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Abstract

In this paper we present a detailed analysis of queueing models with vacations and impatient customers, where the source of impatience is the absence of the server. Instead of the standard assumption that customers perform independent abandonments, we consider situations where customers abandon the system simultaneously. This is, for example, the case in remote systems where customers may decide to abandon the system, when a transport facility becomes available.

Keywords: queueing; vacations; reneging; impatient customers; synchronization; q-hypergeometric series; mean value analysis; stationary distribution.

1 Introduction

Queueing systems with reneging (i.e., impatient customers) have been studied extensively. The main assumption in the literature is that customers perform *independent abandonments*, that is, each one of them sets an impatience clock and abandons the system as soon as the clock expires. For Markovian models, this type of abandonments introduces state-inhomogeneous transition rate matrices, which implies certain difficulties in computing the performance measures. For non-Markovian models, the basic idea is to use the methodology from the study of the $M/G/\infty$ queue. In both cases, however, it seems fair to say that most of the models are analytically untractable.

The study of queueing systems with impatient customers goes back at least to the pioneering papers of Palm (1953, 1957) who studied the $M/M/c$ queue, where the customers have independent exponentially distributed impatience times. Subsequently, Daley (1965), Takacs (1974) and Baccelli et al. (1984) considered various queueing models with general service and/or inter-arrival times and more involved abandonment schemes.

More recently, Boxma and de Waal (1994) studied the $M/M/c$ queue with generally distributed impatience times, while Altman and Borovkov (1997) investigated the stability issue in a retrial queue with impatient customers. In all the aforementioned works, customers become impatient due to the long waiting time already experienced, although the server provides continuously service. The study of reneging within the class of queueing systems with vacations is a new endeavor. Although, there exists a significant number of papers and books on vacation queueing systems (see, e.g., Takagi (1991) and Tian and Zhang (2006)), the reneging feature has not yet received much attention. Only recently, Altman and Yechiali (2006) and Yechiali (2007) considered systems with vacations, where the source of the impatience is the absence of the server. The authors assume that the customers perform independent abandonments, whenever the server is unavailable.

In the present paper, we study two models with vacations, where the customers are impatient but they perform *synchronized abandonments*. These models are motivated by remote systems where customers have to wait for a certain transport facility to abandon the system. Then, whenever the facility visits the system, the present customers decide whether to leave the system or not. Therefore, we have synchronized departures for some of the customers.

The first model is the single-server queue with multiple vacations, where customers decide whether to abandon the system or not when the vacation periods finish. In the second model, we suppose that the abandonments epochs occur according to a Poisson process during vacation periods. At the abandonment epochs, every present customer remains in the system with probability q or abandons the system with probability $p = 1 - q$, independently of the others. The analysis of this model extends the analysis of Altman and Yechiali (2006), in the framework of synchronized abandonments. The new feature of these models with synchronization is the existence of binomial type jumps at the abandonment epochs. Similar models with binomial type transitions have been recently studied by Economou (2004), Economou and Kapodistria (2006), Artalejo et al. (2007) and Economou and Fakinos (2008).

The paper is organized as follows. In Section 2, we describe the dynamics of the models. In Sections 3 and 4 we first study the Markovian case and present, separately, the stationary analysis of the two models. We also obtain more explicit results under various limiting regimes concerning the parameters of the models. In Sections 5 and 6 we proceed with the non-Markovian case, assuming general service and vacation time distributions. The paper concludes with Section 7, where we discuss possible generalizations and extensions.

2 Model description

We consider a queueing system where customers arrive one by one according to a Poisson process at rate λ. Service is provided by a single server who can be in one of two modes: on (active) or off (non-active - on vacation). Customers are served singly when the server is on, while no service is provided when the server is off. The service times are generally distributed according to a distribution $B(t)$, having Laplace-Stieltjes transform (LST) $\tilde{B}(s) = E(e^{-sB})$ and finite first and second moments $E(B)$ and $E(B^2)$, where the random variable B represents the service time. The residual (or equilibrium) service time is denoted by B_e, the distribution $B_e(t)$ of which is given by

$$B_e(t) = \frac{\int_0^t (1 - B(u))du}{E(B)} \ ,$$

with LST

$$\tilde{B}_e(s) = \frac{1 - \tilde{B}(s)}{E(B)s} \ .$$

There is infinite waiting room. Whenever the system becomes empty, the server begins a vacation. We assume multiple vacations, i.e., if the system is still empty at the end of a vacation, the server takes another one. If, on the contrary, there is at least one waiting customer at the end of a vacation, the server starts again to provide service. The vacation times are generally distributed according to a distribution $V(x)$, having LST $\tilde{V}(s) = E(e^{-sV})$ and finite first and second moments $E(V)$ and $E(V^2)$, where the random variable V represents the vacation time. The residual vacation time is denoted by V_e with distribution

$$V_e(t) = \frac{\int_0^t (1 - V(u))du}{E(V)} \ ,$$

and LST

$$\tilde{V}_e(s) = \frac{1 - \tilde{V}(s)}{E(V)s} \ .$$

Regarding the abandonments we consider two models:

