Queueing models for bandwidth-sharing disciplines

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Recall that a wide family of AFS policies achieve stability under the simple condition that no individual link is overloaded, given Poisson arrivals and exponentially distributed service requirements [23]. These stability results imply that flow-level performance measures such as expected transfer delays are finite provided that no individual link is overloaded. However, the derivation of the exact transfer delays and actual user throughputs has proven largely elusive, except in the special case of an unweighted proportional fair bandwidth-sharing policy in certain topologies, such as linear networks. In particular, it is not well understood how the flow-level performance measures depend on the specific choice of the fairness coefficient $\alpha$ and the possible additional weight factors associated with the various classes.

In order to gain further insight in the latter issues, in this chapter we develop approximations for the mean number of users in linear networks operating under AFS policies. The approximations are based on the assumption that one or two of the nodes experience heavy-traffic conditions. In case of just a single ‘bottleneck’ node, we exploit the fact that this node approximately behaves as a two-class DPS queue. The mean number of users can thus be calculated from the results of [63]. In the case that there are two nodes critically loaded, we rely on the following two observations. First, the heavy-traffic results of [85, 91] show that with equal class weights, the joint workload process is asymptotically independent of the fairness coefficient $\alpha$. Second, the joint workload process for a proportional fair policy can be exactly computed from the known distribution of the number of users [135]. Combining these two observations, we obtain simple explicit estimates for the workloads at the two bottleneck nodes, which we also numerically validate. We then develop various approximation methods by using the latter estimates in conjunction with characterizations of invariant states from [85, 91] that relate the number of users of the various classes to the workloads at the various nodes.
The remainder of this chapter is organized as follows. In Section 7.1 we provide a detailed model description and discuss some preliminaries. In Section 7.2 we present some results for the known distribution of the user population for a proportional fair policy, and use these to obtain the Laplace-Stieltjes Transform (LST) of the joint workload process at the various nodes. Section 7.3 reviews the heavy-traffic results of [85, 91], which provide the basis for the approximations that we develop subsequently. In Section 7.4 we focus on the case of a single bottleneck node, and exploit the fact that this node approximately behaves as a two-class DPS model to obtain approximations for the mean number of users. Next, in Section 7.5 we turn the attention to a scenario with two bottleneck nodes, and invoke the principle that the joint workload process can be approximated by the known behavior for a proportional fair policy, provided all classes have equal weights. In conjunction with a few equivalent characterizations of invariant states from [85, 91], the latter principle is then leveraged in Section 7.6 to devise various approximation methods. In Sections 7.7 and 7.8 we discuss various model extensions.

### 7.1 Queueing model

We consider a linear network as depicted in Figure 1.7. The network consists of \( L \) nodes, each with unit service rate. There are \( L + 1 \) classes of users: each class corresponds to a specific route in the network. Class-\( i \) users require service at node \( i \) only, \( i = 1, \ldots, L \), whereas class-0 users require service at all \( L \) nodes simultaneously.

We assume that class-\( i \) users arrive according to a Poisson process of rate \( \lambda_i \), and have exponentially distributed service requirements with mean \( \mu_i^{-1} \), \( i = 0, \ldots, L \). The arrival processes are all independent. The traffic load of class \( i \) is then \( \rho_i = \lambda_i \mu_i^{-1} \). Note that the traffic load at node \( i \) is given by \( \rho_0 + \rho_i \), \( i = 1, \ldots, L \). Let \( n = (n_0, n_1, \ldots, n_L) \) be the state of the network, with \( n_i \) representing the number of class-\( i \) users.

The network operates under an AFS policy. When the network is in state \( n \in \mathbb{N}_0^{L+1}\setminus\{\vec{0}\} \), the service rate \( x_i^* \) allocated to each of the class-\( i \) users is obtained by solving the optimization problem (6.9). Let \( s_i(n) := n_i x_i^* \) denote the total service rate allocated to class \( i \). In [23] it was shown that, for \( i = 1, \ldots, L \),

\[
s_0(n) = \frac{(\kappa_0 n_0^\alpha)^{1/\alpha}}{(\kappa_0 n_0^\alpha)^{1/\alpha} + \left(\sum_{j=1}^{L} \kappa_j n_j^\alpha\right)^{1/\alpha}}; \quad s_i(n) = (1 - s_0(n))1_{n_i > 0}, \tag{7.1}
\]

if \( n \in \mathbb{N}_0^{L+1}\setminus\{\vec{0}\} \), where \( 1_{n_i > 0} = 1 \) if \( n_i > 0 \), and 0 otherwise.

Let \( N(t) \) denote the state of the network at time \( t \). Then \( N(t) \) is a Markov process with transition rates:

\[
q(n, n + e_i) = \lambda_i; \quad q(n, n - e_i) = \mu_i s_i(n), \quad i = 0, \ldots, L,
\]
where $e_i$ denotes the $(i+1)$th unit vector in $\mathbb{R}^{L+1}$. Evidently, $\rho_0 + \rho_i < 1$, $i = 1, \ldots, L$, is a necessary condition for the process $N(t)$ to be ergodic. In [23] it was shown that this condition is in fact also sufficient for every $\alpha \in (0, \infty)$.

In general there are no closed-form expressions available for the steady-state distribution of $N(t)$. However, for the case $\alpha = 1$ and $\kappa_i = \kappa$, an explicit expression has been derived in [135], as will be presented in the next section.

### 7.2 Unweighted proportional fairness

In this section we consider the case $\alpha = 1$ and $\kappa_i = \kappa$, $i = 0, \ldots, L$. The following theorem appeared in slightly different form in [135].

**Theorem 7.2.1** Under the stability condition $\max_{1 \leq i \leq L} \rho_0 + \rho_i < 1$, the process $N(t)$ is reversible, with steady-state distribution given by

$$\pi(n) = C^{-1} \left( \sum_{n_0} L i=0 \prod_{i=0}^{L} \rho_i^{n_i}, \right) \quad (7.2)$$

where the normalization constant $C$ equals

$$C = \frac{(1 - \rho_0)^{L-1}}{\prod_{i=1}^{L} (1 - \rho_0 - \rho_i)}. \quad (7.3)$$

The mean number of class-0 users in steady state is given by

$$E(N_0) = \rho_0 \frac{1}{1 - \rho_0} \left( 1 + \sum_{i=1}^{L} \frac{\rho_i}{1 - \rho_0 - \rho_i} \right)$$

and for $i = 1, \ldots, L$,

$$E(N_i) = \frac{\rho_i}{1 - \rho_0 - \rho_i}.$$

Let $W_i(t)$ denote the workload, i.e., the unfinished amount of work at node $i$ at time $t$, $i = 1, \ldots, L$. Thus $W_i(t)$ consists of the remaining service requirements of all class-0 and class-$i$ users at time $t$. Theorem 7.2.1 enables us to derive the LST of the joint distribution of $W(t) = (W_1(t), \ldots, W_L(t))$ in steady state.

