form

\[
q(x) = \int_{-\infty}^{\infty} f(x') \exp(-\alpha|x - x'|) dx'.
\]

This inspires the ansatz \( q_i = A \sum_j f_j z^{i-j} \), where the value \( z < 1 \) is left to be determined. Substituting the ansatz in Eq. 3.54 gives:

\[
\frac{f_i}{A} = k_2 \sum_{j=-n+1}^{n+1} f_j z^{i+1-j} + k_2 \sum_{j=-i-n}^{i+n} f_j z^{i+1-j} - (1 + 2k_2) \sum_{j=-i-n}^{i+n} f_j z^{i-j}
\]

First we need to get rid of the absolute signs, therefore we split the sums in two parts:

\[
\frac{f_i}{A} = k_2 \left( \sum_{j=-n+1}^{i+1} f_j z^{i+1-j} + \sum_{j=i+2}^{i+n+1} f_j z^{j-i-1} \right)
\]

\[ + k_2 \left( \sum_{j=-i-n}^{i-1} f_j z^{i-j} + \sum_{j=i}^{i+n+1} f_j z^{j-i+1} \right)
\]

\[ - (1 + 2k_2) \left( \sum_{j=-i-n}^{i} f_j z^{i-j} + \sum_{j=i+1}^{i+n} f_j z^{j-i} \right).
\]

All the sums have an \( i \) in the summation index. We can take it out of the sum

\[
\frac{f_i}{A} = k_2 \left( z \sum_{j=-n+1}^{1} f_{i+j} z^{-j} + \frac{1}{z} \sum_{j=2}^{n+1} f_{i+j} z^j \right)
\]

\[ + k_2 \left( \frac{1}{z} \sum_{j=-n+1}^{-1} f_{i+j} z^{-j} + z \sum_{j=0}^{i+n} f_{i+j} z^j \right)
\]

\[ - (1 + 2k_2) \left( \sum_{j=-n}^{0} f_{i+j} z^{-j} + \sum_{j=1}^{n} f_{i+j} z^j \right),
\]
We give a minus sign to the summation index of all the summations running over a negative index

\[
\frac{f_i}{\Lambda} = k_2 \left( z \sum_{j=-1}^{n-1} f_{i-j} z^j + \frac{1}{z} \sum_{j=2}^{n+1} f_{i+j} z^j \right)
+ k_2 \left( \frac{1}{z} \sum_{j=1}^{n+1} f_{i-j} z^j + z \sum_{j=0}^{n+1} f_{i+j} z^j \right)
- (1 + 2k_2) \left( \sum_{j=0}^{n} f_{i-j} z^j + \sum_{j=1}^{n} f_{i+j} z^j \right),
\]

and we sum all left parts the same way, as well as the right parts, and subtract or add the parts that were forgotten or added by the new summation index:

\[
f_i = \left( z + \frac{1}{z} \right) k_2 - (1 + 2k_2) \left( \sum_{j=0}^{n} f_{i-j} z^j + \sum_{j=1}^{n} f_{i+j} z^j \right)
+ k_2 f_{i+1} - k_2 f_{i+1} - k_2 \frac{1}{z} f_i + k_2 z f_i,
\]

This equation is satisfied if

\[
\left( z + \frac{1}{z} \right) = \frac{1 + 2k_2}{k_2} \quad \Rightarrow \quad z = \frac{1 + 2k_2 \pm \sqrt{1 + 4k_2^2}}{2k_2},
\]

and because \(z < 1\):

\[
z = \frac{1 + 2k_2 - \sqrt{1 + 4k_2^2}}{k_2}.
\]

(3.55)

If all the summations vanish, it is easy to see that:

\[
A = \frac{1}{k_2(z - 1/z)}.
\]

(3.56)

So to transform from forces to positions we need to calculate

\[
q_i = \frac{1}{k_2(z - 1/z)} \sum_{i-n}^{i+n} f_j z^{j|i-j|}.
\]

(3.57)

