Chapter 5

Selection of optimal weights in Generalized Processor Sharing

In the previous chapter we derived the delay asymptotics in a GPS queue, which was one of the problems mentioned in Section 1.7. In this chapter we turn to the other problem mentioned. That is, in contrast to the previous chapter, where the GPS weights were assumed to be fixed beforehand, we now analyze the selection of optimal GPS weights.

The problem of mapping the QoS requirements on suitable GPS weights has received little attention in the literature, see the overview in Chapter 1. The results of [60] on the weight setting problem rely on the restrictive assumption of leaky-bucket controlled traffic. The contribution of this chapter is that we extend their results on the weight setting to a general and versatile class of input processes, covering a broad range of correlation structures, viz. the class of Gaussian inputs.

We consider a two-class GPS system with Gaussian traffic sources. The QoS criterion is that the loss probability should be kept below some class-specific value. The large deviations approximations of [132] on GPS for Gaussian inputs are the key tool in our analysis. As a first step, we use these approximations to find the admissible region for class 1 for fixed weights, i.e., all numbers of sources $n_1$, $n_2$ of class 1 and class 2 such that the QoS requirement of class 1 is met. By taking the intersection of the admissible region of both classes, we then obtain the admissible region (of the system), i.e., all combinations of flows that satisfy the QoS for both classes. In the special case of Brownian inputs, we explicitly determine the boundary of the admissible region.

We then explicitly derive the realizable region as the union of the admissible regions over all possible weight values, in case of Brownian inputs. A remarkable finding is that nearly the entire realizable region is achieved by one of the strict priority scheduling disciplines. A further key observation is that the QoS requirements and the buffer thresholds fully determine which class should have high priority, if
such a strict priority policy would be imposed. Importantly, the above two remark-
able conclusions also hold for general Gaussian inputs. In the absence of an explicit
description of the boundary of the realizable region, we have relied on extensive nu-
merical experimentation.

The above results indicate that from an efficiency point of view GPS does not
outperform a simple priority discipline. In other words, it suggests that there is
hardly any efficiency improvement to be achieved by implementing GPS (compared to
priority scheduling), in that the admissible region corresponding to some GPS weight
vector, is contained in the admissible region corresponding to one of the priority cases.
It is worth pointing out one important caveat. By assigning positive weights to all
classes, GPS is capable of protecting a class against starvation when some other class
misbehaves, as opposed to priority scheduling, where the low-priority class may be
excluded from service over substantial time intervals.

The remainder of this paper is organized as follows. In Section 5.1 we describe our
two-class GPS model with Gaussian inputs, and review the Mannersalo-Norros ap-
proximations [132] for loss probabilities, which consist of three regimes. In Section 5.2
the stable region is partitioned into three subsets, each subset corresponding to one
of the three regimes. Using the partitioning of the stable region and the Mannersalo-
Norros approximations, we derive the admissible region in Section 5.3. In Section 5.4
we consider Brownian inputs, and explicitly derive the boundary of the admissible re-
gion and the boundary of the realizable region. In Section 5.5 we perform numerical
analysis. In particular, we consider systems shared by two types of applications with
heterogeneous QoS requirements, and numerically derive the realizable regions.

5.1 Preliminaries

In this section we introduce the notation of the two-class GPS model and discuss
Gaussian sources. Then we present approximations for the overflow probabilities.

5.1.1 Queueing model

We consider a model with two queues that share a server of rate $c$. Traffic of class $i$
is buffered in queue $i$, $i = 1, 2$. The scheduling discipline is GPS, with weight $\phi_i \geq 0$
assigned to class $i$, $i = 1, 2$. Without loss of generality we assume that $\phi_1 + \phi_2 = 1$.
The weight $\phi_i$ determines the guaranteed minimum rate for class $i$. If a class does not
fully use the minimum rate, then the excess capacity becomes available to the other
class.

5.1.2 Gaussian input traffic, overflow probabilities

As our first goal in Sections 5.2 and 5.3 is to characterize the admissible region (for
a given weight vector), we first present the Mannersalo-Norros approximations [132]
for the overflow probabilities for given numbers of sources of both classes.

Let class 1 (class 2) consist of a superposition of \( n_1 \) (\( n_2 \)) i.i.d. flows (or: sources), modeled as Gaussian processes with stationary increments. Clearly, \( n_1, n_2 \in \mathbb{N}_0 \), but for convenience we let \( n_1, n_2 \in \mathbb{R}_+ \). We denote the mean rate and variance function of a single class-\( i \) flow by \( \mu_i > 0 \) and \( \psi_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \), respectively, for \( i = 1, 2 \); this mean rate and variance curve fully characterize the probabilistic behavior of the flow. Hence, if \( A_i(s, t) \) denotes the amount of traffic generated by a single flow of type \( i \) in the interval \([s, t]\), then \( \mathbb{E}A_i(s, t) = \mu_i \cdot (t - s) \) and \( \text{Var}A_i(s, t) = \psi_i(t - s) \).

To guarantee stability we assume that \( n_1\mu_1 + n_2\mu_2 \leq c \) (which we refer to as the ‘capacity constraint’). We impose Assumptions A1-A3 on \( \psi_i(\cdot) \), see Chapter 2.

The derivation of the admissible regions relies on the Mannersalo-Norros approximations [132] for the overflow probabilities; these require Assumptions A1 and A2. On the basis of extensive simulation experiments, Mannersalo & Norros [132] showed the accuracy of their approximations. Assumption A3 is needed in the proofs of some lemmas.

Let \( Q_i \) denote the stationary buffer content in the GPS model of class \( i \), and \( \triangle_i(n_1, n_2) \) the Mannersalo-Norros approximation of \(-\log P(Q_i > B_i)\). Define

\[
\psi(t|n_1, n_2) := \frac{1}{2} \inf_{t \geq 0} \frac{(b_1 + (c - n_1\mu_1 - n_2\mu_2)t)^2}{n_1v_1(t) + n_2v_2(t)}. \quad (5.1)
\]

We impose the following assumption on \( \psi(t|n_1, n_2) \).

**Assumption 5.1.1** For any \((n_1, n_2) \in \mathbb{R}^2_+ \) such that \( n_1\mu_1 + n_2\mu_2 \leq c \), \( \psi(t|n_1, n_2) \) has a unique minimizer \( t^F(n_1, n_2) \).

Clearly, \( t^F(n_1, n_2) \) depends on \((n_1, n_2)\), but for ease of notation, we will denote it by \( t^F \) in the remainder of this chapter. Due to Assumption A2, for any \((n_1, n_2) \in \mathbb{R}^2_+ \) such that \( n_1\mu_1 + n_2\mu_2 \leq c \), \( \lim_{t \to 0} \psi(t|n_1, n_2) = \lim_{t \to \infty} \psi(t|n_1, n_2) = \infty \), and thus a minimizer \( t^F \) of \( \psi(t|n_1, n_2) \) clearly exists, but it is not necessarily unique. We performed extensive numerical experiments with the often used variance functions \( \psi_i(\cdot) \), e.g., fractional Brownian motions, and the Gaussian counterpart of the Anick-Mitra-Sondhi (AMS) [11] model (see also Section 5.5), and observed that \( t^F \) was unique in all considered cases, making this uniqueness assumption a weak assumption. In fact, it turned out to be a non-trivial exercise to find a situation with multiple minimizers, see Figure 5.1 for a rare example with two minimizers. By slightly increasing \((n_1, n_2)\), we see that the minimizing \( t \) jumps from 0.2775 to 32.3631. For a related example, see Section 5 of [124].

Also, define

\[
\phi_2^F := \frac{n_2\mu_2}{c} + \left( \frac{n_2v_2(t^F)}{c t^F (n_1v_1(t^F) + n_2v_2(t^F))} \left(b_1 + (c - n_1\mu_1 - n_2\mu_2)t^F\right) \right). \quad (5.2)
\]
Due to the uniqueness of $t^F$, $\phi_2^F$ is unique as well, and is larger than $n_2\mu_2/c$. Then

$$\Delta_1(n_1, n_2) = \begin{cases} 
(i) & \frac{1}{2} \inf_{t \geq 0} \left( \frac{(b_1 + (\phi_2 c - n_1 \mu_1))t}{n_1 v_1(t)} + \frac{\phi_2 c - n_2 \mu_2}{n_2 v_2(t)} \right) \quad \text{for } \phi_2 \in [0, \frac{n_2 \mu_2}{c}]; \\
(ii) & \frac{1}{2} \inf_{t \geq 0} \left( \frac{b_1 + (\phi_2 c - n_1 \mu_1)t}{n_1 v_1(t)} \right) \quad \text{for } \phi_2 \in [\frac{n_2 \mu_2}{c}, \phi_2^F]; \\
(iii) & \frac{1}{2} \inf_{t \geq 0} \left( \frac{b_1 + (\phi_2 c - n_1 \mu_1)t}{n_1 v_1(t) + n_2 v_2(t)} \right) \quad \text{for } \phi_2 \in [\phi_2^F, 1]. 
\end{cases}$$

The approximations $\Delta_2(n_1, n_2)$ are analogous; evidently, we can now approximate $P(Q_1 > B_1)$ by $\exp(-\Delta_1(n_1, n_2))$. We now heuristically explain the three regimes (i), (ii), (iii). As the first and the third have the easiest explanation we start there, before turning to the second regime.

In regime (i) we have that $\phi_2 c \leq n_2 \mu_2$. That is, the mean traffic rate generated by class 2 exceeds the guaranteed rate of service to class 2 (we call this: class 2 in overload). Therefore, it is very likely that type-2 sources claim their guaranteed service rate $\phi_2 c$ essentially all the time. Hence, overflow in queue 1 resembles overflow in a FIFO queue with service rate $\phi_1 c$. The approximation $\Delta_1(n_1, n_2)$ of regime (i) is based on this principle, cf. [7]. The minimizing $t$ represents the (most likely) length of the interval between the epoch queue 1 starts to build up, until it reaches buffer content $b_1$.