**Theorem 7.2.2** Under the stability condition $\max_{1 \leq i \leq L} \rho_0 + \rho_i < 1$, the LST of $W(t)$ in steady state is given by

$$\tilde{W}(z) \equiv \tilde{W}(z_1, \ldots, z_L) = \left( 1 - \frac{\lambda_0}{\mu_0 + \sum_{j=1}^{L} z_j} \right)^{L-1} \prod_{i=1}^{L} \frac{1 - \rho_0 - \rho_i}{1 - \rho_i} \frac{\lambda_i}{\mu_i + z_i}. \quad (7.4)$$
Substituting (7.2) and invoking that service requirement of a class-$i$ user is also exponentially distributed with mean $\mu_i^{-1}$, $i = 0, \ldots, L$. Therefore $W_i(t)$ is distributed as $\sum_{j=1}^{N_0(t)} B_{0,j} + \sum_{j=1}^{N_i(t)} B_{i,j}$, where $B_{i,j}$ are i.i.d. copies of an exponentially distributed variable with mean $\mu_i^{-1}$, $i = 1, \ldots, L$. Now

$$\tilde{W}(z) = \mathbb{E} \left( e^{-\sum_{i=1}^{L} z_i W_i} \right) = \mathbb{E} \left( e^{-\sum_{i=1}^{L} z_i \sum_{j=1}^{N_0} B_{0,j} - \sum_{i=1}^{L} \left( z_i \sum_{j=1}^{N_i} B_{i,j} \right)} \right).$$

Conditioning on the values of $N_i, i = 0, \ldots, L$, we obtain that $\tilde{W}(z)$ equals

$$\sum_{n_0=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} \pi(n) \mathbb{E} \left( e^{-\sum_{i=1}^{L} z_i \sum_{j=1}^{n_i} B_{i,j}} \right) = \sum_{n_0=0}^{\infty} \cdots \sum_{n_L=0}^{\infty} \pi(n) \left( \frac{\mu_0}{\mu_0 + \sum_{j=1}^{L} z_i} \right)^{n_0} \prod_{i=1}^{L} \left( \frac{\mu_i}{\mu_i + z_i} \right)^{n_i}.$$

Substituting (7.2) and invoking that $\rho_i = \lambda_i \mu_i^{-1}$, we obtain that $\tilde{W}(z)$ is equivalent to

$$C^{-1} \sum_{i=1}^{L} \sum_{n_i=0}^{\infty} \left( \frac{\rho_i \mu_i}{\mu_i + z_i} \right)^{n_i} \sum_{n_0=0}^{\infty} \left( \sum_{j=1}^{L} n_j \right)^{n_0} \left( \frac{\rho_0 \mu_0}{\mu_0 + \sum_{j=1}^{L} z_j} \right)^{-1} \left( \sum_{j=1}^{L} \right)^{n_i}$$

$$= C^{-1} \sum_{i=1}^{L} \sum_{n_i=0}^{\infty} \left( \frac{\lambda_i}{\mu_i + z_i} \right)^{n_i} \left( 1 - \frac{\lambda_0}{\mu_0 + \sum_{j=1}^{L} z_j} \right)^{n_0} \left( 1 - \frac{\lambda_i}{\mu_i + z_i} \right)^{-1} \left( \sum_{j=1}^{L} \right)^{n_i}$$

$$= C^{-1} \sum_{i=1}^{L} \sum_{n_i=0}^{\infty} \left( \frac{\lambda_i}{\mu_i + z_i} \right)^{n_i} \left( 1 - \frac{\lambda_0}{\mu_0 + \sum_{j=1}^{L} z_j} \right)^{n_0} \left( 1 - \frac{\lambda_i}{\mu_i + z_i} \right)^{-1} \left( \sum_{j=1}^{L} \right)^{n_i}$$

$$= \left( 1 - \frac{\lambda_0}{\mu_0 + \sum_{j=1}^{L} z_j} \right) \left( 1 - \rho_0 \right)^{L-1} \left( 1 - \frac{\lambda_i}{\mu_i + z_i} \right). \quad (7.5)$$

The second equality above follows by applying the negative binomial formula:

$$(1-x)^{-d} = \sum_{n=0}^{\infty} \binom{d-1+n}{n} x^n.$$

The final equality follows by substituting (7.3). Rearranging (7.5) finally gives (7.4), and completes the proof. \qed
Remark: We now provide some interpretation for the expression for \( \tilde{W}(z) \) given in Theorem 7.2.2. Consider an M/H\(_2\)/1 queue with arrival rate \( \tilde{\lambda}_0 + \lambda_i \) and service requirements that are exponentially distributed with mean \( 1/\tilde{\mu}_0 \) \((1/\mu_i)\) with probability \( \tilde{\lambda}_0/\tilde{\lambda}_0 + \lambda_i \), where \( \tilde{\lambda}_0 := \lambda_0/L \) and \( \tilde{\mu}_0 := \mu_0/L \). The LST of the workload \( V_i(t) \) in steady state of this M/H\(_2\)/1 queue is given by the well-known Pollaczek-Khinchin formula

\[
\tilde{V}_i(z_i) = \frac{(1 - \rho_0 - \rho_i)z_i}{(\tilde{\lambda}_0 + \lambda_i)B(z_i) + z_i - (\tilde{\lambda}_0 + \lambda_i)},
\]

where

\[
\tilde{B}(z_i) := \frac{\tilde{\lambda}_0}{\tilde{\lambda}_0 + \lambda_i} \frac{\tilde{\mu}_0}{\tilde{\mu}_0 + z_i} + \frac{\lambda_i}{\tilde{\lambda}_0 + \lambda_i} \frac{\mu_i}{\mu_i + z_i}.
\]

Substituting \( \tilde{B}(z_i) \) we find

\[
\tilde{V}_i(z_i) = \frac{1 - \rho_0 - \rho_i}{1 - \frac{\lambda_0}{\mu_0 + z_i} - \frac{\lambda_i}{\mu_i + z_i}}.
\]

Let us assume we have \( L \) of these M/H\(_2\)/1 queues, all independent, indexed by \( i, i = 1, \ldots, L \). Then the joint LST of the workload \( V(t) \) is given by

\[
\tilde{V}(z) \equiv \tilde{V}(z_1, \ldots, z_L) = \prod_{i=1}^{L} \tilde{V}_i(z_i) = \prod_{i=1}^{L} \frac{1 - \rho_0 - \rho_i}{1 - \frac{\lambda_0}{\mu_0 + z} - \frac{\lambda_i}{\mu_i + z}}.
\]

Comparing (7.6) with (7.4) indeed shows some similar terms. Obviously, the two expressions cannot be expected to be identical, because the linear network is different from \( L \) independent M/H\(_2\)/1 queues. Taking \( z_i = z, i = 1, \ldots, L \), (7.4) can however be rewritten as

\[
\tilde{W}(z, \ldots, z) = \left( \frac{1 - \frac{\lambda_0}{\mu_0 + z}}{1 - \rho_0} \right)^{L-1} \prod_{i=1}^{L} \frac{1 - \rho_0 - \rho_i}{1 - \frac{\lambda_0}{\mu_0 + z} - \frac{\lambda_i}{\mu_i + z}}.
\]

The above provides some interpretation for the LST (7.4). It says that \( \tilde{V}(z, \ldots, z) = \tilde{W}(z, \ldots, z)\tilde{U}(z) \), where

\[
\tilde{U}(z) := \left( \frac{1 - \rho_0}{1 - \frac{\lambda_0}{\mu_0 + z}} \right)^{L-1}
\]

is a term that accounts for the dependence and interaction among the \( L \) M/H\(_2\)/1 queues. Note that the LST of the workload \( S(t) \) in steady state in an M/M/1 queue with arrival rate \( \tilde{\lambda}_0 \) and service rate \( \tilde{\mu}_0 \) is given by \( \tilde{S}(z) = (1 - \rho_0)/\left(1 - \frac{\lambda_0}{\mu_0 + z}\right) \).
Hence, \( \tilde{U}(z) \) is the LST of the sum of the workloads in \( L - 1 \) of these M/M/1 queues (all independent). The above shows that
\[
\sum_{i=1}^{L} W_i + \sum_{i=1}^{L-1} U_i \xrightarrow{d} \sum_{i=1}^{L} V_i,
\]
where \( U_i, i = 1, \ldots, L - 1 \) are i.i.d. copies of \( U \), and \( \xrightarrow{d} \) indicates that both sides are equal in distribution.

If \( \alpha \neq 1 \) or \( \kappa_i \neq \kappa \), then there are no explicit expressions available for the steady-state distribution of \( N(t) \).