- Unique Abandonment Epoch (UAE) : Every time the server finishes a vacation, every present customer decides whether to stay in the system with probability q or to abandon it with probability $p = 1 - q$, independently of the others.
- Multiple Abandonment Epochs (MAE) : During server vacations, abandonment epochs occur according to a Poisson process with rate ζ. At these epochs, every present customer remains in the system with probability q or abandons the system with probability $p = 1-q$, independently of the others.

Hence, in either model, the number of customers is reduced according to a binomial distribution at every abandonment epoch. However, the analysis of the UAE model turns out to be much easier than the one of the MAE model. For this reason, in what follows, we describe briefly the results for the UAE model and we provide more details for the analysis of MAE model.

3 Markovian UAE model

We consider the UAE model described in Section 2, where both the service and vacation time distributions are exponential with rate μ and γ, respectively. We denote by $\rho = \frac{\lambda}{\mu}$ the traffic intensity, which is assumed to be less than 1. Then, the system can be described by a continuous-time Markov chain $\{(L(t), I(t)),\ t \geq 0\}$, with state space $\{(0,0)\} \cup \{(n,i) : i = 0,1,\ n = 1,2,\ldots\}$, where $L(t)$ is the number of customers in the system at time t and $I(t)$ expresses the mode of the server at time t (more explicitly, it is equal to 1 if the server is on at that time t and 0 otherwise). Figure 1 shows the state-transition diagram. In the next section we first determine, by application of Little's law and the PASTA property, the mean number of customers in the system and the mean sojourn time.

3.1 Mean value analysis

We suppose that the system is in equilibrium and we define the random variable L to be the number of customers in the system and S to be the sojourn time of a customer. Let also L_i be the *conditional* number of customers in the system, given that the server is in state i, $i = 0, 1$. Further we denote by p_i the probability (or fraction of time) that the server is in state i, $i = 0, 1$.

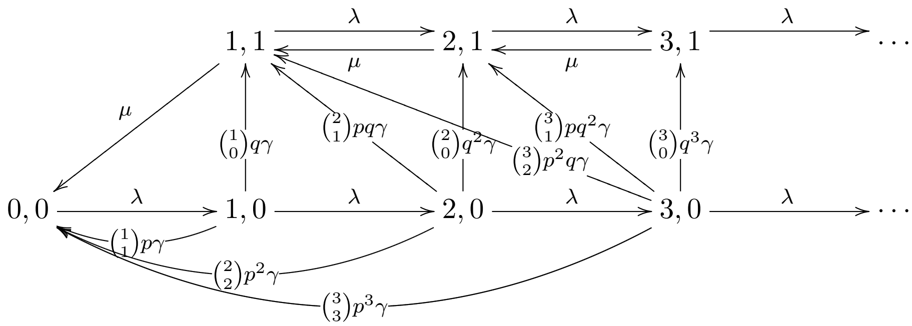

Figure 1: Transition-rate diagram for the UAE model.

Let us consider a tagged arriving customer. Then, by PASTA, the probability that this customer finds the server in state i is p_i. If he finds the server providing service, then his mean sojourn time is $E(L_1)\frac{1}{\mu} + \frac{1}{\mu}$. If he finds the server on vacation, then he first has to wait for the vacation time to expire; the mean residual vacation time is $\frac{1}{\gamma}$. Then, with probability p, he will abandon and, with probability q, he will remain for service, in which case the mean number of customers that he will find in front of him is $qE(L_0)$. Indeed, by PASTA, he sees at his arrival epoch on average $E(L_0)$ customers in the system and each of them will remain for service with probability q. So, in this case, his mean sojourn time is $\frac{1}{\gamma} + p \cdot 0 + q \cdot (qE(L_0)\frac{1}{\mu} + \frac{1}{\mu})$. Hence,

$$E(S) = p_1 \left(E(L_1)\frac{1}{\mu} + \frac{1}{\mu} \right) + p_0 \left(\frac{1}{\gamma} + q^2 E(L_0)\frac{1}{\mu} + q\frac{1}{\mu} \right). \tag{3.1}$$