Regime (iii) requires $\phi_2$ to be at least as large as $\phi_2^F$. It can be verified (by using the explicit formulae for conditional means of Normal random variables) that $\phi_2^F$ is equal to the value of $\phi_2$ for which

$$\mathbb{E}(A_2(-t^F, 0) | A_1(-t^F, 0) + A_2(-t^F, 0) = b_1 + ct^F) = \phi_2 c t^F.$$  \hfill (5.3)

Hence, if $\phi_2 \geq \phi_2^F$, conditioned on the total queue building up $b_1$ in $t^F$ time units, then all this traffic is in queue 1, and queue 2 is essentially empty.
5.2 Partitioning of the stable region

Regime (ii) applies if class 2 is underloaded, but \( \phi_2 \leq \phi_2^F \). When the total queue reaches level \( b_1 \), it is now very likely that the queue of class 2 is non-empty. Hence, an additional constraint must be imposed to keep the buffer content of queue 2 small. The approximation is such that the flows of class 1 generate \( b_1 + \phi_1 ct \), while the class-2 sources generate \( \phi_2 ct \) (i.e., the class-2 sources claim their guaranteed rate). Note that in the approximation it is used that the interval in which the class-2 sources claim rate \( \phi_2 c \) coincides with the interval in which queue 1 builds up. For a refinement of this approximation we refer to [129], which allows scenarios in which the first queue starts to build up before the second queue reaches traffic rate \( \phi_2 c \).

5.2 Partitioning of the stable region

In order to derive the admissible region (for given weights) of the two-class GPS system, we have to determine the admissible region of each class separately and then take the intersection of these two regions. In Sections 5.2 and 5.3, without loss of generality, we focus on the admissible region of the first class (i.e., the set of sources \((n_1, n_2)\) for which the class-1 sources receive the desired QoS), as the second one can be treated in the same fashion. Before the admissible region of the first class can be obtained, which we will do in Section 5.3, we first determine all \((n_1, n_2)\) for which (i) \( \phi_2 \in [0, n_2 \mu_2/c] \), (ii) \( \phi_2 \in (n_2 \mu_2/c, \phi_2^F) \) and (iii) \( \phi_2 \in [\phi_2^F, 1] \), thus partitioning the stable region \( T := \{(n_1, n_2) : n_1 \mu_1 + n_2 \mu_2 \leq c\} \) into three sets. In these three sets we can use the approximation of \( \Delta_1(n_1, n_2) \) presented in Section 5.1.2.

Lemma 5.2.1 Let \( \phi_1 \in (0, 1) \). Then \( T = T_1^i(\phi_1) \cup T_1^{ii}(\phi_1) \cup T_1^{iii}(\phi_1) \) for disjoint non-empty \( T_1^i(\phi_1) \), \( T_1^{ii}(\phi_1) \) and \( T_1^{iii}(\phi_1) \), where

\[
T_1^i(\phi_1) := \left\{ (n_1, n_2) \in T : n_2 \geq \frac{\phi_2 c}{\mu_2} \right\}
\]

\[
T_1^{ii}(\phi_1) := \left\{ (n_1, n_2) \in T : n_2 < \frac{\phi_2 c}{\mu_2}, \quad \frac{b_1 + (\phi_1 c - n_1 \mu_1) t_F}{n_1 v_1(t_F)} > \frac{(\phi_2 c - n_2 \mu_2) t_F}{n_2 v_2(t_F)} \right\};
\]

\[
T_1^{iii}(\phi_1) := \left\{ (n_1, n_2) \in T : n_2 < \frac{\phi_2 c}{\mu_2}, \quad \frac{b_1 + (\phi_1 c - n_1 \mu_1) t_F}{n_1 v_1(t_F)} \leq \frac{(\phi_2 c - n_2 \mu_2) t_F}{n_2 v_2(t_F)} \right\};
\]

such that regime \((j)\) applies in \( T_1^j(\phi_1) \), for \( j \in \{i, ii, iii\} \).

Proof: \( T_1^i(\phi_1) \) follows from the fact that we must have \( \phi_2 \in [0, n_2 \mu_2/c] \). In order to be in \( T_1^{ii}(\phi_1) \) we must have that \( \phi_2 \in (n_2 \mu_2/c, \phi_2^F) \), or equivalently \( n_2 < \phi_2 c/\mu_2 \) and \( \phi_2 < \phi_2^F \). The latter inequality can be rewritten as

\[
\phi_2 < \frac{n_2 \mu_2}{c} + \left( \frac{n_2 v_2(t_F) (b_1 + (c - n_1 \mu_1 - n_2 \mu_2) t_F)}{ct_F (n_1 v_1(t_F) + n_2 v_2(t_F))} \right).
\]
Multiply both sides with $c t^F$, and rearrange the right-hand side to obtain
\[
\phi_2 c t^F < \left( \frac{n_2 v_2(t^F)}{n_1 v_1(t^F)} \left( b_1 + c t^F - n_1 \mu_1 t^F \right) \right) + \frac{n_1 v_1(t^F) n_2 \mu_2 t^F}{n_1 v_1(t^F) + n_2 v_2(t^F)}.
\]
Multiplying both sides with $n_1 v_1(t^F) + n_2 v_2(t^F)$ and collecting ‘equivalent terms’ leads to
\[
n_1 v_1(t^F) \left( \phi_2 c t^F - n_2 \mu_2 t^F \right) < n_2 v_2(t^F) \left( b_1 + \phi_1 c t^F - n_1 \mu_1 t^F \right).
\]
Dividing both sides by $n_1 v_1(t^F)$ and $n_2 v_2(t^F)$ respectively gives
\[
\frac{b_1 + (\phi_1 c - n_1 \mu_1) t^F}{n_1 v_1(t^F)} > \frac{\phi_2 c - n_2 \mu_2) t^F}{n_2 v_2(t^F)}.
\]
(5.4)
The characterization of $T^{iii}_1(\phi_1)$ follows similarly.

In case $\phi_1 \in (0, 1)$, all three sets are non-empty, and this proves the stated. Note that $T = T^{iii}_1(0)$ for $\phi_1 = 0$ and $T = T^{iii}_1(1)$ for $\phi_1 = 1$.

Now consider the boundary between $T^{ii}_1(\phi_1)$ and $T^{iii}_1(\phi_1)$, i.e., combinations of $(n_1, n_2)$ such that (5.4) holds with equality. For most of the $v_i(\cdot)$ curves we considered, this boundary could not be explicitly expressed in terms of a function $f_1(n_2) = n_1$; to compute the boundary, one needs to resort to numerical methods. However, some characteristics of $f_1(\cdot)$ can be derived and are presented in the following lemma.

**Lemma 5.2.2** The following statements can be made about $f(\cdot)$:

(I) $f_1(0) = 0$;

(II) $\phi_1 c / \mu_1$;

(III) $f_1(\cdot)$ only intersects the capacity constraint at $(n_1, n_2) = (\phi_1 c / \mu_1, \phi_2 c / \mu_2)$;

(IV) $f_1(\cdot)$ only intersects the line $n_2 = \phi_2 c / \mu_2$ at $(n_1, n_2) = (\phi_1 c / \mu_1, \phi_2 c / \mu_2)$;

(V) $f_1(\cdot)$ only intersects the $n_1$-axis and $n_2$-axis at $(n_1, n_2) = (0, 0)$.

**Proof:** If $(n_1, n_2) = (0, 0)$, then we have clearly equality in (5.4) (as both sides have value $\infty$), so this gives (I). We continue with (II). Take the point $(n_1, n_2) = (\phi_1 c / \mu_1, \phi_2 c / \mu_2)$. Then it follows that $t^F = \infty$, as $v_i(t)$ is increasing in $t$ by A3, $i = 1, 2$. Plugging $t^F = \infty$ in (5.4), we find that there is equality there, no matter the value of $b_1$. We proceed with (III). Note that $f_1(\cdot)$ is the line where $\Delta_1(n_1, n_2)$ of regimes (ii) and (iii) have equal values. Next define $S := \{(n_1, n_2) | n_1 \mu_1 + n_2 \mu_2 = c\}$. We find that for all $(n_1, n_2) \in S$ we have that $\Delta_1(n_1, n_2)$ of regime (iii) equals 0 (as $v_i(t)$ is increasing in $t$ by A3, $i = 1, 2$). Now note that the only $(n_1, n_2) \in S$ for which $\Delta_1(n_1, n_2) \in S$ for which $\Delta_1(n_1, n_2)$ of regime (ii) equals zero is $(\phi_1 c / \mu_1, \phi_2 c / \mu_2)$. Thus, line $f_1(\cdot)$ only intersects the capacity constraint at $(n_1, n_2) = (\phi_1 c / \mu_1, \phi_2 c / \mu_2)$. We prove (IV) in a similar fashion. If $n_2 = \phi_2 c / \mu_2$, then for regime (ii):

\[
\Delta_1(n_1, \phi_2 c / \mu_2) = \frac{1}{2} \inf_{t \geq 0} \frac{(b_1 + (\phi_1 c - n_1 \mu_1) t)^2}{n_1 v_1(t)}.
\]
whereas for regime (iii):

\[
\Delta_1(n_1, \phi_2 c/\mu_2) = \frac{1}{2} \inf_{t \geq 0} \frac{b_1 + (\phi_1 c - n_1 \mu_1)t}{n_1 v_1(t) + \frac{\phi_2 c}{\mu_2} v_2(t)}.
\]

These two can only be equal if \( n_1 = \phi_1 c/\mu_1 \) as then the optimizer is \( t = \infty \), and we obtain \( \Delta_1(\phi_1 c/\mu_1, \phi_2 c/\mu_2) = 0 \) for regimes (ii) and (iii). We conclude with (V). If \( n_1 = 0 \) or \( n_2 = 0 \) (but not both), then \( \Delta_1(n_1, n_2) \) of regime (ii) equals \( \infty \), whereas \( \Delta_1(n_1, n_2) \) of regime (iii) is bounded. Hence, except for \( (n_1, n_2) = (0, 0) \), \( f_1(\cdot) \) cannot intersect the \( n_1 \)-axis and \( n_2 \)-axis.