### 7.3 Fluid and diffusion models

In this section we discuss the heavy-traffic results of [85, 91], which provide the basis for the approximations developed in Sections 7.5 and 7.6. Define the following fluid scaled processes:
\[
N^k(t) := N(kt)/k \quad \text{and} \quad W^k(t) := W(kt)/k,
\]
where \( W_i(t) = N_0(t)/\mu_0 + N_i(t)/\mu_i, i = 1, \ldots, L \). The fluid model can then be obtained from the original model by letting \( k \to \infty \). For ease of notation, let \( N^{\infty}_0(t) \) be denoted by \( N(t) \), and \( W^{\infty}(t) \) by \( W(N(t)) \). Define
\[
s_0(t) := \frac{(\kappa_0 N_0(t)^{\alpha})^{1/\alpha}}{(\kappa_0 N_0(t)^{\alpha})^{1/\alpha} + (\sum_{l=1}^{L} \kappa_l N_l(t)^{\alpha})^{1/\alpha}}; \quad s_i(t) := (1 - s_0(t)) l_i(t),
\]
for \( i = 1, \ldots, L \), where \( l_i(t) = 1 \) if \( N_i(t) > 0 \), and 0 otherwise, i.e., \( s_i(t) \) denotes the total service rate allocated to class \( i \) at time \( t \), \( i = 0, \ldots, L \). Then the evolution of the workload process can be described as follows:
\[
\frac{d}{dt} N_i(t) = \lambda_i - \mu_i s_i(t), \quad \text{for} \ i = 0, \ldots, L;
\]
\[
N_i(t) \geq 0, \quad \text{for} \ i = 0, \ldots, L.
\]
A fluid model solution is an absolutely continuous function \( \mathcal{N} : [0, \infty) \to \mathbb{R}_+^{L+1} \), such that at each regular point \( t \) for \( \mathcal{N}(\cdot) \) (i.e., a value of \( t \) at which each component of \( \mathcal{N}(\cdot) \) is differentiable), we have that for \( i = 0, \ldots, L \),
\[
\frac{d}{dt} \mathcal{N}_i(t) = \begin{cases} 
\lambda_i - \mu_i s_i(t) & \text{if } \mathcal{N}_i(t) > 0; \\
0 & \text{if } \mathcal{N}_i(t) = 0,
\end{cases}
\]
and for \( i = 1, \ldots, L \),
\[
s_0(t) l_0(t) + \rho_0(1 - l_0(t)) + s_i(t) l_i(t) + \rho_i(1 - l_i(t)) \leq 1.
\]
A state $N$ is called invariant if there is a fluid model solution such that $N(t) = N$ for all $t \geq 0$. Let $J := \{ j \in \{1, \ldots, L \} : \rho_0 + \rho_j = 1 \} \neq \emptyset$ be the set of nodes that are critically loaded.

The following theorem appeared in slightly different form in [91].

**Theorem 7.3.1** The following statements are equivalent:

(i) $N$ is an invariant state;

(ii) $s_i(t) = \rho_i$ for all $i$ such that $N_i > 0$;

(iii) There is a $q \in \mathbb{R}_+^L$ such that

$$
N_0 = \rho_0 \left( \frac{\sum_{j \in J} q_j}{\kappa_0} \right)^{1/\alpha},
$$

for $i \in J$,

$$
N_i = \rho_i \left( \frac{q_i}{\kappa_i} \right)^{1/\alpha},
$$

and for $i \notin J$, $N_i = 0$;

(iv) $N = \triangle(W(N))$, where $\triangle(x)$ is the unique value of $N \in \mathbb{R}_+^{L+1}$ that solves the optimization problem:

$$
\min_{N} F(N) = \frac{1}{\alpha+1} \sum_{i=0}^{L} \lambda_i \kappa_i \mu_i^{-\alpha-1} \left( \frac{N_i}{\lambda_i} \right)^{\alpha+1}
$$

subject to

$$
N_0/\mu_0 + N_i/\mu_i \geq x_i, \quad i \in J
$$

over

$$
N_i \geq 0, \quad i = 0, \ldots, L.
$$

In the remainder of this section we assume that there are $L = 2$ nodes, and that $\kappa_0 = \kappa_1 = \kappa_2 = \kappa$. Furthermore, we assume heavy-traffic conditions at both nodes, i.e., $J = \{1, 2\}$. Define the diffusion scaled processes:

$$
\hat{N}^k(t) := N(k^2 t)/k \quad \text{and} \quad \hat{W}^k(t) := W(k^2 t)/k,
$$

where $W_i(t) = N_0(t)/\mu_0 + N_i(t)/\mu_i$, $i = 1, 2$, as before. In [85] the authors show (under the assumptions mentioned above) that $\hat{W}^k(t)$ converges in distribution to a continuous process $\hat{W}(t)$ as $k \to \infty$. The process $\hat{W}(t)$ is a so-called Semimartingale Reflecting Brownian Motion (SRBM) that lives in the cone

$$
\left\{ w : w_1 = \frac{\rho_0}{\mu_0} \left( \frac{q_1 + q_2}{\kappa} \right)^{1/\alpha} + \frac{\rho_i}{\mu_i} \left( \frac{q_i}{\kappa} \right)^{1/\alpha}, \quad q_1, q_2 \geq 0, \quad i = 1, 2 \right\}.
$$

In [85] it was shown that for all $\alpha \in (0, \infty)$ this is the same as the cone

$$
\left\{ (w_1, w_2) : w_1 \geq 0, \quad w_1 \frac{\rho_0/\mu_0}{(1 - \rho_0)/\mu_1 + \rho_0/\mu_0} \leq w_2 \leq w_1 \frac{(1 - \rho_0)/\mu_2 + \rho_0/\mu_0}{\rho_0/\mu_0} \right\}.
$$
as depicted in Figure 7.1. The state space is an infinite two-dimensional wedge, and the process behaves in the interior of the wedge like a two-dimensional Brownian motion with zero drift and covariance matrix

\[
\begin{pmatrix}
2 \left( \frac{\rho_0}{\mu_0} + \frac{\rho_1}{\mu_1} \right) & 2 \left( \frac{\rho_2}{\mu_2} \right) \\
2 \frac{\rho_1}{\mu_0} & 2 \left( \frac{\rho_2}{\mu_0} + \frac{\rho_2}{\mu_2} \right)
\end{pmatrix}
\]

The process reflects instantaneously at the boundary of the wedge, the angle of reflection being constant along each side. Vertical (horizontal) reflection on the bounding face \( w_2 = w_1 \left( \frac{\rho_2}{\mu_2} \right) \) can be interpreted as a manifestation of so-called entrainment: congestion at node 1 (node 2) prevents node 2 (node 1) from utilizing the full service rate. In [168] it was shown that the process is transient in the cone, i.e., no steady-state distribution exists.

### 7.4 Single bottleneck node

In this section we propose a method for approximating \( E N_i, i = 0, \ldots, L \), in case of a single bottleneck node, i.e., \( |J| = 1 \). In case just a single node, say \( z, z \in \{1, \ldots, L\} \), is critically loaded, statement (iii) of Theorem 7.3.1 suggests that the number of class-\( i \) users, \( i = 1, \ldots, L, i \neq z \), will be negligible compared to the number of
class-0 and class-\(z\) users. Hence, the service rates allocated to the various classes will be predominantly determined by the number of class-0 and class-\(z\) users, and approximately equal

\[
s_0(n) = \frac{(\kappa_0 n^0_0)^{1/\alpha}}{(\kappa_0 n^0_0)^{1/\alpha} + \left(\sum_{j=1}^{L} \kappa_j n_j^0\right)^{1/\alpha}} \approx \frac{(\kappa_0 n^0_0)^{1/\alpha}}{(\kappa_0 n^0_0)^{1/\alpha} + (\kappa_z n^0_z)^{1/\alpha}} = \frac{\kappa^*_0 n_0}{\kappa^*_0 n_0 + \kappa^*_z n_z};
\]

\[
s_i(n) = \frac{\left(\sum_{j=1}^{L} \kappa_j n_j^0\right)^{1/\alpha}}{(\kappa_0 n^0_0)^{1/\alpha} + \left(\sum_{j=1}^{L} \kappa_j n_j^0\right)^{1/\alpha}} \approx \frac{\kappa^*_z n_z}{\kappa^*_0 n_0 + \kappa^*_z n_z}, \quad i = 1, \ldots, L,
\]