Further, Little's law states that

$$E(L) = \lambda E(S), \tag{3.2}$$

where the unconditional $E(L)$ is related to the conditional ones as

$$E(L) = p_0 E(L_0) + p_1 E(L_1). \tag{3.3}$$

Conservation of work gives the relation

$$p_1 = (\lambda p_0 q + \lambda p_1)\frac{1}{\mu}, \tag{3.4}$$

and clearly,

$$p_0 + p_1 = 1. \tag{3.5}$$

Finally, by gluing the periods during which the server is on vacation, we observe that the vacation completion epochs constitute a Poisson process. Hence, by PASTA, we have that $E(L_0)$ coincides with the mean number of customers in the system just before a vacation time finishes, which is equal to the mean number of Poisson (λ) arrivals in a vacation time. Thus,

$$E(L_0) = \lambda\frac{1}{\gamma}. \tag{3.6}$$

Now we have sufficiently many equations for the unknown mean values. Solution of (3.1)-(3.6) yields the following result.

Theorem 3.1 *The mean sojourn time is given by*

$$E(S) = \frac{1}{1-\rho p}\left(\frac{1}{\gamma} + (q^2-1)\frac{\rho}{\gamma}\right) + \frac{q}{1-\rho p} \times \frac{1}{\mu(1-\rho)}, \tag{3.7}$$

and the fraction of time the server is inactive and active, respectively,

$$p_0 = \frac{1-\rho}{1-\rho p}, \quad p_1 = \frac{\rho q}{1-\rho p}. \tag{3.8}$$

In the next section we focus on the determination of the equilibrium distribution of the Markov chain $\{(L(t), I(t)),\, t \geq 0\}$.

3.2 Equilibrium distribution

Let $\{\pi(n,i):\ i = 0,1 \text{ and } n \geq i\}$, denote the equilibrium distribution. We define the probability generating functions (PGFs) $\Pi_0(z)$ and $\Pi_1(z)$ of the equilibrium distribution by

$$\Pi_0(z) = \sum_{n=0}^{\infty} \pi(n,0) z^n \quad \text{and} \quad \Pi_1(z) = \sum_{n=1}^{\infty} \pi(n,1) z^n, \quad |z| \leq 1.$$

The set of balance equations is given as follows,

$$\lambda\pi(0,0) = \mu\pi(1,1) + \gamma\sum_{j=1}^{\infty}\binom{j}{j} p^j \pi(j,0) \tag{3.9}$$

$$(\lambda+\gamma)\pi(n,0) = \lambda\pi(n-1,0),\ n \geq 1 \tag{3.10}$$

$$(\lambda+\mu)\pi(1,1) = \mu\pi(2,1) + \gamma\sum_{j=1}^{\infty}\binom{j}{j-1} q p^{j-1}\pi(j,0) \tag{3.11}$$

$$\begin{aligned}(\lambda+\mu)\pi(n,1) &= \mu\pi(n+1,1) + \lambda\pi(n-1,1) \\ &\quad + \gamma\sum_{j=n}^{\infty}\binom{j}{j-n} q^n p^{j-n}\pi(j,0),\ n \geq 2.\end{aligned} \tag{3.12}$$

Provided $\rho < 1$, this set of equations, together with the normalization equation

$$\pi(0,0) + \sum_{n=1}^{\infty}(\pi(n,0) + \pi(n,1)) = 1,$$

has a unique solution. This solution is presented in the next theorem.

Theorem 3.2 *Provided $\rho < 1$, the equilibrium state distribution $\pi(n,i)$ is given by*

$$\pi(0,0) = \frac{\gamma}{\gamma+\lambda} \times \frac{1-\rho}{1-\rho p} \tag{3.13}$$

$$\pi(n,0) = \pi(0,0)\left(\frac{\lambda}{\lambda+\gamma}\right)^n, \quad n \geq 0 \tag{3.14}$$

$$\pi(n,1) = \begin{cases} \pi(0,0)\frac{(\lambda+\gamma)q}{\gamma+(\lambda-\mu)q}\left(\rho^n - \left(\frac{\lambda q}{\gamma+\lambda q}\right)^n\right), & \text{if } \gamma \neq (\mu-\lambda)q \\ \pi(0,0)(\rho p + q)n\rho^n, & \text{if } \gamma = (\mu-\lambda)q \end{cases}, \quad n \geq 1. \tag{3.15}$$

Proof. By iterating (3.10) we obtain (3.14) yielding

$$\Pi_0(z) = \frac{\lambda+\gamma}{\lambda+\gamma-\lambda z}\pi(0,0). \tag{3.16}$$

By multiplying (3.11) by z and (3.12) by z^n and adding for all $n = 1, 2, \ldots$ we obtain

$$(\lambda+\mu)\Pi_1(z) = \lambda z\Pi_1(z) + \frac{\mu}{z}(\Pi_1(z) - z\pi(1,1)) + \gamma\Pi_0(p+qz) - \gamma\Pi_0(p). \tag{3.17}$$