In our numerical experiments with the often used variance functions \( v_i(\cdot) \), e.g., fractional Brownian motions, the Gaussian counterpart of the AMS model, and others as presented in [7], we observed that \( f_1(\cdot) \) is strictly increasing, as depicted in Figure 5.2.

### 5.3 Analysis of the admissible region

In this section we analyze the admissible region of the first class (for given weights), i.e., all combinations of \( (n_1, n_2) \) that satisfy \( \Delta_1(n_1, n_2) \geq \delta_1 \), for some \( \delta_1 > 0 \). We show that this set consists of three disjoint subsets: \( S_1(\phi_1) = S_{i}^1(\phi_1) \cup S_{ii}^1(\phi_1) \cup S_{iii}^1(\phi_1) \), with \( S_{j}^i(\phi_1) \subset T_j^i, j \in \{i, ii, iii\} \), which we derive below. Finally, we present our main result that characterizes the boundary of \( S_1(\phi_1) \). Again we concentrate on \( S_1(\phi_1) \), but of course \( S_2(\phi_1) \) can be treated analogously, thus determining the admissible region \( S(\phi_1) := S_1(\phi_1) \cap S_2(\phi_1) \).
5.3.1 Region $S^1_i(\phi_1)$

We define $S^1_i(\phi_1)$ as the subset of $T^1_i(\phi_1)$ (see Section 5.2), for which $\triangle_1(n_1, n_2) \geq \delta_1$. That is,

$$\triangle_1(n_1, n_2) = \frac{1}{2} \inf_{t \geq 0} \frac{(b_1 + (\phi_1 c - n_1 \mu_1) t)^2}{n_1 v_1(t)} \geq \delta_1.$$ 

Rearranging and collecting terms yields

$$n_1 \leq \max \left\{ n_1 : \forall t \geq 0 : X_t n_1^2 + Y_t n_1 + Z_t \geq 0 \right\},$$

where

$$X_t := \mu_1^2 t^2; \quad Y_t := -2b_1 \mu_1 t - 2\phi_1 \mu_1 t^2 - 2\delta_1 v_1(t); \quad Z_t := b_1^2 + \phi_1^2 c^2 t^2 + 2b_1 \phi_1 c t.$$ 

This eventually leads to

$$n_1 \leq n^Q_1 := \max \left\{ n_1 : n_1 \leq \inf_{t \geq 0} -Y_t - \sqrt{Y_t^2 - 4X_t Z_t} \right\},$$

and

$$n_1 \leq \max \left\{ n_1 : n_1 \geq \inf_{t \geq 0} -Y_t + \sqrt{Y_t^2 - 4X_t Z_t} \right\} = \infty.$$ 

Clearly, $n_1 \leq \infty$ always holds, so this constraint is redundant. It is noted that $\triangle_1(\phi_1 c/\mu_1, n_2)$ of regime $(i)$ equals 0, as it is minimized for $t = \infty$ by A3. Since we require that $\triangle_1(n_1, n_2) \geq \delta_1 > 0$, this implies that $n^Q_1 < \phi_1 c/\mu_1$. An example of a set $S^1_i(\phi_1)$ is depicted in Figure 5.3 (top, left).

5.3.2 Region $S^2_i(\phi_1)$

In this regime $S^2_i(\phi_1)$ consists of all combinations $(n_1, n_2)$ in $T^2_i(\phi_1)$ such that

$$\triangle_1(n_1, n_2) = \frac{1}{2} \inf_{t \geq 0} \left( \frac{(b_1 + (\phi_1 c - n_1 \mu_1) t)^2}{n_1 v_1(t)} + \frac{(\phi_2 c - n_2 \mu_2)^2 t^2}{n_2 v_2(t)} \right) \geq \delta_1.$$ 

Proceeding in the same manner as above, this reduces to

$$n_2 \leq g_1(n_1) := \inf_{t \geq 0} -Y_t - \sqrt{Y_t^2 - 4X_t Z_t} \frac{2X_t}{2X_t},$$

(5.6)
5.3 Analysis of the admissible region

Figure 5.3: The typical partitioning of the admissible region of the first queue $S_1(\phi_1)$. Top, left: $S_1(\phi_1)$. Top, right: $S_1^{ii}(\phi_1)$. Bottom: $S_1^{iii}(\phi_1)$.

where

$$X_t := \frac{\mu_2^2 t^2}{v_2(t)};$$
$$Y_t := \frac{(b_1 + (\phi_1 c - n_1 \mu_1) t)^2}{n_1 v_1(t)} - \frac{2 \phi_2 c \mu_2 t^2}{v_2(t)} - 2 \delta_1;$$
$$Z_t := \phi_2^2 c^2 t^2 / v_2(t).$$

As $g_1(\cdot)$ plays an important role in describing the boundary of $S_1(\phi_1)$, the remainder of this subsection is devoted to some structural properties of $g_1(\cdot)$. First notice that

$$\frac{1}{2} \inf_{t \geq 0} \left( \frac{(b_1 + (\phi_1 c - n_1 \mu_1) t)^2}{n_1 v_1(t)} + \frac{(\phi_2 c - n_2 \mu_2)^2 t^2}{n_2 v_2(t)} \right) \geq \frac{1}{2} \inf_{t \geq 0} \frac{(b_1 + (\phi_1 c - n_1 \mu_1) t)^2}{n_1 v_1(t)} + \frac{1}{2} \inf_{t \geq 0} \frac{(\phi_2 c - n_2 \mu_2)^2 t^2}{n_2 v_2(t)};$$
the first part of the right-hand side of the last equation coincides with the loss probability of regime (i). By definition all $n_1 \leq n_1^{Q_1}$ satisfy the loss constraint of regime (i). Hence, all $n_1 \leq n_1^{Q_1}$ \forall (n_2)$ satisfy the loss constraint of regime (ii) as well. One can easily see that if $n_2 = \phi_2c/\mu_2$, then the loss probability of the middle regime reduces to that of the first regime. Thus, this implies that $g_1(n_1^{Q_1}) = \phi_2c/\mu_2$ and that $g_1(\cdot)$ is only defined on the interval $[n_1^{Q_1}, \infty)$.

**Lemma 5.3.1** $g_1(\cdot)$ is strictly decreasing on the interval $[n_1^{Q_1}, \phi_1c/\mu_1]$.

**Proof:** First note that $g_1(\cdot)$ corresponds to all possible combinations $(n_1, n_2)$ for which $\Delta_1(n_1, n_2)$ of regime (ii) equals $\delta_1$, i.e., $\Delta_1(n_1, n_1(n_1)) = \delta_1$. Consider $(n_1, n_2) = (a, b)$, with $g_1(a) = b$, where $n_1^{Q_1} < a < \phi_1c/\mu_1$ and $b < \phi_2c/\mu_2$, or equivalently

$$\inf_{t \geq 0} \left( \frac{(b_1 + (\phi_1c - \phi_1c)t)^2}{2av_1(t)} + \frac{(\phi_2c - b\mu_2)^2t^2}{2bv_2(t)} \right) = \delta_1.$$  

Let an optimizer be denoted by $t^\circ$. Now, consider the point $(n_1, n_2) = (a + \epsilon_a, b + \epsilon_b)$, with $\epsilon_a \in (0, \phi_1c/\mu_1 - a)$ and $\epsilon_b \in (0, \phi_2c/\mu_2 - b)$. Clearly,

$$\inf_{t \geq 0} \left( \frac{(b_1 + (\phi_1c - (a + \epsilon_a)c\mu_1)t)^2}{2(a + \epsilon_a)v_1(t)} + \frac{(\phi_2c - (b + \epsilon_b)\mu_2)^2t^2}{2(b + \epsilon_b)v_2(t)} \right) \leq \frac{(b_1 + (\phi_1c - (a + \epsilon_a)c\mu_1)t^\circ)^2}{2(a + \epsilon_a)v_1(t^\circ)} + \frac{(\phi_2c - (b + \epsilon_b)\mu_2)^2(t^\circ)^2}{2(b + \epsilon_b)v_2(t^\circ)} < \delta_1.$$  

Thus, we have that $\Delta_1(a + \epsilon_a, b + \epsilon_b) < \delta_1$, implying that it is impossible that $g_1(a + \epsilon_a) = b + \epsilon_b$. In the same manner we can also prove that it is impossible that $g_1(a + \epsilon_a) = b + \epsilon_b$, and $g_1(a - \epsilon_a) = b - \epsilon_b$, with $\epsilon_a \in (0, a - n_1^{Q_1}) \land \epsilon_b \in (0, b)$, but not both 0. Hence, there must exist a value $x > b$ such that we have $\Delta_1(a - \epsilon_a, x) = \delta_1$, i.e., $g_1(a - \epsilon_a) = x$, and there must exist a value $y < b$ such that we have $\Delta_1(a + \epsilon_a, y) = \delta_1$, i.e., $g_1(a + \epsilon_a) = y$. This proves that $g_1(\cdot)$ must be a strictly decreasing function of $n_1$ on the interval $[n_1^{Q_1}, \phi_1c/\mu_1]$.