where \(\kappa^*_0 = \kappa_0^{1/\alpha}\) and \(\kappa^*_z = \kappa_z^{1/\alpha}\). Thus, node \(z\) roughly behaves as a DPS model with relative weights \(\kappa^*_0\) and \(\kappa^*_z\) for classes 0 and \(z\), respectively. The results of [63] then imply that \(E N_0\) and \(E N_z\) satisfy the set of linear equations

\[
EN_0 - \rho_0 EN_0 - \kappa^*_z \frac{\lambda_z EN_0 + \lambda_0 EN_z}{\kappa^*_0 \mu_0 + \kappa^*_z \mu_z} \approx \rho_0;
\]

\[
EN_z - \rho_z EN_z - \kappa^*_0 \frac{\lambda_0 EN_0 + \lambda_z EN_z}{\kappa^*_0 \mu_0 + \kappa^*_z \mu_z} \approx \rho_z,
\]

from which we deduce that

\[
EN_0 \approx \frac{\rho_0}{1 - \rho_0 - \rho_z} \left(1 + \frac{\mu_0 \rho_z (\kappa^*_z - \kappa^*_0)}{\kappa^*_0 \mu_0 (1 - \rho_0) + \kappa^*_z \mu_z (1 - \rho_z)}\right);
\]

\[
EN_z \approx \frac{\rho_z}{1 - \rho_0 - \rho_z} \left(1 + \frac{\mu_z \rho_0 (\kappa^*_0 - \kappa^*_z)}{\kappa^*_0 \mu_0 (1 - \rho_0) + \kappa^*_z \mu_z (1 - \rho_z)}\right).
\]

Let \(\tilde{E N}_i\) denote the approximation for \(E N_i\), \(i = 0, z\). Then

\[
\tilde{\pi} := \frac{\kappa^*_0 \tilde{E N}_0 + \kappa^*_z \tilde{E N}_z}{\kappa^*_0 \kappa_0 n_0^0 + \kappa^*_z \kappa_z n_z^0}
\]

can be regarded as an approximation for the service rate allocated to classes \(i = 1, \ldots, L, i \neq z\). The number of class-\(i\) users, \(i = 1, \ldots, L, i \neq z\), will approximately behave as in an M/M/1 queue with arrival rate \(\lambda_i\) and service rate \(\mu_i \tilde{\pi}\). This gives the approximation

\[
E N_i \approx \frac{\rho_i}{\tilde{\pi} - \rho_i}, \quad i = 1, \ldots, L, \quad i \neq z.
\]

Note that the values of \(\kappa_i, i = 1, \ldots, L, i \neq z\), do not appear in this approximation. This suggests that the weights of classes that do not traverse the bottleneck node, will tend to have limited impact on the flow-level performance.

We now discuss the numerical experiments that we conducted to examine the accuracy of the above-described method. We first test this approach for a linear
network with $L = 2$ nodes, $\alpha = 1$, and $\kappa_i = \kappa$, $i = 0, 1, 2$, for which we have exact expressions for $EN_i$, $i = 0, 1, 2$, see Theorem 7.2.1. We fix $\rho_0 = 0.6$ and $\rho_1 = 0.39$, so that node 1 is highly loaded ($z = 1$), and vary the value of $\rho_2$. Note that in case of equal weights, the approximations only depend on the traffic characteristics through the class loads, and not on the specific values of the $\lambda_i$ and $\mu_i$. The results are presented in Table 7.1, and indicate that the approximations are remarkably accurate. As could be expected, the smaller the value of $\rho_2$, the better the approximations.

In case $\alpha \neq 1$ or $\kappa_i \neq \kappa$, there are no exact expressions available for $EN_i$, $i = 0, 1, 2$, and we need to resort to simulation experiments to investigate the accuracy of the approximations. Throughout this chapter, the simulation numbers are obtained as
averages over 10000 busy periods. We choose the same setting as above, but with 
\( \kappa_0 = 2, \kappa_1 = 0.5 \) and \( \kappa_2 = 1 \). In this case the approximations do depend on the specific 
values of the \( \mu_i \). We consider two scenarios: in Scenario 1 we take \( \mu_0 = \mu_1 = \mu_2 = 1 \), 
while in Scenario 2 we set \( \mu_0 = 0.75, \mu_1 = 1 \) and \( \mu_2 = 1.5 \). The results are presented 
in Tables 7.2 and 7.3.

Note that the approximations for \( \mathbb{E}N_0 \) and \( \mathbb{E}N_1 \) do not depend on the presence 
of class-2 users, and are in particular independent of the value of \( \rho_2 \). Further observe 
that if \( \alpha \to \infty \), then \( \kappa_i^0, \kappa_i^1 \to 1 \), and as a consequence \( \mathbb{E}N_i \approx \rho_i/(1 - \rho_0 - \rho_1) \), 
\( i = 0, 1 \). The results are surprisingly accurate, even if node 2 is also relatively highly 
loaded \( (\rho_0 + \rho_2 = 0.9) \). Note that \( \mathbb{E}N_2^{sim} \) is increasing in \( \rho_2 \), as could be expected. 
The influence of \( \rho_2 \) on \( \mathbb{E}N_0^{sim} \) and \( \mathbb{E}N_1^{sim} \) is more subtle, as closer inspection of 
Tables 7.2 and 7.3 demonstrates. It might be natural to expect that increasing \( \rho_2 \) 
would also have an adverse impact on \( \mathbb{E}N_0^{sim} \) and \( \mathbb{E}N_1^{sim} \). As the value of \( \rho_2 \) and 
the number of class-2 users increases, however, the service rate \( s_0(n) \) will decrease, 
whereas the service rate \( s_2(n) \) will increase. The resulting increase in the number 
of class-0 users will have the counteracting effect of decreasing \( s_2(n) \), and conversely 
the expected decrease in the number of class-2 users will have the opposite effect of 
increasing \( s_0(n) \). Because of these interacting effects, the net impact basically remains 
unpredictable, and as Tables 7.2 and 7.3 reveal, \( \mathbb{E}N_0^{sim} \) and \( \mathbb{E}N_1^{sim} \) do not necessarily 
change in a monotone manner as the value of \( \rho_2 \) increases.

### 7.5 Two bottleneck nodes and equal weights: workload invariance

In this section we consider the scenario that there are two nodes critically loaded, i.e., 
\( |J| = 2 \). Since the nodes can be indexed arbitrarily, we may assume without loss of 
generality that \( J = \{1, 2\} \). Also, suppose that \( \kappa_i = \kappa, i = 0, \ldots, L \).

Let \( W(t) \) be the workload process associated with the two bottleneck nodes. The 
results from [85, 91] as reviewed in Section 7.3 indicate that the behavior of \( W(t) \) is 
asymptotically independent of the value of \( \alpha \). In particular, this suggests that the 
behavior of the workload process can be approximated by the known distribution for 
\( \alpha = 1 \). In order to examine this hypothesis, we calculated the mean workload (using 
Theorem 7.2.2)

\[
\mathbb{E}W_i^{\text{exact}}(\alpha = 1) \equiv \mathbb{E}W_i^{\text{exact}}(1) = \frac{\lambda_i/\mu_i^2}{1 - \rho_0 - \rho_i} + \frac{\lambda_0/\mu_0^2}{1 - \rho_0} \left( 1 + \sum_{j=1}^{L} \frac{\rho_j}{1 - \rho_0 - \rho_j} \right), \quad (7.7)
\]

with \( i = 1, 2 \), and compared it with simulation for the case of \( L = 2 \) nodes, \( \rho_0 + \rho_1 = \rho_0 + \rho_2 = 0.99 \), and \( \mu_i = \kappa_i = 1, i = 0, 1, 2 \). We also considered the asymmetric case 
\( \rho_0 + \rho_1 = \rho_0 + \rho_2 = 0.99, \kappa_i = 1, i = 0, 1, 2, \mu_0 = 0.75, \mu_1 = 1 \) and \( \mu_2 = 1.5 \).
Flow-level performance of linear networks

Table 7.4: Testing whether $W(t)$ is independent of $\alpha$. Left (Right): the symmetric (asymmetric) case.

