Noise in quantum and classical computation & non-locality

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Citation for published version (APA):

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Appendix B

Convex hull of all 1-qubit Clifford operations

We now show that the Clifford polytope $P$ defined in (5.8) is equivalent to the polytope defined by (5.10) and (5.11). In principle, this proof can be carried out by a computer, using for example the software cplex [46]. We give an explicit proof.

Define the polyhedron

\[ Q := \{ S \in \mathbb{R}^{3 \times 3} \mid \langle F, S \rangle \leq 1 \text{ for all } F \in \mathcal{F} \}, \]

where $\mathcal{F}$ is as in (5.11). Our objective is to show the equality $P = Q$.

First, let us prove the easy inclusion $P \subseteq Q$. For this, let $C \in \mathcal{C}$ and $F \in \mathcal{F}$ be of the form $F = C_1 B C_2$ with $C_1, C_2 \in \mathcal{C}$ and $B \in \{ B_1, B_1^T, B_2 \}$. Then, $\langle F, C \rangle = \langle B, C_1^T C C_2^T \rangle$. As $C_1^T C C_2^T \in \mathcal{C}$, it suffices to verify that $\langle B, C \rangle \leq 1$ for any $C \in \mathcal{C}$ and $B = B_1, B_2$. (We have used here the fact that $\mathcal{C}$ is a group which is closed under transposing matrices.) For $C \in \mathcal{C}$, the inequality $\langle B_1, C \rangle \leq 1$ is obvious and the inequality $\langle B_2, C \rangle \leq 1$ can be checked by direct inspection.

The reverse inclusion $Q \subseteq P$ follows from the following result.

B.0.1. Theorem. Any facet of the polytope $P$ is defined by an inequality of the form $\langle F, S \rangle \leq 1$ where $F \in \mathcal{F}$.

The rest of the Appendix is devoted to the proof of this result. We first need to go in more detail into the structure of the Clifford matrices.

B.1 Preliminaries about the Clifford matrices

Each matrix $C \in \mathcal{C}$ corresponds to a “signed permutation” $(\sigma, s)$, where $\sigma \in \text{Sym}(3)$ and $s \in \{ \pm 1 \}^3$. Namely, $C$ has nonzero entries precisely at the $(\sigma(i), i)$-positions with $C_{\sigma(i), i} = s_i$ for $i = 1, 2, 3$; we then also denote $C$ as $C_{\sigma, s}$. The
Odd permutations

<table>
<thead>
<tr>
<th>Even permutations</th>
<th>Odd permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1 = (1, 2, 3)$ $C_{\sigma_1}$:</td>
<td>$\sigma_4 = (1, 3, 2)$ $C_{\sigma_4}$:</td>
</tr>
<tr>
<td>$\begin{pmatrix} * &amp; 0 &amp; 0 \ 0 &amp; * &amp; 0 \ 0 &amp; 0 &amp; * \end{pmatrix}$</td>
<td>$\begin{pmatrix} * &amp; 0 &amp; 0 \ 0 &amp; * &amp; 0 \ 0 &amp; 0 &amp; * \end{pmatrix}$</td>
</tr>
<tr>
<td>$\sigma_2 = (2, 3, 1)$ $C_{\sigma_2}$:</td>
<td>$\sigma_5 = (2, 1, 3)$ $C_{\sigma_5}$:</td>
</tr>
<tr>
<td>$\begin{pmatrix} * &amp; 0 &amp; 0 \ 0 &amp; * &amp; 0 \ 0 &amp; 0 &amp; * \end{pmatrix}$</td>
<td>$\begin{pmatrix} * &amp; 0 &amp; 0 \ 0 &amp; * &amp; 0 \ 0 &amp; 0 &amp; * \end{pmatrix}$</td>
</tr>
<tr>
<td>$\sigma_3 = (3, 1, 2)$ $C_{\sigma_3}$:</td>
<td>$\sigma_6 = (3, 2, 1)$ $C_{\sigma_6}$:</td>
</tr>
<tr>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; * \ 0 &amp; * &amp; 0 \ * &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; * &amp; 0 \ * &amp; 0 &amp; 0 \ * &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

Table 1

The following observation can be directly verified and will be useful for the proof.