The accuracy of Eq. 3.57 depends on \(n\). For \(k_2 = 1\) and \(\gamma = 0.5\) machine precision (for double precision floating point calculations) is reached at around \(n = 40\). In Figure 3.13(b) the positions calculated from the forces of Fig. 3.13(a) are shown. The forces shown in Fig. 3.13(a) and forces calculated from Eq. 3.53 and the positions of 3.13(b) are indistinguishable.
Figure 3.16: Illustration of bond percolation on a square grid. The solid lines denote connections between grid points, and there is no connection between grid points if there is a dashed line. A connectivity parameter $p$ determines the probability of a connection between grid points. Above a critical value $p_c$ of this parameter an infinitely sized grid will always have a spanning connection between the left and the right side, below this parameter there will never be one. Around this parameter, finitely sized grids behave critically. Left: standard percolation, both the upper and the lower path connect the left and the right side. Right: directed percolation. The connections can only be made in one direction, and so the upper path does not connect left to right, while the lower path does.

Appendix 3.E An outlook on the BK model of multicontact friction; directed percolation

In this chapter we have shown that the BK model of multicontact friction exhibits a solitary state for $k_2 \lesssim 1.5$ with $\gamma \approx 0.5$. We have also shown that all observables can be calculated analytically in this solitary state. This is very useful since macroscopic observables of systems that are not in the solitary state, but are in this range of parameters, are nonetheless very close to the ones we can calculate from the solitary state approximation. However, although we know everything about the solitary state we are not yet able to fully understand it. In particular, the question remains how a system ends up in the solitary state, and why the solitary state does not arise spontaneously for $k_2 \gg 1.5$. We have partially answered this question at the end of Appendix 3.C.3, by noting that for larger $k_2$ the demands on the forces become stricter. However, the existence of a threshold value of $k_2$ does not follow from the force rules. In this appendix we suggest that the answer to this problem could be found in the relation with the model of directed percolation, illustrated in Fig. 3.16.
Figure 3.5 shows that for low values of $k_2$ the solitary state spans the whole system, while for high values the solitary state stays localized. This seems to suggest that the transition from a collective state to a solitary state is comparable to a percolation transition, where $k_2$ takes the role of the connectivity parameter. If it exists, the value of $k_2$ below which an infinitely large system will eventually reach the solitary state, and above which it will never do, can then be identified as the critical connectivity parameter in percolation theory, and the system should behave critically around this parameter, with critical exponents that can be measured to identify its universality class.

Our model should be compared to percolation in two dimensions, with the block index number as the first dimension and time as the second. The fact that the collective state cannot go “back in time” furthermore suggests that if ever, the Burridge-Knopoff model should be related to directed percolation. Directed percolation was proposed to describe the flow of fluids through a porous material, driven along a specific direction (see figure 3.16b). Below a certain connectivity, the porous material is macroscopically impermeable to fluids, and increasing the connectivity above this value critically increases the permeability to a nonzero value. In our case, the solitary state can arise spontaneously from the collective states, but not vice versa (at zero temperature). This suggests that the collective state, depicted as dark patches in Figure 3.5, can be identified as the propagating fluid. So the left panel of Fig. 3.5 can be thought of as an impermeable material, whereas the right panel depicts a permeable material.

Although many systems seem to be examples of directed percolation, no experimental system has of yet been found that falls in the same universality class as directed percolation. The BK model of multicontact friction could be a candidate. Grassberger and Barkema [16] suggested that the calculations reported in Fig. 3.8, where the collective state is imposed at the open boundaries, has remarkable similarities to a directed percolation system that is driven at the edges perpendicular to the flow direction. The size distribution of the depth of percolation from the sides should be determined, and compared to critical surface exponents of a theoretical directed percolation model. Unfortunately we have not pursued this line of research, although we feel it could be very interesting.
3.6 References

[15] The distance a block slips in the initial collective stick-slip regime, is the maximum distance slipped by any block in the system. This distance is given by Eq. 3.11 with \( k_2 = 0 \), so that \( \omega = \sqrt{1 - \gamma^2} \).