In Section 5.2 we remarked that for the often used variance functions, the function $f_1(\cdot)$, which separates regime (ii) from regime (iii), is increasing on the interval $[0, \phi_2c/\mu_2]$, with $f(\phi_2c/\mu_2) = \phi_1c/\mu_1$. As $g_1(\cdot)$ is strictly decreasing on the interval $[n_1^{Q_1}, \phi_1c/\mu_1]$, with $g_1(n_1^{Q_1}) = \phi_2c/\mu_2$, we conjecture that $f_1(\cdot)$ and $g_1(\cdot)$ intersect at a unique point $(n_1, n_2) = (n_1^1, n_2^1)$, with $n_1^{Q_1} < n_1^1 < \phi_1c/\mu_1$ and $n_2^1 < \phi_2c/\mu_2$; in Section 5.4 we will show that for Brownian motion inputs this claim is true. We also validated this conjecture by performing numerous numerical experiments with other Gaussian inputs. In none of these cases a counter example could be found. Then a typical shape of the region $S_1^1(\phi_1)$ would be like Figure 5.3 (top, right).
5.3 Analysis of the admissible region

5.3.3 Region \( S^{iii}(\phi_1) \)

\( S^{iii}(\phi_1) \) consists of all combinations of \((n_1, n_2)\) in \( T^{iii}(\phi_1) \) such that

\[
\Delta_1(n_1, n_2) = \frac{1}{2} \inf_{t \geq 0} \frac{(b_1 + (c - n_1 \mu_1 - n_2 \mu_2)t)^2}{n_1 v_1(t) + n_2 v_2(t)} \geq \delta_1.
\]

Once again, standard rewriting yields

\[
n_2 \leq h_1(n_1) = \inf_{t \geq 0} \frac{-Y_t - \sqrt{Y_t^2 - 4X_tZ_t}}{2X_t},
\]

where

\[
X_t := \mu_2^2 t^2;
Y_t := 2n_1 \mu_1 \mu_2^2 - 2\delta_1 v_2(t) - 2b_1 \mu_2 t - 2c_1 t^2;
Z_t := b_1^2 + 2b_1 c t + c_1^2 t^2 + n_2^2 \mu_2^2 t^2 - 2b_1 n_1 \mu_1 t - 2c_1 n_1 t^2 - 2\delta_1 n_1 v_1(t).
\]

Let \( n_1^{\text{max}} \) denote the value of \( n_1 \) that solves \( h_1(n_1) = 0 \). The following lemma states some properties of \( h_1(\cdot) \).

**Lemma 5.3.2** \( h_1(\cdot) \) is strictly decreasing on the interval \([0, n_1^{\text{max}}]\) and tighter than the capacity constraint. Furthermore, \( g_1(n_1) \geq h_1(n_1) \) for all \( n_1 \in [n_1^Q, n_1^{\text{max}}] \).

**Proof:** The proof of the first statement is similar to Lemma 5.3.1. We now show that \( h_1(\cdot) \) is tighter than the capacity constraint. If \( n_1 \mu_1 + n_2 \mu_2 = c \) (capacity constraint), then \( \Delta_1(n_1, n_2) \) of regime (iii) equals 0, as the optimizer is \( t^F = \infty \), due to A3. Note that the line \( h_1(\cdot) \) are all \((n_1, n_2)\) such that \( \Delta_1(n_1, n_2) \) of regime (iii) equals \( \delta_1 \), with \( \delta_1 > 0 \). Hence, the capacity constraint cannot be part of \( h_1(\cdot) \), implying that \( h_1(\cdot) \) is tighter than the capacity constraint, i.e., it lies below the capacity constraint.

We proceed with the proof of the last statement. We first prove that \( \Delta_1(n_1, n_2) \) of regime (ii) is at least as large as the one of regime (iii), and then we use this to show that \( g_1(n_1) \geq h_1(n_1) \) for all \( n_1 \in [n_1^Q, n_1^{\text{max}}] \). Let \( a_1 := b_1 + (\phi_1 c - n_1 \mu_1) t \), \( a_2 := (\phi_2 c - n_2 \mu_2) t \), \( v_1 := n_1 v_1(t) \) and \( v_2 := n_2 v_2(t) \). It can be seen that it suffices to prove that for all \( t \geq 0 \),

\[
\frac{a_1^2}{v_1} + \frac{a_2^2}{v_2} \geq \frac{(a_1 + a_2)^2}{v_1 + v_2}.
\]

Rearranging (5.8) yields \( a_1^2 v_2^2 + a_2^2 v_1^2 - 2 a_1 a_2 v_1 v_2 \geq 0 \), which is equivalent to \( (a_1 v_2 - a_2 v_1)^2 \geq 0 \), thus proving \( \Delta_1(n_1, n_2) \) of regime (ii) is at least as large as the one of regime (iii). Note that there is equality if \( a_1 v_2 = a_2 v_1 \), so in that case \( \Delta_1(n_1, n_2) \) of regimes (ii) and (iii) coincide and they have the same optimizer \( t^F \). Recall from Section 5.2 that \( a_1 v_2 = a_2 v_1 \), with \( t = t^F \), corresponds to the line \( f_1(\cdot) \).
By definition \( g_1(n_1) (h_1(n_1)) \) is the value of \( n_2 \) where \( \Delta_1(n_1, n_2) \) of regime (ii) (iii) equals \( \delta_1 \). Let \( \Delta_1(n_1, n_2) \) of regime \( x \), with \( x \in \{i, ii, iii\} \), be denoted by \( \Delta_i^x(n_1, n_2) \). Since we proved that \( \Delta_1(n_1, n_2) \) of regime (ii) is at least as large as the one of regime (iii), we have that
\[
\Delta_i^{iii}(n_1, g_1(n_1)) \leq \Delta_i^{ii}(n_1, g_1(n_1)) = \delta_1 = \Delta_i^{ii}(n_1, h_1(n_1)).
\]
In the same manner as Lemma 5.3.1, we can prove that \( \Delta_i^{ii}(n_1, n_2) \) decreases in \( n_2 \) for fixed \( n_1 \), given that \( n_1 \mu_1 + n_2 \mu_2 \leq c \), implying that \( g_1(n_1) \geq h_1(n_1) \) for all \( n_1 \in [n_1^{Q_1}, n_1^{\max}] \).

By definition, for \( (n_1, n_2) = (f_1(n_2), n_2) \) the approximations of \( \Delta_1(n_1, n_2) \) are equal for regimes (ii) and (iii) (see previous lemma). Hence, if \( f_1(\cdot) \) and \( g_1(\cdot) \) intersect at \( (n_1^n, n_2^n) \) (see Section 5.3.2), then this is also the point where \( f_1(\cdot) \) and \( h_1(\cdot) \) intersect. Figure 5.3 (bottom) illustrates the region \( S_1^{iii}(\phi_1) \).

5.3.4 Region \( S_1(\phi_1) \)

\( S_1(\phi_1) \) can be obtained by taking the union of the three described regions, i.e.,
\[
S_1(\phi_1) = S_1^{i}(\phi_1) \cup S_1^{ii}(\phi_1) \cup S_1^{iii}(\phi_1).
\]
We now state our main result, which follows from Sections 5.3.1-5.3.3.

**Theorem 5.3.3** The boundary of the admissible region of the first queue, \( S_1(\phi_1) \), is defined as follows:

\[
\begin{align*}
0 \leq n_1 &\leq n_1^{Q_1} : \quad n_2 = (c - n_1 \mu_1)/\mu_2; \\
n_1^{Q_1} < n_1 &< n_1^{i} : \quad n_2 = g_1(n_1); \\
n_1^{i} \leq n_1 &\leq n_1^{\max} : \quad n_2 = h_1(n_1).
\end{align*}
\]

5.4 Brownian inputs

For most Gaussian inputs that satisfy A1-A3 the boundary of \( S(\phi_1) \) cannot be explicitly computed; consequently, in those cases one has to rely on numerical techniques (as will be done in the numerical examples in Section 5.5). For the ‘canonical model’ with Brownian inputs though, we have succeeded in finding closed-form expressions for the boundary. As indicated in [126], Brownian motions can be used to approximate weakly-dependent traffic streams, cf. also the celebrated ‘Central Limit Theorem in functional form’. We let the variance functions be characterized through \( v_i(t) = \lambda_i t \), with \( \lambda_i > 0 \), \( i = 1, 2 \).

5.4.1 Region \( S_1(\phi_1) \)

It is a matter of straightforward calculus to show that \( t_F = b_i/(c - n_1 \mu_1 - n_2 \mu_2) \).
Now, the Mannersalo-Norros approximation reduces to the following. The critical
weight \( \phi_2^F \) equals
\[
1 - \frac{n_1 \lambda_1 - n_2 \lambda_2}{n_1 \lambda_1 + n_2 \lambda_2} \left( 1 - \frac{n_1 \mu_1 + n_2 \mu_2}{c} \right) - \frac{n_1 \mu_1}{c}. \tag{5.9}
\]
Then we get the approximations
\[
\triangle_1(n_1, n_2) = \left\{
\begin{array}{ll}
  (i) & 2b_1 \frac{\phi_1 c - n_1 \mu_1}{n_1 \lambda_1} \\
  (ii) & \frac{1}{2} \left( (b_1 + (\phi_1 c - n_1 \mu_1))^2 \right) + \frac{(\phi_2 c - n_2 \mu_2)^2}{n_2 \lambda_2} \\
  (iii) & 2b_1 \frac{c - n_1 \mu_1 - n_2 \mu_2}{n_1 \lambda_1 + n_2 \lambda_2}
\end{array}\right.
\]
with the 'critical time scale' \( t^* \) given by
\[
\frac{b_1}{\sqrt{(\phi_1 c - n_1 \mu_1)^2 + (\phi_2 c - n_2 \mu_2)^2 n_1 \lambda_1 n_2 \lambda_2}}. \tag{5.10}
\]
In [126] it was shown that the resulting expressions are 'asymptotically exact' in the many-sources regime.