6. Observation. Let $\sigma \in \text{Sym}(3)$. Then, $\sum_{C \in C_\sigma} C = 0$. Moreover, for any position $(\sigma(i), i)$ corresponding to a nonzero entry for matrices in $C_\sigma$ and for $d \in \{\pm 1\}$, there exist $C, C' \in C_\sigma$ with $C + C' = 2dE_{\sigma(i), i}$, which implies $dE_{\sigma(i), i} \in P$. Thus, $\pm E_{i,j} \in P$ for any $i, j = 1, 2, 3$.

We now proceed with the proof of Theorem B.0.1. Let $\langle F, S \rangle \leq b$ be an inequality defining a facet of $P$, where $F \in \mathbb{R}^{3 \times 3}$ and $b \in \mathbb{R}$. That is, the inequality $\langle F, S \rangle \leq b$ is valid for $P$, which means that $\langle F, S \rangle \leq b$ holds for any $S \in P$, and the set

$$R_F := \{ C \in C \mid \langle F, C \rangle = b \}$$

contains nine affinely independent matrices. Without loss of generality, we may assume that $b = 1$. Indeed, $b \geq 0$ since $0 \in P$. Moreover, $b > 0$ for, otherwise, we would have $F_{ij} = 0$ for all $i, j = 1, 2, 3$, implying $F = 0$, in view of Observation 6. Thus, by rescaling, we can now assume that the facet is of the form $\langle F, S \rangle \leq 1$. We sometimes speak of the “facet $F$” for short. Our objective is to show that $F = C_1BC_2$ for some $C_1, C_2 \in C$, $B \in \{B_1, B_1^T, B_2\}$. 

Condition $\det(C) = 1$ translates into $s_1s_2s_3 = \text{sign}(\sigma)$; that is, $s_1s_2s_3 = 1$ if $\sigma$ is an even permutation (i.e., one of $\sigma_1 := (1, 2, 3), \sigma_2 := (2, 3, 1), \sigma_3 := (3, 1, 2)$) and $s_1s_2s_3 = -1$ if $\sigma$ is an odd permutation (i.e., one of $\sigma_4 := (1, 3, 2), \sigma_5 := (2, 1, 3), \sigma_6 := (3, 2, 1)$). Thus the set $C$ of Clifford matrices is naturally partitioned into six subclasses

$$C = \bigcup_{\sigma \in \text{Sym}(3)} C_\sigma,$$

where $C_\sigma := \{ C_{\sigma,s} \mid s \in \{\pm 1\}^3, s_1s_2s_3 = \text{sign}(\sigma) \}$

with $|C_\sigma| = 4$. For convenience we display in the table below the six subclasses $C_\sigma$; the nonzero entries are indicated by $\ast$. 

Appendix B. Convex hull of all 1-qubit Clifford operations
B.2. The case $|\mathcal{R}_F \cap \mathcal{C}_\sigma| = 3$ for some $\sigma \in \text{Sym}(3)$

Call $F, F' \in \mathbb{R}^{3 \times 3}$ equivalent if $F' = C_1 FC_2$ for some $C_1, C_2 \in \mathcal{C}$. Then, as $\mathcal{C}$ is a group, $\langle F', S \rangle \leq 1$ defines a facet of $P$ if and only if $\langle F, C \rangle \leq 1$ does. Moreover, $\mathcal{R}_F = \{ C_1 \mathcal{R}_F C_2 \mid C \in \mathcal{R}_F \}$. This property will be used repeatedly throughout the proof as it permits to exploit symmetry and to reduce the number of cases we need to check.

The proof is based on a detailed inspection of the structure of the set $\mathcal{R}_F$. We begin with collecting several properties of the matrix $F$ and the set $\mathcal{R}_F$.

7. Observation. $|\mathcal{R}_F \cap \mathcal{C}_\sigma| \leq 3$ for any $\sigma \in \text{Sym}(3)$.

Proof: If $\mathcal{C}_\sigma \subseteq \mathcal{R}_F$, then $\langle F, C \rangle = 1$ for any $C \in \mathcal{C}_\sigma$, which implies $4 = \sum_{C \in \mathcal{C}_\sigma} \langle F, C \rangle$, contradicting the fact that $\sum_{C \in \mathcal{C}_\sigma} C = 0$ by Observation 6. ■