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>$\rho_1 = \rho_2$</th>
<th>$\alpha$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$\rho_0$</th>
<th>$\rho_1 = \rho_2$</th>
<th>$\alpha$</th>
<th>$X_1$</th>
<th>$X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.60</td>
<td>1</td>
<td>0.001</td>
<td>0.019</td>
<td>0.3</td>
<td>0.60</td>
<td>1</td>
<td>0.006</td>
<td>-0.009</td>
</tr>
<tr>
<td>0.5</td>
<td>0.49</td>
<td>1</td>
<td>0.006</td>
<td>0.020</td>
<td>0.5</td>
<td>0.49</td>
<td>1</td>
<td>-0.046</td>
<td>-0.033</td>
</tr>
<tr>
<td>0.7</td>
<td>0.29</td>
<td>1</td>
<td>-0.015</td>
<td>-0.024</td>
<td>0.7</td>
<td>0.29</td>
<td>1</td>
<td>0.048</td>
<td>0.042</td>
</tr>
<tr>
<td>0.3</td>
<td>0.69</td>
<td>2</td>
<td>-0.042</td>
<td>-0.047</td>
<td>0.3</td>
<td>0.69</td>
<td>2</td>
<td>-0.065</td>
<td>-0.077</td>
</tr>
<tr>
<td>0.5</td>
<td>0.49</td>
<td>2</td>
<td>-0.041</td>
<td>-0.056</td>
<td>0.5</td>
<td>0.49</td>
<td>2</td>
<td>-0.025</td>
<td>-0.038</td>
</tr>
<tr>
<td>0.7</td>
<td>0.29</td>
<td>2</td>
<td>-0.039</td>
<td>-0.049</td>
<td>0.7</td>
<td>0.29</td>
<td>2</td>
<td>-0.030</td>
<td>-0.040</td>
</tr>
<tr>
<td>0.3</td>
<td>0.69</td>
<td>5</td>
<td>-0.027</td>
<td>-0.065</td>
<td>0.3</td>
<td>0.69</td>
<td>5</td>
<td>-0.046</td>
<td>-0.036</td>
</tr>
<tr>
<td>0.5</td>
<td>0.49</td>
<td>5</td>
<td>-0.005</td>
<td>0.003</td>
<td>0.5</td>
<td>0.49</td>
<td>5</td>
<td>-0.055</td>
<td>-0.057</td>
</tr>
<tr>
<td>0.7</td>
<td>0.29</td>
<td>5</td>
<td>-0.058</td>
<td>-0.069</td>
<td>0.7</td>
<td>0.29</td>
<td>5</td>
<td>-0.037</td>
<td>-0.035</td>
</tr>
<tr>
<td>0.3</td>
<td>0.69</td>
<td>$\infty$</td>
<td>-0.007</td>
<td>-0.011</td>
<td>0.3</td>
<td>0.69</td>
<td>$\infty$</td>
<td>-0.028</td>
<td>-0.022</td>
</tr>
<tr>
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<td>0.49</td>
<td>$\infty$</td>
<td>-0.043</td>
<td>-0.063</td>
<td>0.5</td>
<td>0.49</td>
<td>$\infty$</td>
<td>-0.048</td>
<td>-0.076</td>
</tr>
<tr>
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<td>$\infty$</td>
<td>-0.061</td>
<td>-0.055</td>
<td>0.7</td>
<td>0.29</td>
<td>$\infty$</td>
<td>-0.003</td>
<td>-0.009</td>
</tr>
</tbody>
</table>

Define

$$X_i := \frac{E W_i^\text{sim}(\alpha)}{E W_i^\text{exact}(1)} - 1, \quad i = 1, 2.$$ 

The results, summarized in Table 7.4, indicate that the mean workload for $\alpha = 1$ indeed provides a reasonably accurate approximation for a wide range of $\alpha$ values. Note that $X_i$ should be equal to 0 for all cases with $\alpha = 1$. In most cases with $\alpha > 1$, $E W_i^\text{exact}(1)$ is larger than $E W_i^\text{sim}(\alpha)$, and thus seems to yield a conservative approximation. Below we provide an explanation for this observation. In preparation for that, we first present the following proposition.

Proposition 7.5.1 For fixed $n = (n_0, \ldots, n_L)$ and $\kappa_i = \kappa, i = 0, \ldots, L$, the service rate $s_0(n)$ allocated to class-0 users is increasing in $\alpha$.

Proof: For fixed $n = (n_0, \ldots, n_L)$ and $\kappa_i = \kappa, i = 0, \ldots, L$, we obtain from (7.1) that

$$s_0(n) = \frac{n_0}{n_0 + (\sum_{j=1}^{L} n_j^\kappa)^{1/\alpha}}. \quad (7.8)$$

Equivalently, we have to prove that $(\sum_{j=1}^{L} n_j^\alpha)^{1/\alpha}$ is decreasing in $\alpha$. First note that

$$n_1^\alpha + \cdots + n_L^\beta < (n_1^\alpha + \cdots + n_L^\alpha)^r$$

for all $r > 1$. Therefore,

$$\left( \sum_{i=1}^{L} n_i^\alpha \right)^{1/\beta} = \left( \sum_{i=1}^{L} n_i^{\alpha r} \right)^{1/\alpha r} = (n_1^{\alpha r} + \cdots + n_L^{\alpha r})^{1/\alpha r} < (n_1^\alpha + \cdots + n_L^\alpha)^{r/\alpha} = \left( \sum_{i=1}^{L} n_i^\alpha \right)^{1/\alpha},$$
for all \( \beta > \alpha \), which proves the stated.

Now observe that the workload at each of the nodes is minimized (sample-pathwise in fact) when class 0 receives priority over classes 1 and 2. Since the capacity allocated to class-0 users is increasing in \( \alpha \), it is thus plausible that more generally the mean workload \( E W^\text{exact}_i(\alpha) \) decreases as function of \( \alpha \), which implies that \( X_i \) is smaller than 0 for \( \alpha > 1 \), \( i = 1, 2 \). This provides an explanation for the negative values in Table 7.4. Below we show that the latter property can in fact be rigorously proved using Proposition 7.5.1 and stochastic coupling arguments.

Denote by \( r_i(t) \) the instantaneous service rate allocated to class \( i \) at time \( t \), i.e., \( r_i(t) = s_i(N(t)) \) if \( N_i(t) > 0 \), and otherwise \( r_i(t) = 0 \), \( i = 0, \ldots, L \). Denote by \( R_i(t) := \int_0^t r_i(u) du \) the cumulative amount of service received by class \( i \) during the time interval \([0, t]\), \( i = 0, \ldots, L \). Denote by \( B_{i,n} \) the service requirement of the \( n \)-th arriving class-\( i \) user, \( i = 0, \ldots, L \). Denote by \( C_i(s) := \sup\{n : \sum_{m=1}^n B_{i,m} < s\} \) the number of class-\( i \) service completions as function of the amount of service received by class \( i \), \( i = 0, \ldots, L \), assuming a FIFO service discipline. Thus \( D_i(t) = C_i(R_i(t)) \) represents the number of class-\( i \) service completions during the time interval \([0, t]\), \( i = 0, \ldots, L \). Denote by \( A_i(t) \) the number of class-\( i \) users arriving during the time interval \([0, t]\), \( i = 0, \ldots, L \). Denote by \( Q_i(t) := \sum_{m=1}^{A_i(t)} B_{i,m} \) the amount of class-\( i \) work arriving during the time interval \([0, t]\), \( i = 0, \ldots, L \). Denote by \( V_i(t) \) the amount of class-\( i \) work at time \( t \), \( i = 0, \ldots, L \).

Since the service requirements are exponentially distributed, the stochastic behavior of the network does not depend on the service discipline within classes, as long as that discipline is not based on any knowledge of the actual realizations of the service requirements. We may therefore assume that the service discipline within classes is FIFO.

Consider the behavior of the network under two AFS policies with parameters \( \beta \) and \( \gamma \) for the same realizations of the arrival processes and service requirements. We attach \( \beta \) and \( \gamma \) as superscripts to the various quantities associated with the two policies.