Let us first derive the function \( f_1(\cdot) \). Recall from Section 5.2 that \( f_1(\cdot) \) is equivalent to all pairs of \((n_1, n_2)\) that satisfy (5.4) with equality. Plugging in the expression for \( t^F \) and some rearranging yields
\[
n_1 = \frac{c \lambda_2 (1 + \phi_1) - n_2 \lambda_2 \mu_2}{\phi_2 c \lambda_1 n_2} + 2 \lambda_2 \mu_1 - \lambda_1 \mu_2 =: f_1(n_2). \tag{5.11}
\]
It can easily be verified that \( f_1(0) = 0 \) and \( f_1(\phi_2 c/\mu_2) = \phi_1 c/\mu_1 \). The following lemma states some properties of \( f_1(\cdot) \); define
\[
\xi := \frac{(1 + \phi_1) \mu_1}{\phi_1 \mu_2}. \tag{5.12}
\]

**Lemma 5.4.1** \( f_1(\cdot) \) is continuous and has a continuous derivative on the interval \([0, \phi_2 c/\mu_2] \). Furthermore, \( f_1(\cdot) \) is concave on \([0, \phi_2 c/\mu_2] \) if \( \lambda_1 < \xi \lambda_2 \); \( f_1(\cdot) \) is convex on \([0, \phi_2 c/\mu_2] \) if \( \lambda_1 > \xi \lambda_2 \); \( f_1(\cdot) \) has a constant positive derivative on \([0, \phi_2 c/\mu_2] \) if \( \lambda_1 = \xi \lambda_2 \) and this derivative has the value \( \phi_1 \mu_2 / (\phi_2 \mu_1) \).

**Proof:** For any given \( \alpha, \beta \) and \( \gamma \) such that \( \beta \neq -\gamma n_2 \), note that
\[
\frac{d^2}{dn_2^2} \frac{1 - \alpha n_2}{\beta n_2 + \gamma} = \frac{-2 \beta (\alpha \beta + \gamma)}{(\beta + \gamma n_2)^3} =: p(n_2). \tag{5.13}
\]
It is clear that \( p(n_2) \) changes sign only at \( n_2 = -\beta/\gamma \). Now, let
\[
\alpha = \frac{\lambda_2 \mu_2}{c \lambda_2 (1 + \phi_1)}; \quad \beta = \frac{\phi_2 c \lambda_1}{c \lambda_2 (1 + \phi_1)}; \quad \gamma = \frac{2 \lambda_2 \mu_1 - \lambda_1 \mu_2}{c \lambda_2 (1 + \phi_1)}.
\]
Then, due to (5.11), \( f''(n_2) = p(n_2) \), and therefore \( f''(n_2) \) changes sign only at

\[
n_2 = -\frac{\beta}{\gamma} = \frac{\phi_2 c \lambda_1}{\lambda_1 \mu_2 - 2 \lambda_2 \mu_1}.
\]

(5.14)

Note that expression (5.14) does not lie in \([0, \phi_2 c/\mu_2]\), so \( f_1(\cdot) \) is either convex or concave on this interval. From (5.13) we conclude that there is concavity when \( \lambda_1 < \xi \lambda_2 \) (corresponding to \( \alpha \beta > -\gamma \)), and convexity otherwise. \( \square \)

Subsequently, in order to fully characterize the areas \( S_1^i(\phi_1), S_2^i(\phi_1), S_1^{ii}(\phi_1) \), we now derive \( n_1^{Q_1}, g_1(\cdot) \) and \( h_1(\cdot) \). We do this by relying on (5.5), (5.6) and (5.7), respectively. This yields

\[
\frac{2 \phi_1 c b_1}{2 b_1 \mu_1 + \delta_1 \lambda_1} =: n_1^{Q_1};
\]

(5.15)

\[
\frac{(\phi_2 c - n_2 \mu_2)^2 b_1^2}{n_2 \lambda_2 (\delta_1^2 \lambda_1 + 2 b_1 \delta_1 \mu_1)} + \frac{2 \phi_1 c b_1 \delta_1}{\delta_1^2 \lambda_1 + 2 b_1 \delta_1 \mu_1} =: g_1^{-1}(n_2);
\]

(5.16)

\[
\frac{2 c b_1}{2 b_1 \mu_2 + \delta_1 \lambda_2} - n_1 \frac{2 b_1 \mu_1 + \delta_1 \lambda_1}{2 b_1 \mu_2 + \delta_1 \lambda_2} =: h_1(n_1).
\]

(5.17)

Note that \( h_1(\cdot) \) is linear in \( n_1 \) and that

\[
h_1(n_1^{\text{max}}) = h_1 \left( \frac{2 c b_1}{2 b_1 \mu_1 + \delta_1 \lambda_1} \right) = 0.
\]

Due to Lemma 5.4.1, \( f_1(\cdot), g_1(\cdot) \) and \( h_1(\cdot) \) have a unique intersection point \((n_1, n_2)\) given by

\[
(n_1^{\text{opt}}, n_2^{\text{opt}}) = \left( \frac{c b_1 (\delta_1 \lambda_2 (1 + \phi_1) + 2 b_1 \mu_2 \phi_1)}{(\delta_1 \lambda_1 + 2 b_1 \mu_1) (\delta_1 \lambda_2 + b_1 \mu_2)} \right) \cdot \phi_2 c b_1.
\]

(5.18)

Now we have all the ingredients to describe the boundary of \( S_1(\phi_1) \) explicitly. The admissible region of the second queue can be treated analogously. Both are depicted in Figure 5.4.

### 5.4.2 Region \( S(\phi_1) \)

A combination \((n_1, n_2)\) is contained in \( S(\phi_1) \) if it satisfies the QoS requirements for both classes. That is, if it is contained in \( S_1(\phi_1) \cap S_2(\phi_1) \). In this subsection we characterize the boundary of \( S(\phi_1) \). In the analysis below the ratios \( b_1/b_2 \) and \( \delta_1/\delta_2 \) turn out to be crucial. We therefore introduce \( b := b_1/b_2 \) and \( d := \delta_1/\delta_2 \). Let us first mention some useful facts.

**Lemma 5.4.2** If \( b < d \) \((b > d)\) then \( h_2^{-1}(n_1) > h_1(n_1) \) \((h_2^{-1}(n_1) < h_1(n_1))\) for all \( n_1 \) that satisfy \( h_2^{-1}(n_1) \geq 0 \) and \( h_1(n_1) \geq 0 \).
5.4 Brownian inputs

Proof: We only prove the claim for \( b < d \), as the claim for \( b > d \) follows analogously. We know that

\[
\begin{align*}
  h_1(n_1) &= \frac{2cb_1}{2b_1\mu_2 + \delta_1\lambda_2} - n_1 \frac{2b_1\mu_1 + \delta_1\lambda_1}{2b_3\mu_2 + \delta_1\lambda_2}, \\
  h_2^{-1}(n_1) &= \frac{2cb_2}{2b_2\mu_2 + \delta_2\lambda_2} - n_1 \frac{2b_2\mu_1 + \delta_2\lambda_1}{2b_2\mu_2 + \delta_2\lambda_2}.
\end{align*}
\]

Now, \( h_2^{-1}(0) > h_1(0) \) implies that

\[
\frac{2cb_2}{2b_2\mu_2 + \delta_2\lambda_2} > \frac{2cb_1}{2b_1\mu_2 + \delta_1\lambda_2} \quad \text{or} \quad b < d,
\]

but we also have that \( h_2(0) > h_1^{-1}(0) \) implies that

\[
\frac{2cb_2}{2b_2\mu_1 + \delta_2\lambda_1} > \frac{2cb_1}{2b_1\mu_1 + \delta_1\lambda_1} \quad \text{or} \quad b < d.
\]

Since \( h_1(\cdot) \) and \( h_2^{-1}(\cdot) \) are linear, this proves the stated for \( b < d \). Note that \( h_1(\cdot) \) and \( h_2^{-1}(\cdot) \) are identical if \( b = d \). \( \Box \)

**Lemma 5.4.3** If \( b < d/2 \) (\( b > 2d \)) then \( n_1^{Q_1} < n_1^{I_2} \) (\( n_1^{Q_1} > n_1^{I_2} \)) and \( n_2^{Q_2} > n_2^{I_2} \) (\( n_2^{Q_2} < n_2^{I_2} \)). If \( d/2 \leq b \leq 2d \) then \( n_1^{Q_1} \geq n_1^{I_2} \) and \( n_2^{Q_2} \geq n_2^{I_2} \).

**Proof:** We only prove the claim for \( b < d/2 \), as the claims for \( b > 2d \) and \( d/2 \leq b \leq 2d \) follow in a similar fashion. Use the explicit expressions for \( n_1^{Q_1} \) and \( n_1^{I_2} \). Thus, \( n_1^{Q_1} < n_1^{I_2} \) is equivalent to

\[
\frac{2\phi_1cb_1}{\delta_1\lambda_1 + 2b_1\mu_1} < \frac{\phi_1cb_2}{\delta_2\lambda_1 + b_2\mu_1} \quad \text{or} \quad 2\delta_2b_1\lambda_1 + 2b_1b_2\mu_1 < \delta_1b_2\lambda_1 + 2b_1b_2\mu_1.
\]
Omitting common terms and some rearranging directly yields $b < d/2$.

Likewise, it holds that $n_2^{Q_2} > n_2^1$ if $b < d/2$, since

$$\frac{2\phi_2 cb_2}{\delta_2 \lambda_2 + 2b_2 \mu_2} > \frac{\phi_2 cb_1}{\delta_1 \lambda_2 + b_1 \mu_2} \quad \text{or} \quad 2\delta_1 b_2 \lambda_2 + 2b_1 b_2 \mu_2 > \delta_2 b_1 \lambda_2 + 2b_1 b_2 \mu_2,$$

reduces to $b < 2d$.

Combining the two previous lemmas leads to the conclusion that we have to distinguish between three cases: (a) $b < d/2$, (b) $d/2 \leq b \leq 2d$ and (c) $b > 2d$. Below we show that the shape of the boundary of $S(\phi_1)$ depends on (a), (b) or (c) if $\phi_1 \in (0, 1)$. First we characterize the boundary of $S(\phi_1)$ for $\phi_1 = 0$ and $\phi_1 = 1$. The boundary of $S(0)$ is given by

$$0 \leq n_1 \leq n_1^O : \quad n_2 = n_2^{Q_2};$$

$$n_1^O < n_1 < n_1^{\max} : \quad n_2 = h_1(n_1),$$

where $n_2^{Q_2}$ is evaluated at $\phi_1 = 0$, and $n_1^O := h_1^{-1}(n_2^{Q_2})$. The boundary of $S(1)$ is

$$0 \leq n_1 \leq n_1^{Q_1} : \quad n_2 = h_2^{-1}(n_1),$$

where $n_1^{Q_1}$ is evaluated at $\phi_1 = 1$.

