Appendix A

Some more facts about Linear algebra

In the following we will present some more facts about linear algebra, which might help as a reminder. We use the same notation as in Section 2.1 and continue from there.

**Linear independence** A set $|\phi_1\rangle, \ldots, |\phi_m\rangle \in \mathbb{C}^d$ is called linearly independent if the only way to choose $\alpha_i \in \mathbb{C}$ such that $\sum_{i=1}^m \alpha_i |\phi_i\rangle = 0$ is to choose $\alpha_i = 0$ for all $i$.

**Rank** The rank of a matrix $A \in \mathbb{C}^{d \times d}$ is the largest number of linearly independent rows of $A$.

**Inverse matrix** If for some matrix $A \in \mathbb{C}^{d \times d}$ there exists some $B \in \mathbb{C}^{d \times d}$ with the property that $AB = I$ then we call $B$ the inverse of $A$ and denote it by $A^{-1}$. Note that if $AB = I$ then also $BA = I$ [55]. $A \in \mathbb{C}^{d \times d}$ is invertible if and only if $A$ has full rank $d$.

**Unitary matrix** A matrix $A$ is called unitary if $AA^\dagger = I$. The following conditions for $A \in \mathbb{C}^{d \times d}$ are equivalent:

1. $A$ is unitary

2. $\forall \phi, \psi \in \mathbb{C}^d : \langle A\phi, A\psi \rangle = \langle \phi, \psi \rangle$ (inner-product preserving)

3. $\forall \phi \in \mathbb{C}^d : ||A\phi|| = ||\phi||$ (norm-preserving),

with $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ as defined in Section 2.1.
Unitary diagonalization  A matrix $A \in \mathbb{C}^{d \times d}$ can be unitarily diagonalized, if there is some matrix $U \in \mathbb{C}^{d \times d}$ and a matrix $\Lambda$ whose off-diagonal entries are all zero with the property that

$$A = U^\dagger \Lambda U.$$  

The values on the diagonal of $\Lambda$ are called the eigenvalues of $A$ and the columns of $U$ the corresponding eigenvectors.

A matrix $A$ is called normal if $AA^\dagger = A^\dagger A$. It turns out that precisely all normal matrices can be diagonalized in this way.

Hermiticity  A matrix $A$ is called hermitian if $A = A^\dagger$. Note that every hermitian matrix is normal and therefore can be unitarily diagonalized as $A = U^\dagger \Lambda U$ as above. From $A = A^\dagger$ it follows that $U^\dagger \Lambda U = U^\dagger \Lambda^\dagger U$, and then further $\Lambda = \Lambda^\dagger$. This means that hermitian matrices only have real eigenvalues.

Tensor products  If $A = \mathbb{C}^{a \times a}$ and $B = \mathbb{C}^{b \times b}$, then

$$A \otimes B := \mathbb{C}^{ab \times ab}$$

is called the tensor product of $A$ and $B$.

For elements $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we define

$$A \otimes B = \begin{pmatrix} A_{11}B & \ldots & A_{1a}B \\ \vdots & \ddots & \vdots \\ A_{a1}B & \ldots & A_{aa}B \end{pmatrix}$$

as the tensor product of $A$ and $B$. The tensor product enjoys many nice properties, for example

$$(A \otimes B)^* = A^* \otimes B^*$$

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

We will write $A^A \otimes B^B$ if it is otherwise not clear from the context on which space $A$ and $B$ act.

(partial) Trace  The trace of a matrix $A \in \mathbb{C}^{d \times d}$ is

$$\text{Tr}(A) = \sum_{i=1}^{d} A_{ii},$$

i.e., it is the sum of all entries on the diagonal of $A$. Note that for unitarily diagonalizable matrices $\text{Tr}(A)$ is the sum of all its eigenvalues. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we define the operator

$$\text{Tr}_A(A \otimes B) = B \cdot \text{Tr}(A).$$
Requiring that $\text{Tr}_A(\cdot)\,$ is linear uniquely defines this operator. Note that $\text{Tr}_A(\cdot)\,$ maps from $A \otimes B$ to $B$. The operation $\text{Tr}_A(\cdot)\,$ is called the \textit{partial trace over $A$} or just “tracing out system $A$”.

Similarly, one can define $\text{Tr}_B(\cdot)$ to be the unique linear operator with the property that

$$\text{Tr}_B(A \otimes B) = A \cdot \text{Tr}(B).$$