**Proposition 7.5.2** Suppose that the system is empty at time \( t = 0 \). If \( \beta \leq \gamma \), then \( W_i^\beta(t) \geq W_i^\gamma(t) \) for all \( t \geq 0 \), \( i = 1, \ldots, L \).

**Proof:** Below we will prove that if \( \beta \leq \gamma \), then (i) \( N^\beta_0(t) \geq N^\gamma_0(t) \), (ii) \( R^\beta_0(t) \leq R^\gamma_0(t) \), and (iii) \( R^\beta_i(t) + R^\gamma_i(t) \leq R^\gamma_i(t) + R^\gamma_i(t) \) for all \( t \geq 0 \), \( i = 1, \ldots, L \). Note that \( V_i^\beta(t) = Q_i^\beta(t) - R_i^\beta(t), i = 0, \ldots, L, j = \beta, \gamma \), so that \( R^\beta_0(t) + R^\gamma_0(t) \leq R^\gamma_0(t) + R^\gamma_0(t) \) implies that \( W^\beta_0(t) = V_0^\beta(t) + V_0^\gamma(t) \geq V_0^\gamma(t) + V_0^\gamma(t) = W^\gamma_0(t) \) for all \( t \geq 0 \), \( i = 1, \ldots, L \).

First note that \( N^\beta_i(t) = A_i^\beta(t) - D_i^\beta(t) \), with \( D_i(t) = C_i^\gamma(R_i^\gamma(t)), i = 0, \ldots, L, j = \beta, \gamma \), i.e., inequality (i) follows from (ii), and it suffices to prove that inequalities...
The numerical results presented in the previous section indicate that

(ii) and (iii) hold. Below we assume that inequality (ii) or (iii) does not hold, and we show that this results in a contradiction. Let $u > 0$ be the first time epoch at which one of the two inequalities is violated. First assume that inequality (ii) is the first one to be violated, i.e., $R_i^0(u) + r_j^0(u) > r_j^0(u)$ (with strict inequality), but $R_i^0(u) + R_i^0(u) < R_i^0(u) + R_j^0(u)$, $i = 1, \ldots, L$. Clearly, then $N_i^0(u) = N_j^0(u)$, and, using Proposition 7.5.1, it follows that $N_i^0(u) < N_j^0(u)$ for some $j = 1, \ldots, L$, because otherwise we would have $r_j^0(u) \leq r_j^0(u)$. This implies that $R_i^0(u) > R_j^0(u)$, and thus $R_i^0(u) + R_j^0(u) > R_i^0(u) + R_j^0(u)$, which contradicts the initial assumption. Next, assume that inequality (iii) is the first one to be violated, i.e., $R_i^0(u) + R_j^0(u) = R_i^0(u) + R_j^0(u)$ and $r_j^0(u) + r_j^0(u) > r_j^0(u) + r_j^0(u)$ for some $j = 1, \ldots, L$, but $R_i^0(u) \leq R_i^0(u)$. It follows that $N_i^0(u) = 0$, because otherwise we would have $r_i^0(u) + r_j^0(u) = 1 \geq r_i^0(u) + r_j^0(u)$. This implies that $R_i^0(u) \leq R_j^0(u)$, and thus $R_i^0(u) = R_i^0(u)$. $R_j^0(u) = R_j^0(u)$. $R_i^0(u) = R_i^0(u)$ (as $R_i^0(u) + R_j^0(u) = R_i^0(u) + R_j^0(u)$ and $R_i^0(u) \leq R_i^0(u)$ by assumption), and $R_i^0(u) \leq R_j^0(u)$ for all $i = 1, \ldots, L$, $i \neq j$, as well. Consequently, $N_i^0(u) = N_i^0(u)$, $N_i^0(u) = N_i^0(u) = 0$ and $N_i^0(u) \geq N_i^0(u)$ for all $i = 1, \ldots, L$, $i \neq j$. This means that $r_i^0(u) \leq r_i^0(u)$, and thus, since $r_j^0(u) = r_j^0(u) = 0$, $r_j^0(u) + r_j^0(u) \leq r_i^0(u) = r_i^0(u) + r_j^0(u)$, which contradicts the initial assumption. Hence, we have proven that if $\beta \leq \gamma$, then inequalities (i), (ii) and (iii) hold, and therefore this proves the stated. $\square$

7.6 Two bottleneck nodes and equal weights: approximations

In this section we develop three methods for approximating $\mathbb{E}N_i$, $i = 0, 1, 2$. Recall that we suppose that $J = \{1, 2\}$ and $\kappa_i = \kappa$, $i = 0, \ldots, L$. The various methods differ in some technical details, but they all rely on the insights from the heavy-traffic results as reviewed in Section 7.3. In Section 7.6.4 we present approximations for $\mathbb{E}N_i$, $i = 3, \ldots, L$.

7.6.1 Method 1

The numerical results presented in the previous section indicate that $\mathbb{E}W_i^{\text{exact}}(\alpha)$ is nearly constant in $\alpha \in (0, \infty)$, provided that the load at nodes 1 and 2 is sufficiently high. In particular, it is approximately equal to the known value for $\alpha = 1$ as given by (7.7). Further observe that $\mathbb{E}W_i^{\text{exact}}(\alpha) = \mathbb{E}N_0/\mu_0 + \mathbb{E}N_i/\mu_i$, $i = 1, \ldots, L$. Thus, we obtain

$$
\mathbb{E}N_0/\mu_0 + \mathbb{E}N_i/\mu_i \approx \frac{\lambda_i/\mu_i^2}{1 - \rho_0 - \rho_i} + \frac{\lambda_0/\mu_0^2}{1 - \rho_0} \left(1 + \sum_{j=1}^{L} \frac{\rho_j}{1 - \rho_0 - \rho_j}\right), \quad i = 1, 2, \quad (7.9)
$$

i.e., a set of two approximately linear equations with three unknowns. If we can find one additional constraint, then we should be able to determine $\mathbb{E}N_i$, $i = 0, 1, 2$ (as
Now observe that Theorem 7.3.1 shows that an invariant state $N$ in the fluid model can be expressed as

$$N_0 = \rho_0 \left( \frac{q_1 + q_2}{k_0} \right)^{1/\alpha}; \quad N_i = \rho_i \left( \frac{q_i}{k_i} \right)^{1/\alpha}, \quad i = 1, 2, \quad q \in \mathbb{R}_+^2.$$ 

This suggests the following approximation for $(E_N, E_{N_1}, E_{N_2})$:

$$\int_{q_1=0}^{\infty} \int_{q_2=0}^{\infty} \left( \rho_0 \left( \frac{q_1 + q_2}{k} \right)^{1/\alpha}, \rho_1 \left( \frac{q_1}{k} \right)^{1/\alpha}, \rho_2 \left( \frac{q_2}{k} \right)^{1/\alpha} \right) \mathrm{d}P(Q_1 < q_1, Q_2 < q_2),$$

which is equivalent to

$$\frac{1}{k^{1/\alpha}} \left( \rho_0 E \left( (Q_1 + Q_2)^{1/\alpha} \right), \rho_1 E \left( Q_1^{1/\alpha} \right), \rho_2 E \left( Q_2^{1/\alpha} \right) \right).$$

Using the additional approximation

$$(E_{N_0}, E_{N_1}, E_{N_2}) \approx \gamma \left( \rho_0 (EQ_1 + EQ_2)^{1/\alpha}, \rho_1 (EQ_1)^{1/\alpha}, \rho_2 (EQ_2)^{1/\alpha} \right), \quad (7.10)$$

with $\gamma$ some multiplicative constant, and substituting (7.10) in (7.9) then yields a system of two equations with two unknowns. Numerically solving this system yields $\gamma^\alpha EQ_i, \ i = 1, 2$, from which we can obtain $E N_i, \ i = 0, 1, 2$, using (7.10). Note that

$$E \left( (Q_1 + Q_2)^{1/\alpha} \right) \leq E \left( Q_1^{1/\alpha} \right) + E \left( Q_2^{1/\alpha} \right)$$

if $\alpha \in (1, \infty)$, which would provide an upper bound for $E N_0$ relative to $E N_i, \ i = 1, 2$. Likewise, if $\alpha \in (0, 1)$, then this would give a lower bound for $E N_0$ relative to $E N_i, \ i = 1, 2$.