**Remark:** One can easily show that $S(0) \subset S(1)$ if $b < d$, $S(1) \subset S(0)$ if $b > d$ and $S(0) = S(1)$ if $b = d$.

In the following we show that there are different generic shapes of the boundary of $S(\phi_1)$, $\phi_1 \in (0, 1)$, within each of the three cases.

**Case $b < d/2$**

It can easily be seen that the boundary of $S(\phi_1)$ has four possible shapes in this case (see Figure 5.5). The shape of the boundary $(a_1, (a_2), (a_3)$ or $(a_4)$) depends on the value of $\phi_1$, but each shape occurs as will be shown in the following lemmas. Let $n_1^X$ be the solution of $g_1(n_1) = h_2^{-1}(n_1)$. Furthermore, let $n_1^W$ solve $g_1(n_1) = g_2^{-1}(n_1)$, and let $n_2^W = g_1(n_1^W)$. Finally, define $n_1^X := g_1^{-1}(n_2^{Q_2})$.

**Lemma 5.4.4** The boundary of $S(\phi_1)$ has shape $(a_1)$ if $\phi_1 \in [X_3, 1)$, where

$$X_3 := \frac{\delta_2 \lambda_2 (\delta_1 \lambda_1 + 2b_1 \mu_1)}{\delta_2 \lambda_2 (\delta_1 \lambda_1 + 2b_1 \mu_1) + 2 \lambda_1 \mu_2 (\delta_1 b_2 - \delta_2 b_1)}.$$  \hspace{1cm} (5.19)

**Proof:** In order to have shape $(a_1)$ we must have that $h_2^{-1}(n_1^{Q_1}) \geq \phi_2 c/\mu_2$ for some value of $\phi_1 \in (0, 1)$. That is,

$$\frac{2cb_2}{\delta_2 \lambda_2 + 2b_2 \mu_2} - \frac{2\phi_1 cb_1 (\delta_2 \lambda_1 + 2b_2 \mu_1)}{(\delta_2 \lambda_2 + 2b_2 \mu_2)(\delta_1 \lambda_1 + 2b_1 \mu_1)} \geq \frac{(1 - \phi_1)c}{\mu_2}.$$  \hspace{1cm} (5.20)
One can easily show that this reduces to a constraint of the form $-A + B\phi_1 \geq 0$, with $A, B > 0$. For $\phi_1 = 0$ the left-hand side of the constraint (5.20) is equivalent to

$$\frac{2c b_2}{\delta_2 \lambda_2 + 2b_2 \mu_2} - \frac{c}{\mu_2},$$

which is smaller than 0 (assuming that $\delta_2, \lambda_2 > 0$). Hence, the constraint is not satisfied. For $\phi_1 = 1$ the left-hand side of (5.20) equals

$$\frac{2c \lambda_1 (\delta_1 b_2 - \delta_2 b_1)}{(\delta_2 \lambda_2 + 2b_2 \mu_2)(\delta_1 \lambda_1 + 2b_1 \mu_1)}.$$
which is larger than 0 if \( b/d < 1 \), which is true, as we required \( b < d/2 \). Thus, since the constraint is a linear function of \( \phi_h \), there must be value of \( \phi_1 \in (0, 1) \) for which \( h_2^{-1}(n_1^{Q_1}) = \phi_2 c/\mu_2 \). Straightforward calculus shows there is equality for \( \phi_1 = X_3 \). □

**Lemma 5.4.5** The boundary of \( S(\phi_1) \) has shape \((a_4)\) if \( \phi_1 \in (0, X_1) \), where

\[
X_1 := \frac{\delta_3 \delta_2 \lambda_2 \mu_1}{\delta_3 \delta_2 \lambda_2 \mu_1 + 2 \delta_1 \delta_2 \lambda_1 (\delta_2 \lambda_2 + 2 b_2 \mu_2)}.
\] (5.21)

**Proof:** The proof is analogous to that of Lemma 5.4.4. Shape \((a_4)\) occurs if there exists a value of \( \phi_1 \in (0, 1) \) for which \( g_1^{-1}(n_2^{Q_2}) \geq \phi_1 c/\mu_1 \). This constraint can be rewritten as \( A - B \phi_1 \geq 0 \), with \( A, B > 0 \). Since it is satisfied for \( \phi_1 = 0 \), but not for \( \phi_1 = 1 \), there exists a unique value of \( \phi_1 \in (0, 1) \), \( X_1 \), such that there is equality, i.e., \( g_1^{-1}(n_2^{Q_2}) = \phi_1 c/\mu_1 \). □

**Lemma 5.4.6** The boundary of \( S(\phi_1) \) has shape \((a_3)\) if \( \phi_1 \in (X_1, X_2) \), where \( X_2 \) is the value of \( \phi_1 \) such that \( n_2^{l_2} = h_2^{-1}(n_1^{l_2}) = g_1(n_1^{l_2}) \).

**Proof:** The shape of the boundary is like \((a_3)\) if \( h_2^{-1}(n_1^{l_2}) < g_1(n_1^{l_2}) \) and if \( g_1^{-1}(n_2^{Q_2}) < \phi_1 c/\mu_1 \). The latter constraint is satisfied if \( \phi_1 > X_1 \) (Lemma 5.4.5). In contrast to the latter constraint, the former constraint does not reduce to a constraint that is a linear function of \( \phi_1 \). It can be shown that there exists a unique value of \( \phi_1 \), \( X_2 \), such that \( h_2^{-1}(n_2^{l_2}) = g_1(n_1^{l_2}) \). The expression of \( X_2 \) is not presented here (as it is quite complicated); it depends on the parameters \( \delta_1, \delta_2, b_1, b_2, \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \). Now, the constraint is satisfied for all \( \phi_1 \in [0, X_2) \).

We now show that \( X_2 \in (X_1, X_3) \). First recall that \( g_1(n_1) \) is defined on the interval \((n_1^{Q_1}, n_1^{l_1^*})\) in \( S_1(\phi_1) \), whereas \( h_2^{-1}(n_1) \) is defined on the interval \([0, n_1^{l_2}]\) in \( S_2(\phi_1) \). Therefore, if \( g_1(\cdot) \) and \( h_2^{-1}(\cdot) \) are part of the boundary of \( S(\phi_1) \), then they are defined on (parts of) the mentioned intervals. If \( \phi_1 \in (0, X_1] \), then \( g_1(\cdot) \) is defined on the interval \((n_1^X, n_1^{l_1^*})\), with \( n_1^X > \phi_1 c/\mu_1 \) (see shape \((a_4)\)). By definition \( n_1^{l_2} < \phi_1 c/\mu_1 \), so this implies that \( g_1(\cdot) \) and \( h_2^{-1}(\cdot) \) cannot intersect if \( \phi_1 \in (0, X_1] \). Furthermore, if \( \phi_1 \in [X_3, 1) \), then \( h_2^{-1}(n_1) > g_1(n_1) \) for all \( n_1 \in (n_1^{Q_1}, \min\{n_1^{l_1}, n_1^{l_2}\}) \) (see shape \((a_1)\)), so \( X_2 \notin [X_3, 1) \). Hence, we conclude that \( 0 < X_1 < X_2 < X_3 < 1 \) □

**Lemma 5.4.7** The boundary of \( S(\phi_1) \) has shape \((a_2)\) if \( \phi_1 \in [X_2, X_3) \).

**Proof:** One observes shape \((a_2)\) if \( h_2^{-1}(n_1^{Q_1^*}) < \phi_2 c/\mu_2 \) and \( h_2^{-1}(n_1^{l_2}) \geq g_1(n_1^{l_2}) \). From Lemmas 5.4.4 and 5.4.6 we know that this coincides with \( \phi_1 < X_3 \) and \( \phi_1 \geq X_2 \) respectively. □
5.4 Brownian inputs

Proposition 5.4.8 If \( b < d/2 \), then the boundary of \( S(\phi_1) \) has
shape \((a_4)\) for \( 0 < \phi_1 \leq X_1 \);
shape \((a_3)\) for \( X_1 < \phi_1 < X_2 \);
shape \((a_2)\) for \( X_2 \leq \phi_1 < X_3 \);
and shape \((a_1)\) for \( X_3 \leq \phi_1 < 1 \).

Here \( X_1 \) is the value of \( \phi_1 \) such that \( g_{1}^{-1}(n_2^{Q_2}) = \phi_1c/\mu_1 \), \( X_2 \) is the value of \( \phi_1 \) such that \( n_2^1 = h_{2}^{-1}(n_1^{Q_1}) = g_1(n_1^1) \), and \( X_3 \) is the value of \( \phi_1 \) that solves \( h_{2}^{-1}(n_1^{Q_1}) = \phi_2c/\mu_2 \).

Case \( d/2 \leq b \leq 2d \)

As proved in Lemma 5.4.3, this criterion leads to \( n_1^{Q_1} \geq n_2^{Q_2} \geq n_2^1 \). Now, the boundary of \( S(\phi_1) \) can have three shapes ((b1), (b2) and (b3)). Shape (b1) is depicted in Figure 5.6 (top, left). Shape (b2) corresponds to \((a_3)\), and (b3) to \((a_4)\).

As in the case of \( b < d/2 \), one can easily prove that each shape is observed. The proofs are omitted as they are similar to the proofs of Lemmas 5.4.4 and 5.4.5. We directly state the following proposition.