8. Observation. If $F_{ij} = d \in \{-1, 1\}$, then all $C \in \mathcal{C}$ with $C_{ij} = d$ belong to $\mathcal{R}_F$.

Proof: Let $C \in \mathcal{C}$ with $C_{ij} = d$. There exists $C' \in \mathcal{C}$ with $C + C' = 2dE_{ij}$. Summing $\langle F, C \rangle \leq 1$ and $\langle F, C' \rangle \leq 1$ yields $\langle F, C + C' \rangle \leq 2$. As $\langle F, C + C' \rangle = 2dF_{ij} = 2$, we have the equalities $\langle F, C \rangle = \langle F, C' \rangle = 1$, which implies $C \in \mathcal{R}_F$. ■

9. Observation. Let $C \neq C' \in \mathcal{R}_F \cap \mathcal{C}_\sigma$ (for some $\sigma \in \text{Sym}(3)$) and assume that $C_{\sigma(i),i} = C'_{\sigma(i),i} = d \in \{\pm 1\}$ for some $i \in \{1, 2, 3\}$. Then, $F_{\sigma(i),i} = d$ and $F_{\sigma(j),j} + \text{sign}(\sigma)dE_{\sigma(k),k} = 0$ with $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$.

Proof: Equality $F_{\sigma(i),i} = d$ follows from the fact that $C + C' = 2dE_{\sigma(i),i}$. Then, $1 = \langle F, C \rangle$ implies $0 = F_{\sigma(j),j}C_{\sigma(j),j} + F_{\sigma(k),k}C_{\sigma(k),k}$. Using $C_{\sigma(i),i}C_{\sigma(j),j}C_{\sigma(k),k} = \text{sign}(\sigma)$, we find $F_{\sigma(j),j} + \text{sign}(\sigma)dE_{\sigma(k),k} = 0$. ■

Our last observation is an easy corollary of the former two observations.

10. Observation. If $F_{\sigma(i),i} = d \in \{\pm 1\}$ (for some $\sigma \in \text{Sym}(3)$), then $F_{\sigma(j),j} + \text{sign}(\sigma)dE_{\sigma(k),k} = 0$, where $\{i, j, k\} = \{1, 2, 3\}$.

One can verify that, for $F = B_1, B_1^T$, $|\mathcal{R}_F \cap \mathcal{C}_\sigma| = 2$ for all $\sigma \in \text{Sym}(3)$ while, for $F = B_2, |\mathcal{R}_F \cap \mathcal{C}_\sigma| = 3$ for $\sigma = \sigma_1, \sigma_2$. Based on this observation we now distinguish two cases: Either, $|\mathcal{R}_F \cap \mathcal{C}_\sigma| \leq 2$ for all $\sigma \in \text{Sym}(3)$ (in which case we show that $F$ is equivalent to $B_1$ or $B_1^T$), or $|\mathcal{R}_F \cap \mathcal{C}_\sigma| = 3$ for some $\sigma \in \text{Sym}(3)$ (in which case we show that $F$ is equivalent to $B_2$).

B.2 The case $|\mathcal{R}_F \cap \mathcal{C}_\sigma| = 3$ for some $\sigma \in \text{Sym}(3)$

Using symmetry, we may assume that $|\mathcal{R}_F \cap \mathcal{C}_\sigma| = 3$ for the (odd) permutation $\sigma = \sigma_4$. We prove this in detail to show how this kind of symmetry argument
works. To shorten notation we will use +,− for 1 respectively −1 whenever we explicitly write out matrices.

Define the matrices

\[
C_1 = \begin{pmatrix}
-0 & 0 & 0 \\
0 & 0 & - \\
0 & - & 0
\end{pmatrix}, \quad C_2 = \begin{pmatrix}
0 & + & 0 \\
0 & 0 & + \\
+ & 0 & 0
\end{pmatrix}, \quad C_3 = \begin{pmatrix}
0 & 0 & + \\
0 & 0 & + \\
0 & + & 0
\end{pmatrix}
\] (B.1)

lying, resp., in \(C_\sigma_4, C_\sigma_3, C_\sigma_2\).

Our assumption is that \(|R_F \cap C_{\sigma_i}| = 3\) for some \(i = 1, \ldots, 6\); we show that one can replace \(F\) by another equivalent facet \(F'\) in such a way that \(i = 4\) holds.