We tested this approach by comparing the results with simulation figures. We took the same simulation parameters as in the previous section. The results are presented in Tables 7.5 and 7.6. Throughout this chapter, $E N_i^{M_j}$ denotes the approximation of $E N_i$ that is obtained by using Method $j$. Note that in Table 7.5 we have $E N_i^{M_1} = E N_i^{M_2}$ by symmetry. The tables indicate that Method 1 gives reasonably accurate estimates for $E N_i$, particularly $E N_0$. Note that Method 1 is fast as well: it suffices to solve a system of two equations with two unknowns.

### 7.6.2 Method 2

We now discuss a second method for approximating $E N_i, \ i = 0, 1, 2$. Again, we start from Equation (7.9) as in Method 1. The difference with Method 1 is that we now use statement $(iv)$ (instead of $(iii)$) of Theorem 7.3.1. Statement $(iv)$ implies that a
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\[ \rho_0 = \rho_1 = \rho_2 \alpha \]

\[
\begin{array}{cccccccc}
\rho_0 & \rho_1 & \alpha & \text{EN}_0^{\text{sim}} & \text{EN}_1^{\text{sim}} & \text{EN}_2^{\text{sim}} & \text{EN}_1^{\text{sim}} = \text{EN}_2^{\text{sim}} \\
0.3 & 0.69 & 1 & 60.20 & 59.80 & 68.58 & 70.81 & 68.77 \\
0.5 & 0.49 & 1 & 100.27 & 99.33 & 48.61 & 50.66 & 48.67 \\
0.7 & 0.29 & 1 & 135.01 & 138.07 & 29.19 & 27.67 & 28.60 \\
0.3 & 0.69 & 2 & 50.21 & 48.95 & 72.98 & 72.36 & 79.02 \\
0.5 & 0.49 & 2 & 86.98 & 87.42 & 54.90 & 52.79 & 60.58 \\
0.7 & 0.29 & 2 & 126.62 & 128.91 & 33.59 & 33.38 & 37.76 \\
0.3 & 0.69 & 5 & 47.86 & 42.83 & 77.25 & 72.40 & 85.75 \\
0.5 & 0.49 & 5 & 88.24 & 79.86 & 58.97 & 60.25 & 68.14 \\
0.7 & 0.29 & 5 & 120.76 & 122.49 & 33.59 & 34.38 & 44.18 \\
0.3 & 0.69 & \infty & 49.75 & 39.02 & 77.94 & 80.25 & 89.55 \\
0.5 & 0.49 & \infty & 82.62 & 74.83 & 59.02 & 60.25 & 73.17 \\
0.7 & 0.29 & \infty & 120.81 & 117.92 & 35.64 & 36.67 & 48.75 \\
\end{array}
\]

Table 7.5: Results for Method 1: the symmetric case.

\[
\begin{array}{cccccccc}
\rho_0 & \rho_1 & \alpha & \text{EN}_0^{\text{sim}} & \text{EN}_1^{\text{sim}} & \text{EN}_2^{\text{sim}} & \text{EN}_1^{\text{sim}} = \text{EN}_2^{\text{sim}} \\
0.3 & 0.69 & 1 & 59.40 & 59.75 & 70.06 & 68.77 & 67.58 & 68.65 \\
0.5 & 0.49 & 1 & 95.19 & 99.22 & 45.80 & 48.70 & 48.46 & 48.55 \\
0.7 & 0.29 & 1 & 143.78 & 137.93 & 31.01 & 28.66 & 29.58 & 28.48 \\
0.3 & 0.69 & 2 & 50.17 & 51.12 & 71.83 & 80.27 & 73.27 & 85.75 \\
0.5 & 0.49 & 2 & 90.58 & 90.72 & 55.61 & 60.04 & 56.49 & 65.56 \\
0.7 & 0.29 & 2 & 127.79 & 131.91 & 33.86 & 36.68 & 33.78 & 40.51 \\
0.3 & 0.69 & 5 & 49.75 & 45.99 & 75.98 & 80.27 & 73.27 & 85.75 \\
0.5 & 0.49 & 5 & 85.94 & 85.16 & 56.51 & 67.46 & 61.03 & 76.68 \\
0.7 & 0.29 & 5 & 127.15 & 127.71 & 35.20 & 42.27 & 39.39 & 48.90 \\
0.3 & 0.69 & \infty & 50.21 & 43.75 & 77.31 & 80.08 & 83.68 & 100.64 \\
0.5 & 0.49 & \infty & 88.24 & 82.88 & 59.16 & 60.04 & 61.03 & 76.68 \\
0.7 & 0.29 & \infty & 130.52 & 126.06 & 35.64 & 36.67 & 40.41 & 52.22 \\
\end{array}
\]

Table 7.6: Results for Method 1: the asymmetric case.

workload vector \( w = (w_1, w_2) \) uniquely determines a state vector \( n \) that solves the optimization problem:

\[
\begin{align*}
\min_{n} & \quad F(n_0, n_1, \ldots, n_L) \\
\text{subject to} & \quad n_0/\mu_0 + n_i/\mu_i \geq w_i, \quad i = 1, 2 \\
& \quad n_i \geq 0, \quad i = 0, \ldots, L.
\end{align*}
\]

The method now works as follows. We determine the vector \( (\text{EN}_0, \text{EN}_1, \ldots, \text{EN}_L) \) that minimizes the function \( F(\text{EN}_0, \text{EN}_1, \ldots, \text{EN}_L) \) subject to the constraints in (7.9). Note that \( \text{EN}_i = 0, i = 3, \ldots, L. \)

As it turns out, Methods 1 and 2 result in similar approximations for \( \text{EN}_i, i = 0, 1, 2. \) This is not too surprising: the only difference between the methods is that we use statement (iii) in one case, and (iv) in the other. However, statements (iii) and (iv) are in fact equivalent in case of heavy traffic, so both methods should roughly agree when the load is sufficiently high.

Remark: Method 2 uses the mean workloads to approximate the mean number of
users. However, we can potentially improve the accuracy of the approximation if we use the distribution of the workloads, which is also asymptotically independent of $\alpha$ in heavy traffic. The resulting approximation is then given by

$$\mathbb{E}N_i = \sum_{n \geq 0} \Delta_i (w(n)) \pi(n), \quad i = 0, 1, 2,$$

where $w_i(n) = n_0/\mu_0 + n_i/\mu_i$, $i = 1, 2$, $\Delta(x)$ is as in Theorem 7.3.1, and $\pi(n)$ is given by (7.2). This will typically result in a different approximation for $\mathbb{E}N_i$ than Method 2, since the optimization problem (7.11) is non-linear. The disadvantage is that it is very time-consuming.

### 7.6.3 Method 3

This method is similar to both previous methods, i.e., we again start from Equation (7.9) to obtain a set of two equations with three unknowns $\mathbb{E}N_{i}$, $i = 0, 1, 2$. Statement (ii) of Theorem 7.3.1 provides an additional equation, which allows us to numerically solve the above system of equations. First note from (7.8) that

$$\sum_{n \geq 0} \frac{n_0}{n_0 + (\sum_{l=1}^{L} n_l^\alpha)^{1/\alpha}} \tilde{\pi}(n) = \sum_{n \geq 0} s_0(n) \tilde{\pi}(n) = \rho_0,$$

where $\tilde{\pi}(n)$ is the steady-state distribution of $N(t)$ in case $\alpha \in (0, \infty)/\{1\}$. The additional equation is then obtained by replacing the latter equation by the approximation

$$\frac{\mathbb{E}N_0}{\mathbb{E}N_0 + (\sum_{l=1}^{L} \mathbb{E}N_l^\alpha)^{1/\alpha}} \approx \frac{\mathbb{E}N_0}{\mathbb{E}N_0 + (\mathbb{E}N_0^\alpha + \mathbb{E}N_2^\alpha)^{1/\alpha}} = \rho_0.$$

We numerically solved the above system of equations for both the symmetric and asymmetric scenarios considered in the previous section. The results are presented in Tables 7.7 and 7.8. Note that the approximations obtained from Method 3 slightly differ from those of Methods 1 and 2. This may be explained from the fact that statement (ii) of Theorem 7.3.1 (for $i = 0$) is only partly satisfied.