Proposition 5.4.9 If \( d/2 \leq b \leq 2d \), then the boundary of \( S(\phi_1) \) has
shape \((b_4)\) for \( 0 < \phi_1 \leq Y_1 \);
shape \((b_3)\) for \( Y_1 < \phi_1 < Y_2 \);
and shape \((b_2)\) for \( Y_2 \leq \phi_1 < 1 \).

Here \( Y_1 \) is the value of \( \phi_1 \) such that \( g_{1}^{-1}(n_2^{Q_2}) = \phi_1c/\mu_1 \) and \( Y_2 \) coincides with the value of \( \phi_1 \) such that \( g_{2}^{-1}(n_1^{Q_1}) = \phi_2c/\mu_2 \).

Case \( b > 2d \)

The last case is the counterpart of the first case. Therefore, the proofs are also omitted in the following. Now, \( n_1^{Q_1} > n_2^{Q_2} \) and \( n_2^{Q_2} < n_2^1 \). Define \( n_2^Y := h_{1}^{-1}(n_2^{Q_2}) \), and let \( n_2^Y \) be the solution of \( g_{2}^{-1}(n_1) = h_{1}(n) \). There are four possible shapes of \( S(\phi_1) \), \( \phi_1 \in (0, 1) \). Shapes \((c_1)\) and \((c_2)\) are depicted in Figure 5.6. Shape \((c_3)\) corresponds to \((a_3)\), and \((c_4)\) to \((b_1)\).

Proposition 5.4.10 If \( b > 2d \), then the boundary of \( S(\phi_1) \) has
shape \((c_4)\) for \( 0 < \phi_1 \leq Z_1 \);
shape \((c_2)\) for \( Z_1 < \phi_1 \leq Z_2 \);
shape \((c_3)\) for \( Z_2 < \phi_1 < Z_3 \);
and shape \((c_4)\) for \( Z_3 \leq \phi_1 < 1 \).

Here \( Z_1 \) corresponds to the value of \( \phi_1 \) such that \( h_{1}^{-1}(n_2^{Q_2}) = \phi_1c/\mu_1 \), \( Z_2 \) is the value of \( \phi_1 \) that solves \( n_2^1 = h_{1}^{-1}(n_2^{Q_2}) = g_1(n_1^1) \) and \( Z_3 \) is the value of \( \phi_1 \) such that \( g_{2}^{-1}(n_1^{Q_1}) = \phi_2c/\mu_2 \).
5.4.3 The realizable region

Let \( R \) denote the realizable region, i.e.,
\[
R := \bigcup_{\phi_1 \in [0, 1]} S(\phi_1). \tag{5.22}
\]

In the following we show that we do not always need all values of \( \phi_1 \in [0, 1] \) to compose \( R \). We now state our main result.

**Theorem 5.4.11** The realizable region \( R \) can be obtained as follows:
\[
b < d/2 : \quad R = \bigcup_{\phi_1 \in (0, X_2) \cup \{1\}} S(\phi_1);
\]
\[
d/2 \leq b \leq d : \quad R = \bigcup_{\phi_1 \in (0, 1]} S(\phi_1);
\]

Figure 5.6: Top, from left to right: shape \((b_1)\) and \((c_1)\). Bottom: shape \((c_2)\).
\[ d < b \leq 2d : \quad R = \bigcup_{\phi_1 \in [0,1)} S(\phi_1); \]
\[ b > 2d : \quad R = \bigcup_{\phi_1 \in \{0\} \cup (Z_2,1)} S(\phi_1). \]

**Proof:** We only prove the first statement, as the other three statements can be proved in a similar fashion. We already remarked above that \( S(0) \subset S(1) \) in case \( b < d/2 \). Furthermore, in this case we also have that \( S(\phi_1) \subset S(1) \) for all values of \( \phi_1 \in [X_2,1) \).

To see this, compare the boundaries \((a_1)\) and \((a_2)\) with the boundary of \( S(1) \), and recall that \( h_{b-1}^{-1}(\cdot) > h_1(\cdot) \) if \( b < d/2 \). What is left to prove is that we need all values of \( \phi_1 \in (0, X_2) \) to compose \( R \) if \( b < d/2 \). Note that the boundary of \( S(\phi_1) \) has shape \((a_1)\) if \( \phi_1 \in (X_1, X_2) \), implying that \( S(\phi_1) \) contains \((n_1^W, n_2^W)\), with \( h_{b-1}^{-1}(n_1^W) < n_2^W \), which cannot be part of \( S(1) \). From Lemma 5.4.6 it follows that for all values of \( \phi_1 \in (X_1, X_2) \), \( n_1^W \) \((n_2^W)\) increases (decreases) as \( \phi_1 \) increases (but not linearly), implying that we need all values of \( \phi_1 \in (X_1, X_2) \) to compose \( R \) if \( b < d/2 \). Likewise, shape \((a_4)\) is seen if \( \phi_1 \in (0, X_1) \). The point \((n_1^X, n_2^Q)\) will then be contained in \( S(\phi_1) \), which cannot be contained in \( S(1) \) either. From Lemma 5.4.5 it follows that as \( \phi_1 \) increases in the corresponding interval, \( n_1^W \) \((n_2^Q)\) linearly increases (decreases), implying that we also need all values of \( \phi_1 \in (0, X_1) \) to compose \( R \) if \( b < d/2 \), thus proving the first statement. \( \square \)

Using Theorem 5.4.11, the boundary of \( R \) can now also be determined. Since there are four possible cases in Theorem 5.4.11, it follows that the boundary of \( R \) can have four different generic shapes. Below we discuss each one of these. Let us first introduce some notation. From now on, we write \( z(\phi_1) = z \) if \( z \) depends on \( \phi_1 \). Note that \( \phi_2 = 1 - \phi_1 \), thus if an expression contains \( \phi_2 \), we can easily rewrite it as function of \( \phi_1 \).

**Case** \( b < d/2 \)

Theorem 5.4.11 shows that we need all values of \( \phi_1 \in (0, X_2) \) and \( \phi_1 = 1 \) to compose \( R \) in this case. Straightforward calculus shows that all values of \( \phi_1 \in (0, X_2) \) contribute to the boundary of \( R \) in the following way:

\[ \phi_1 \in (0, X_1): \quad (n_1, n_2) = (g_1^{-1}(n_2^Q(\phi_1)), n_2^Q(\phi_1)); \quad (5.23) \]
\[ \phi_1 \in (X_1, X_2): \quad (n_1, n_2) = (n_1^W(\phi_1), n_2^W(\phi_1)); \quad (5.24) \]
with

\[ n_1^W := g_1^{-1}(n_2^Q(0)) > 0; \quad n_2^Q(0) = h_2^{-1}(0); \]
\[ g_1^{-1}(n_2^Q(X_1)) = n_1^W(X_1); \quad n_2^Q(X_1) = n_2^W(X_1); \]
\[ n_2^W(X_2) = h_2^{-1}(n_1^W(X_2)). \]
Selection of optimal weights in Generalized Processor Sharing

It can be shown that (5.23) corresponds to a function \( n_2 = k_1(n_1) \) which linearly decreases as \( n_1 \) increases. Furthermore, (5.24) corresponds to a function \( n_2 = k_2(n_1) \) which non-linearly decreases as \( n_1 \) increases. Recall that the boundary of \( S(1) \) is given by the line \( h_2^{-1}(\cdot) \) on some predefined interval, so also the contribution of \( \phi_1 = 1 \) to the boundary of \( R \) can easily be derived. Moreover, \( k_1(\cdot), k_2(\cdot) \) and \( h_2^{-1}(\cdot) \) perfectly connect, as one can show that

\[
\left. \frac{\partial n_2^W(\phi_1)}{\partial \phi_1} \right|_{\phi_1 = X_1} = \frac{\partial n_2^W(\phi_1)}{\partial \phi_1} \left|_{\phi_1 = X_1} \right. \quad ; \quad \left. \frac{\partial n_2^W(\phi_1)}{\partial \phi_1} \right|_{\phi_1 = X_2} = \frac{\partial h_2^{-1}(n_1)}{\partial n_1}.
\]

see Figure 5.7 (top, left) for an illustration. We are now able to describe the boundary of \( R \), which follows from the above.

**Proposition 5.4.12** If \( b < d/2 \), then the boundary of \( R \), denoted by \( r_1 \), is continuous.

The approach that is required to derive the boundary of \( R \) in the other cases is very similar to that one in the current case. Therefore, we leave out the details in the remaining three cases.

**Case** \( d/2 \leq b \leq d \)

Let \( n_2 = k_3(n_1) \) and \( n_2 = k_4(n_1) \) be functions that correspond to the following equations, respectively:

\[
\phi_1 \in (Y_1, Y_2) : \quad (n_1, n_2) = (n_1^W(\phi_1), n_2^W(\phi_1));
\]

\[
\phi_1 \in [Y_2, 1) : \quad (n_1, n_2) = (n_1^Q_1(\phi_1), g_2^{-1}(n_1^Q_1(\phi_1))).
\]

It can be shown that \( k_3(\cdot) \) is a non-linearly decreasing function, whereas \( k_4(\cdot) \) is a linearly decreasing function. Furthermore, it can be shown that \( k_1(\cdot), k_3(\cdot) \) and \( k_4(\cdot) \) connect perfectly, see Figure 5.7 (top, right). Recalling that \( Y_1 = X_1 \), we now have all the ingredients to describe the boundary.

**Proposition 5.4.13** If \( d/2 \leq b \leq d \), then the boundary of \( R \), denoted by \( r_2 \), is continuous.

**Case** \( d < b \leq 2d \)

We directly state the result on the boundary of \( R \), since it is very similar to the previous case.

**Proposition 5.4.14** If \( d < b \leq 2d \), then the boundary of \( R \), denoted by \( r_3 \), is continuous.
5.4 Brownian inputs

Let $n_2 = k_5(n_1)$ be a function that corresponds to the following:

$$\phi_1 \in (Z_2, Z_3) : \quad (n_1, n_2) = (n_1^W(\phi_1), n_2^W(\phi_1)).$$

Recalling that $Z_3 = Y_2$, we now directly state the following proposition.