For this, suppose first \(i = 2, 3\). As the mapping \(X \mapsto XC_i\) maps \(C_{\sigma_i}\) to \(C_{\sigma_1}\), we can replace the facet \(F\) by \(F' := FC_i\) and then we find \(|R_{F'} \cap C_{\sigma_1}| = 3\) since \(R_{F'} = R_FC_i\). Thus we may assume \(|R_F \cap C_{\sigma_1}| = 3\). As the mapping \(X \mapsto XC_1\) maps \(C_{\sigma_1}\) to \(C_{\sigma_4}\), replacing the facet \(F\) by \(F' := FC_1\), we find \(|R_{F'} \cap C_{\sigma_4}| = 3\). Thus we can now assume \(|R_F \cap C_{\sigma_i}| = 3\) for some \(i = 4, 5, 6\). If \(i = 5\), as the mapping \(X \mapsto XC_3\) maps \(C_{\sigma_5}\) to \(C_{\sigma_4}\), replace \(F\) by \(F' := FC_3\); if \(i = 6\), the mapping \(X \mapsto XC_2\) maps \(C_{\sigma_6}\) to \(C_{\sigma_4}\) and one can replace \(F\) by \(F' := FC_2\); in both cases we get back to the case when \(|R_{F'} \cap C_{\sigma_4}| = 3\).

Thus we now assume \(|R_F \cap C_{\sigma_4}| = 3\). Moreover, we may assume that the following matrices from \(C_{\sigma_4}\)

\[
\begin{pmatrix}
+ & 0 & 0 \\
0 & 0 & - \\
0 & + & 0
\end{pmatrix}, \quad \begin{pmatrix}
- & 0 & 0 \\
0 & 0 & + \\
+ & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
- & 0 & 0 \\
0 & 0 & - \\
0 & + & 0
\end{pmatrix}
\] (B.2)

belong to \(R_F\). (To see this, replace if necessary \(F\) by \(FC\), where \(C \in C_{\sigma_1}\).) Using Observation 9, we obtain \(F_{11} = -1, F_{23} = -1, F_{32} = 1\). From this we get by Observation 8 that also the matrices

\[
\begin{pmatrix}
- & 0 & 0 \\
0 & - & 0 \\
0 & 0 & +
\end{pmatrix}, \quad \begin{pmatrix}
0 & + & 0 \\
0 & 0 & - \\
+ & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & + \\
0 & - & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & - \\
0 & 0 & - \\
0 & 0 & 0
\end{pmatrix}
\] (B.3) (B.4) (B.5)

belong to \(R_F\). By Observation 9, we also obtain \(F_{22} = F_{33}, F_{12} = F_{31}\) and \(F_{13} = -F_{21}\).

1. Claim. There exists also an even permutation \(\sigma \in \text{Sym}(3)\) for which \(|R_F \cap C_\sigma| = 3\).
**Proof:** Assume for contradiction that, for $i = 1, 2, 3$, the set $\mathcal{R}_F \cap C_{\sigma_i}$ contains only the respective two matrices from (B.3)-(B.5). Choose a subset $B \subseteq \mathcal{R}_F$ consisting of nine affinely independent matrices and such that $\mathcal{R}_F \cap C_{\sigma_4} \subseteq B$. We have $|B \cap C_{\sigma_1}| \leq 1$, since the two matrices in (B.3) are affinely dependent with the last two matrices in (B.2). Similarly, $|B \cap C_{\sigma_2}| \leq 1$, $|B \cap C_{\sigma_3}| \leq 1$. As $|B| = 9$, we deduce that $|B \cap C_{\sigma_5}| \geq 2$ or $|B \cap C_{\sigma_6}| \geq 2$. Say, $C \neq C' \in \mathcal{R}_F \cap C_{\sigma_5}$. Then $C$ and $C'$ have the same nonzero entry $d \in \{-1, 1\}$ in some position $(k, l)$. By Observation 9 this yields $F_{kl} = d$. Now, there is also an even permutation $\sigma$ for which $k = \sigma(l)$. By Observation 8 we then deduce that at least two matrices from $C_{\sigma}$ must be in $\mathcal{R}_F$, which contradicts our assumption. The other case $|B \cap C_{\sigma_6}| \geq 2$ goes analogously.}

It is sufficient to consider the case $|\mathcal{R}_F \cap C_{\sigma_1}| = 3$. Indeed, if $|\mathcal{R}_F \cap C_{\sigma_2}| = 3$, then one may replace $F$ by $C_3 FC_4$ with $C_3$ as in (B.1) and

$$C_4 := \begin{pmatrix} 0 & 0 & + \\ - & 0 & 0 \\ 0 & - & 0 \end{pmatrix},$$

since the mapping $X \mapsto C_3 X C_4$ maps $C_{\sigma_2}$ to $C_{\sigma_1}$ and preserves the set of three matrices from (B.2), as well as the set of 6 matrices from (B.3)-(B.5) (namely, (B.3) → (B.5) → (B.4) → (B.3)). One can handle the case when $|\mathcal{R}_F \cap C_{\sigma_3}| = 3$ in the same way.