### 7.6.4 Approximation for non-bottleneck nodes

In the previous subsections we presented three methods for approximating the mean number of users at the bottleneck nodes. We now provide an approximation for the number of users at the remaining nodes, i.e., $\mathbb{E}N_i$, $i = 3, \ldots, L$. The method is similar in nature as the one presented in Section 7.4 for the case of a single bottleneck node. Let $\mathbb{E}N_i$ denote the approximations obtained for $\mathbb{E}N_i$, $i = 0, 1, 2$. In view of (7.8), define

$$s_0 := \frac{\mathbb{E}N_i}{\mathbb{E}N_0 + (\mathbb{E}N_1^\alpha + \mathbb{E}N_2^\alpha)^{1/\alpha}}$$
as an approximation for the service rate allocated to class 0. As before, the number of class-i users, $i = 3, \ldots, L$, will roughly behave as in an M/M/1 queue with arrival rate $\lambda_i$ and service rate $\mu_i(1 - s_0)$. This gives the approximation

$$EN_i \approx \frac{\rho_i}{1 - s_0 - \rho_i}, \quad i = 3, \ldots, L.$$  

Remark: For the linear network (see Figure 1.7) it was shown in [27] that BFS is equivalent to unweighted proportional fairness, i.e., $\alpha = 1$ and $\kappa_i = \kappa$, $i = 0, \ldots, L$. Note that the steady-state distribution in Theorem 7.2.1 indeed only depends on the loads, and not on any higher-order traffic characteristics. In [24] it was shown that BFS provides a good approximation for unweighted proportional fairness and unweighted max-min fairness. The results of this section, though, illustrate that the accuracy of the BFS approximation for unweighted max-min fairness degrades in heavy-traffic conditions.
7.7 Unequal service rates

In the previous sections we assumed that each of the nodes had unit service rate. In this section we assume that node $i$ has service rate $c_i$, $i = 1, \ldots, L$, and indicate how the results of the previous sections can be generalized.

In case the service rates are not all equal, it can be verified that there exists no closed-form expression for the AFS allocation $s_i(n)$, $i = 0, \ldots, L$. The following proposition presents bounds on $s_i(n)$. First define

$$c_{\min} := \min_{i=1,\ldots,L} c_i; \quad c_{\max} := \max_{i=1,\ldots,L} c_i;$$

$\bar{s}_0(n) := \frac{(\kappa_0 n_0^\alpha)^{1/\alpha}}{(\kappa_0 n_0^\alpha)^{1/\alpha} + \left(\sum_{j=1}^L \kappa_j \left(\frac{n_j c_{\min}}{c_j}\right)^{\alpha}\right)^{1/\alpha}} c_{\min}$

and

$$\underline{s}_0(n) := \frac{(\kappa_0 n_0^\alpha)^{1/\alpha}}{(\kappa_0 n_0^\alpha)^{1/\alpha} + \left(\sum_{j=1}^L \kappa_j \left(\frac{n_j c_{\max}}{c_j}\right)^{\alpha}\right)^{1/\alpha}} c_{\max}.$$

**Proposition 7.7.1** If $n \neq 0$ then, for $i = 1, \ldots, L$,

$$\underline{s}_i(n) \leq s_i(n) \leq \bar{s}_i(n); \quad (c_i - \underline{s}_i(n)) 1_{n_i > 0} \leq s_i(n) \leq (c_i - \bar{s}_i(n)) 1_{n_i > 0}.$$

**Proof:** In order to obtain $s_i(n)$, we first need to solve the optimization problem (6.9). If $n_0 > 0$, then it is straightforward to show that the optimizer $x_0^n$ satisfies $f(x_0^n) = g(x_0^n)$, where

$$f(x_0) := \kappa_0 x_0^{-\alpha}; \quad g(x_0) := \sum_{j=1}^L \kappa_j n_j^\alpha (c_j - n_0 x_0)^{-\alpha}.$$

As mentioned above, in general there does not exist a closed-form expression for $x_0^n$ that satisfies $f(x_0^n) = g(x_0^n)$ However, note that

$$g(x_0) \geq \sum_{j=1}^L \kappa_j n_j^\alpha \left(\frac{c_j}{c_{\max}} n_0 x_0\right)^{-\alpha} = \sum_{j=1}^L \kappa_j n_j^\alpha \left(\frac{c_j}{c_{\max}} (c_{\max} - n_0 x_0)\right)^{-\alpha} := \underline{g}(x_0).$$

Also, we have that

$$g(x_0) \leq \sum_{j=1}^L \kappa_j n_j^\alpha \left(\frac{c_j}{c_{\min}} n_0 x_0\right)^{-\alpha} = \sum_{j=1}^L \kappa_j n_j^\alpha \left(\frac{c_j}{c_{\min}} (c_{\min} - n_0 x_0)\right)^{-\alpha} := \overline{g}(x_0).$$

The value of $x_0$ for which $n_0 f(x_0)$ equals $n_0 g(x_0)$ is $\overline{s}_0(x)$, and the value of $x_0$ for which $n_0 f(x_0)$ equals $n_0 \underline{g}(x_0)$ is $\underline{s}_0(x)$, i.e., we find that $\underline{s}_0(n) = n_0 x_0^n \leq \underline{s}_0(n)$.
if \( n_0 > 0 \). Next use that \( s_i(n) = c_i - s_0(n) \) if \( n_i > 0 \), and the bounds on \( s_0(n) \), to find bounds on \( s_i(n) \), \( i = 1, \ldots, L \). Clearly, if \( n_i = 0 \), then by definition \( s_i(n) = 0 \), \( i = 0, \ldots, L \), which is also supported by the bounds.

Proposition 7.7.1 shows that the bounds are tight if \( c_i = c \), \( i = 1, \ldots, L \), i.e., if each node has service rate \( c \). By setting \( \rho_i = \lambda_i / (\mu_i c_{\min}) \) or \( \rho_i = \lambda_i / (\mu_i c_{\max}) \), \( i = 0, \ldots, L \), we may use the same techniques of the previous sections to derive approximations for the mean number of users of each class, given that one or two of the nodes are critically loaded. Clearly, the smaller the difference between \( c_{\max} \) and \( c_{\min} \), the better the approximations will be.

7.8 Discussion

In Section 7.6 we devised approximations for the mean number of users, based on the assumption that two of the nodes operate under heavy-traffic conditions and that all classes have equal weights. It is substantially more difficult to handle the cases in which there are 1) two nodes critically loaded and not all class weights are equal, or 2) more than two bottleneck nodes. Although the mean number of users can still be related to the mean workloads in these scenarios, the joint workload process at these nodes is no longer independent of the fairness coefficient \( \alpha \). In addition, even for a weighted proportional fair policy the workload distribution is no longer known. Hence, we cannot apply the three methods presented in Section 7.6 for approximating the mean number of users.

One option to obtain conservative estimates in case 2) would be to use the property that the workload for an unweighted AFS policy, with \( \alpha \) larger than one, is smaller than for an unweighted proportional fair policy as mentioned in Section 7.5. Alternatively, as in Section 7.3, we can approximate the workload process by an SRBM living in a cone that now does depend on the fairness coefficient \( \alpha \). Subsequently, we can derive the steady-state distribution of the process, thus having an approximation for the mean workloads. If we succeeded in this, then we could obtain approximations for the mean number of users by applying one of the three methods. However, it turns out to be extremely hard to derive the steady-state distribution of an SRBM living in a multi-dimensional cone, see [76]. The latter suggests that it is also hard to determine the steady-state distribution of the approximation for the workload process, if possible at all.