**Proposition 5.4.15** If $b > 2d$, then the boundary of $R$, denoted by $r_4$, is continuous.
Although Theorem 5.4.11 shows that we need a range of weights to obtain $R$, the above results suggest that almost all of $R$ is obtained by the priority scheduling discipline, e.g., $\phi_1 = 0$ or $\phi_1 = 1$. This observation is established by comparing the boundary of $R$ with the boundaries of $S(0)$ and $S(1)$, and showing that at least one of these two boundaries almost matches with the boundary of $R$. In particular, in case $b \leq d$, the admissible region $S(1)$ covers most of $R$, whereas in case $b > d$ the region $S(0)$ approximates $R$. We further explore this issue in the next section.

5.5 Numerical analysis

In this section we numerically compute the boundary of the realizable region for two realistic examples of Gaussian inputs, with very diverse parameter settings. As the inputs are non-Brownian, the boundary of the admissible region (and thus the realizable region) has to be obtained numerically. The goal is to compare the realizable region with the admissible region corresponding to the priority cases. Denoting by $|R|$, $|S(0)|$ and $|S(1)|$ the number of different pairs $(n_1, n_2)$, $n_1, n_2 \in \mathbb{N}_0$, that are contained in sets $R$, $S(0)$ and $S(1)$, respectively, we define

$$O_i \equiv \frac{|S(i)|}{|R|} \quad \text{and} \quad O_2 \equiv \frac{|S(0)|}{|R|},$$

(5.25)

i.e., $O_i$ is a measure that indicates what fraction of the realizable region can be obtained by prioritizing class $i$, $i = 1, 2$. Recall that $S(0) \subseteq R$ and $S(1) \subseteq R$, hence $O_1, O_2 \in [0, 1]$. The following examples illustrate that either $S(0)$ or $S(1)$ (or both) covers most of the realizable region (as was the case for Brownian inputs, see Section 5.4), i.e., either $O_1$ or $O_2$ (or both) is almost equal to 1.

5.5.1 Example 1

Consider two traffic classes sharing a total capacity $(c)$ of 10 Mbps. The first class consists of data traffic, whereas the second class corresponds to voice traffic. Traffic of the first class is modeled as fractional Brownian motion, i.e., $v_1(t) = \alpha t^{2H_1}$, with $H_1 \in (0, 1)$. The mean traffic rate $\mu_1$ is 0.2 Mbps and its variance function is given by $v_1(t) = 0.0025t^{2H_1}$. In measurement studies it was frequently found that $H_1$ lies between, say, 0.7 and 0.85. Below we take $H_1 \in \{0.5, 0.65, 0.8, 0.95\}$.

Traffic of the second class corresponds to the Gaussian counterpart of the AMS model. In the AMS model work arrives from sources in bursts which have peak rate $h$ and exponentially distributed lengths with mean $\beta^{-1}$. After each burst, the source switches off for a period that is exponentially distributed with mean $\lambda^{-1}$. The variance curve of a single source is

$$v_2(t) = \frac{2\lambda\beta h^2}{(\lambda + \beta)^2} \left( t - \frac{1}{\lambda + \beta} (1 - \exp(-(\lambda + \beta)t)) \right).$$

(5.26)
We first choose $h = 0.032$, $\lambda = 1/0.65$ and $\beta = 1/0.352$ in (5.26), in line with the parameters for coded voice used in [162]. Hence, the mean traffic rate of a source of class 2 ($\mu_2$) is 0.011 Mbps. Note that traffic of class 1 is LRD (i.e., the autocorrelations are non-summable), whereas the traffic of class 2 is SRD.

Table 5.1 shows the values of the performance measures $O_1$ and $O_2$ for multiple combinations of $b_1$, $b_2$, $\delta_1$, $\delta_2$ and $H_1$. Note that $\delta_i = 13.8$ ($\delta_i = 6.9$) corresponds to an overflow probability of $10^{-6}$ ($10^{-3}$) for class $i$, $i = 1, 2$. Table 5.1 shows that either $O_1$ or $O_2$ (or both) is approximately equal to 1. Hence, this implies that most of $R$ can be obtained by giving priority to class 1 or 2. In case of $O_1 \approx 1$ and $O_2 \approx 1$, it does not really matter which class to prioritize, in the sense that the realizable region is almost completely obtained by applying one of the priority strategies.

One can expect that $O_1$ and $O_2$ are sensitive to the traffic characteristics. In order to investigate this, we performed several experiments. Table 5.2 shows the values of $O_1$ and $O_2$ for different values of $\mu_1$, given that $b_1 = 0.1$, $b_2 = 0.5$, $\delta_1 = 6.9$, $\delta_2 = 13.8$, $\mu_1 = 0.2$, $h = 0.032$, $\lambda = 1/0.65$ and $\beta = 1/0.352$. The results show that $O_1$ and $O_2$ are only mildly affected by $\mu_1$.

Subsequently, we replace $\lambda$ and $\beta$ by $\alpha \lambda$ and $\alpha \beta$, respectively, in (5.26), with
Table 5.3: Sensitivity of $O_1$ and $O_2$ with respect to $H_1$ and $\alpha$. (Example 1).

In addition to the parameter values presented in Tables 5.1-5.3, we have considered many other parameter values for the $B_i$s, $\delta_i$s, $\mu_i$s, $H_1$, $\alpha$, and $c$. The result that priority strategies cover nearly the entire realizable region appears to remain valid.

### 5.5.2 Example 2

In this example we let the two traffic classes share a total capacity of 100 Mbps. Both traffic classes consist of data traffic, where the first class has a high access rate and the second class has a somewhat lower access rate. Recall that the data rate of the class-$i$ user access-channel in a network is known as the access rate of the class-$i$ user, $i = 1, 2$. The speed of the access-channel determines how fast (or the maximum rate) the class-$i$ user can inject data into a network. The variance functions are given by $0.5625 t^{2H_1}$ and $0.0025 t^{2H_2}$, so both classes are modeled as fractional Brownian motions.

Table 5.4 shows the values of $O_1$ and $O_2$ for different combinations of $b_1$, $b_2$, $\delta_1$, $\delta_2$ and $H_1$, assuming that $H_2 = 0.8$, $\mu_1 = 3$ and $\mu_2 = 0.2$. Table 5.5 shows the values of $O_1$ and $O_2$ for different combinations of $\mu_1$ and $\mu_2$, assuming that $b_1 = 1$, $b_2 = 4$, $\delta_1 = 6.9$, $\delta_2 = 18.4$, $H_1 = 0.8$ and $H_2 = 0.65$. Finally, Table 5.6 shows the values of $O_1$ and $O_2$ for different combinations of $H_1$ and $H_2$, assuming that $b_1 = 1$, $b_2 = 4$, $\delta_1 = 6.9$, $\delta_2 = 18.4$, $H_1 = 0.8$ and $H_2 = 0.65$.
5.5 Numerical analysis

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</table>

Table 5.4: Sensitivity of $O_1$ and $O_2$ with respect to $b_1$, $b_2$, $\delta_1$, $\delta_2$ and $H_1$. (Example 2).

$\delta_1 = 6.9$, $\delta_2 = 18.4$, $\mu_1 = 3$ and $\mu_2 = 0.2$.

Tables 5.4-5.6 show again that $R$ is nearly fully covered by $S(0)$ and/or $S(1)$. We have experimented with other parameter values, and observed that this claim was still valid in virtually all situations considered.

5.5.3 Discussion

In Section 5.4.3 we showed that, in case of Brownian inputs, $R$ is accurately approximated by $S(1)$ if $b \leq d$, and by $S(0)$ otherwise. Therefore, if the ratio of the buffer thresholds is less than or equal to the ratio of the (exponential) decay rates of the overflow probabilities, then one should select $(\phi_1, \phi_2) = (1, 0)$, otherwise $(\phi_1, \phi_2) = (0, 1)$. That is, if $b \leq d$ ($b > d$) then one should prioritize class 1 (2). Interestingly, this criterion does not involve the characteristics of the sources. The numerical analysis presented in this section (as well as the additional numerical experiments that we performed) suggest that for other Gaussian sources there is a similar criterion. However, it is in general not given by $b \leq d$ ($b > d$) as is the case for Brownian inputs; it seems that the traffic characteristics of the two classes should be taken into account as well, as illustrated in Tables 5.1-5.6.

In the scenario that one class has bursty traffic and loose QoS requirements, whereas for the other class it is the reverse (smooth traffic and stringent QoS requirements), we can give an argument that may informally explain why nearly the entire realizable region is achievable by strict priority scheduling strategies. In that case the buffer asymptotics of the bursty traffic class will not be affected by the weights (may be even completely insensitive), as long as the traffic intensity of the smooth traffic class (defined as the ratio of the aggregate input rate of the smooth traffic class to the service rate $c$) does not exceed its weight. The latter will necessar-
Selection of optimal weights in Generalized Processor Sharing

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$O_1$</th>
<th>$O_2$</th>
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<tr>
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<tr>
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<td>0.952</td>
<td>0.987</td>
</tr>
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</table>

Table 5.5: Sensitivity of $O_1$ and $O_2$ with respect to $\mu_1$ and $\mu_2$. (Example 2).

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$O_1$</th>
<th>$O_2$</th>
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<td>0.855</td>
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<td>0.990</td>
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<tr>
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<td>0.992</td>
<td>0.876</td>
</tr>
<tr>
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<td>0.990</td>
<td>0.877</td>
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<tr>
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<td>0.95</td>
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</tbody>
</table>

Table 5.6: Sensitivity of $O_1$ and $O_2$ with respect to $H_1$ and $H_2$. (Example 2).

ily hold, as otherwise the smooth traffic class would be negatively influenced by the bursty traffic class. This insensitivity implies that there is little lost by simply giving strict priority to the smooth traffic class. In other scenarios there does not seem to be a clear intuitive explanation.