The set $\mathcal{R}_F$ already contains the matrices

$$D_1 := \begin{pmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & + \end{pmatrix}, \quad D_2 := \begin{pmatrix} - & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{pmatrix}$$

from $C_{\sigma_1}$ (displayed in (B.3)). The remaining two matrices of $C_{\sigma_1}$ are

$$D_3 := \begin{pmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{pmatrix}, \quad D_4 := \begin{pmatrix} + & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{pmatrix}.$$

If $D_4 \in \mathcal{R}_F$, one may replace the facet $F$ by $F' := D_2 F D_1$ to obtain that $D_1, D_2, D_3 \in \mathcal{R}_F$, since the mapping $X \mapsto D_2 X D_1$ maps the set $\{D_1, D_2, D_4\}$ to $\{D_1, D_2, D_3\}$ and leaves the set of 3 matrices from (B.2) invariant as well as the set of 6 matrices from (B.3)-(B.5). Thus we may assume that $D_3 \in \mathcal{R}_F$.

By Observation 9, we find that $F_{33} = F_{22} = 1$. As $F_{32} = 1$, Observation 10 implies that $F_{31} = F_{13}$. Similarly, $F_{33} = 1$ implies that $F_{12} = F_{21}$. Putting all equations together we obtain $F_{12} = F_{21} = -F_{13} = -F_{31} = -F_{12}$, implying they are all zero. Thus

$$F = \begin{pmatrix} - & 0 & 0 \\ 0 & + & - \\ 0 & + & + \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} B_2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (B.7)
B.3 The case $|\mathcal{R}_F \cap \mathcal{C}_\sigma| \leq 2$ for all $\sigma \in \text{Sym}(3)$

Let again $B \subseteq \mathcal{R}_F$ consist of nine affinely independent matrices. As $|\mathcal{R}_F| \geq 9$, $|\mathcal{R}_F \cap \mathcal{C}_\sigma| = 2$ for at least three permutations $\sigma$. W.l.o.g. we can assume that two of those permutations are odd permutations and that they are equal, say, to $\sigma_4$ and $\sigma_6$ (replacing if necessary $F$ by an equivalent facet). Further we may assume $\mathcal{R}_F$ contains the following two matrices of $\mathcal{C}_{\sigma_4}$:

$$
\begin{pmatrix}
-0 & 0 & 0 \\
0 & 0 & - \\
0 & - & 0 \\
\end{pmatrix},
\begin{pmatrix}
-0 & 0 & 0 \\
0 & 0 & + \\
0 & + & 0 \\
\end{pmatrix} \in \mathcal{R}_F. \tag{B.8}
$$

This can be seen using the following two mappings $X \mapsto X C_2$ (with $C_2$ defined as in (B.1)) and $X \mapsto C X$ (with $C \in \mathcal{C}_{\sigma_1}$) which permit to map any subset of size 2 of $\mathcal{C}_{\sigma_4}$ to any other such subset and which preserve $\mathcal{C}_{\sigma_6}$ as well. We choose the basis $\mathcal{B}$ containing the two matrices of (B.8). From Observation 9 we find $F_{11} = -1$ and $F_{23} = -F_{32} \neq \pm 1$; the latter inequality follows from the fact that $|\mathcal{R}_F \cap \mathcal{C}_{\sigma_4}| = 2$ combined with Observation 8. As $F_{11} = -1$, by Observation 8,

$$
\begin{pmatrix}
-0 & 0 & 0 \\
0 & 0 & - \\
0 & + & 0 \\
\end{pmatrix},
\begin{pmatrix}
-0 & 0 & 0 \\
0 & 0 & + \\
0 & + & 0 \\
\end{pmatrix} \in \mathcal{R}_F \cap \mathcal{C}_{\sigma_1} \tag{B.9}
$$

and Observation 9 implies $F_{22} = F_{33} \neq \pm 1$. At most one of the two matrices in (B.9) belongs to $\mathcal{B}$ since they are affinely dependent with the matrices in (B.8). Say, $|\mathcal{B} \cap \mathcal{C}_{\sigma_1}| = 1$.

Let us now examine which two matrices of $\mathcal{C}_{\sigma_6}$ belong to $\mathcal{R}_F$. Set

$$
C_5 := \begin{pmatrix} + & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{pmatrix} \in \mathcal{C}_{\sigma_1}, \quad X_1 := \begin{pmatrix} 0 & 0 & - \\ 0 & - & 0 \\ - & 0 & 0 \end{pmatrix} \in \mathcal{C}_{\sigma_6}.
$$

The two mappings $X \mapsto X C_5$ and $X \mapsto C_5 X$ preserve the set of matrices in (B.8) and permit to map any other matrix of $\mathcal{C}_{\sigma_6}$ to the matrix $X_1$. Therefore we can assume w.l.o.g. that $X_1 \in \mathcal{R}_F \cap \mathcal{C}_{\sigma_6}$. The second matrix of $\mathcal{R}_F \cap \mathcal{C}_{\sigma_6}$ does not have entry $-1$ at the position $(2,2)$ since, otherwise, $F_{22} = -1$ contradicting an earlier claim. Hence the second matrix in $\mathcal{R}_F \cap \mathcal{C}_{\sigma_6}$ is

$$
X_2 := \begin{pmatrix} 0 & 0 & + \\ 0 & + & 0 \\ - & 0 & 0 \end{pmatrix}, \quad \text{or} \quad X_3 := \begin{pmatrix} 0 & 0 & - \\ 0 & + & 0 \\ + & 0 & 0 \end{pmatrix}.
$$

1. Consider first the case when $X_2 \in \mathcal{R}_F \cap \mathcal{C}_{\sigma_6}$. Then, $F_{31} = -1$ and $F_{22} = -F_{13} \neq \pm 1$. As $F_{31} = -1$, we have

$$
\begin{pmatrix} 0 & + & 0 \\ 0 & 0 & - \\ - & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & - & 0 \\ 0 & 0 & + \\ - & 0 & 0 \end{pmatrix} \in \mathcal{R}_F \cap \mathcal{C}_{\sigma_3} \tag{B.10}
$$
B.3. The case $|\mathcal{R}_F \cap C_\sigma| \leq 2$ for all $\sigma \in \text{Sym}(3)$

and $F_{12} = F_{23} \neq \pm 1$. As $\mathcal{B}$ contains at most three of the matrices $X_1, X_2$ and in (B.10), we must have $|\mathcal{B} \cap C_{\sigma_2}| = 2$ or $|\mathcal{B} \cap C_{\sigma_5}| = 2$. We obtained earlier that $F_{33} = F_{22} = -F_{13} \neq \pm 1$ and $F_{12} = F_{23} = -F_{32} \neq \pm 1$. In other words, the second and third columns of $F$ contain no entry $\pm 1$. On the other hand, the two matrices from $\mathcal{B} \cap C_{\sigma_i} (i = 2, 5)$ have one common nonzero entry which therefore is located in the first column, at the position $(2, 1)$. This implies $F_{21} = \pm 1$.

(a) If $F_{21} = 1$, then Observation 10 implies $F_{12} = F_{33}$ and $F_{13} = -F_{32}$.

Combining with the former relations on entries of $F$, we find

$$F = \begin{pmatrix} - & 0 & 0 \\ + & 0 & 0 \\ - & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (B.11)

(b) If $F_{21} = -1$, then in the same way we find

$$F = \begin{pmatrix} - & 0 & 0 \\ - & 0 & 0 \\ - & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (B.12)

In both cases we find that $F$ is equivalent to $B_1$.

2. Consider now the case when $X_3 \in \mathcal{R}_F \cap C_{\sigma_6}$. Then, $F_{13} = -1$, $F_{22} = -F_{31} \neq \pm 1$,

$$\begin{pmatrix} 0 & 0 & - \\ - & 0 & 0 \\ 0 & + & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & - \\ + & 0 & 0 \\ 0 & - & 0 \end{pmatrix} \in \mathcal{R}_F \cap C_{\sigma_2}$$  \hspace{1cm} (B.13)

and $F_{21} = F_{32} \neq \pm 1$. In the same way as in the first case one finds that $F$ is equivalent to $B_1^T$.

This proves that the facet description of polytope $P$ from Lemma 5.4.1 is indeed given by (5.10) and (5.11).