Solving (3.17) for $\Pi_1(z)$ and plugging (3.16) yields

$$\Pi_1(z) = \frac{q\rho z(\gamma+\lambda)}{(\gamma+\lambda q(1-z))(1-\rho z)}\pi(0,0). \tag{3.18}$$

Expanding (3.18) in partial fractions and using the geometric series leads to (3.15). Finally, (3.13) follows from the normalization equation. ∎

Remark 3.1 From equations (3.16) and (3.18) we obtain

$$\begin{aligned} E(L) &= \sum_{n=1}^{\infty} n(\pi(n,0)+\pi(n,1)) \\ &= \Pi_0'(1) + \Pi_1'(1) \\ &= \pi(0,0)\frac{\lambda+\gamma}{\gamma}\times\frac{\lambda}{\gamma} + \pi(0,0)\frac{(\lambda+\gamma)q\rho}{\gamma(1-\rho)}\times\frac{\gamma(1-\rho)+\rho\gamma+\lambda q(1-\rho)}{\gamma(1-\rho)} \\ &= p_0\frac{\lambda}{\gamma} + p_1\frac{\gamma+\lambda q(1-\rho)}{\gamma(1-\rho)}, \end{aligned}$$

which, after application of Little's law (3.2), yields expression (3.7) for the mean sojourn time.

Remark 3.2 Let $X(\alpha)$ denote an exponential random variable with rate α and $Y(j,\alpha)$ denote an Erlang random variable consisting of j phases with rate α. Let us consider a tagged arriving customer. Then, by the PASTA property, he finds the system in state (n,i) with probability $\pi(n,i)$. If he finds the system in state $(n,1)$, then his sojourn time is $Y(n+1,\mu)$. If he finds the system in state $(n,0)$, then with probability p his sojourn time will be $X(\gamma)$, since the tagged customer abandons the system, and with probability $\binom{n}{j}p^{n-j}q^{j+1}$ his sojourn time will be $X(\gamma)+Y(j+1,\mu)$, for $j = 0,1,\ldots,n$, where the r.v. $X(\gamma)$ and $Y(j+1,\mu)$ are independent. Hence, by using the geometric form of the equilibrium distribution (3.13)-(3.15), we have that the LST of the sojourn time $\tilde{S}(s) = E(e^{-sS})$ can be represented as

$$\tilde{S}(s) = p_0 p\frac{\gamma}{\gamma+s} + p_0 q\frac{\gamma}{\gamma+s}\times\frac{\gamma\mu}{\gamma\mu+(\gamma+q\lambda)s} + p_1\frac{\mu-\lambda}{\mu-\lambda+s}\times\frac{\gamma\mu}{\gamma\mu+(\gamma+q\lambda)s}.$$

This shows that the sojourn time S is a mixture of $X(\gamma)$, $X(\gamma)+X(\frac{\gamma\mu}{\gamma+q\lambda})$, $X(\mu-\lambda)+X(\frac{\gamma\mu}{\gamma+q\lambda})$ with mixing probabilities $p_0 p$, $p_0 q$ and p_1, respectively.

4 Markovian MAE model

We now consider the MAE model described in Section 2, where both the service and vacation time distributions are exponential with rate μ and γ, respectively. This system can be described by the same continuous-time Markov chain $\{(L(t), I(t)),\ t \geq 0\}$ as for the UAE model, but, of course, with different transition rates. The state-transition diagram is given in figure 2.

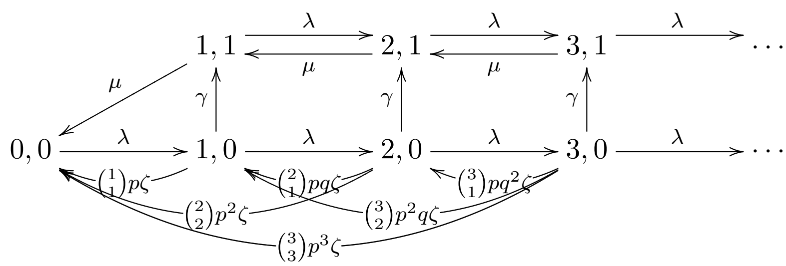

Figure 2: Transition-rate diagram for the MAE model.

4.1 Mean value analysis

Let us, again, consider a tagged arriving customer. Then, by PASTA, the probability that this customer finds the server in state i is p_i. If he finds the server providing service, then his mean sojourn time is $E(L_1)\frac{1}{\mu} + \frac{1}{\mu}$. If he finds the server on vacation, then he first has to wait for the vacation time to expire before servicing starts and, while waiting, he may decide to abandon at one of the abandonment opportunities. Let $E(V^*)$ be his mean time in the system till the end of the vacation; note that $E(V^*)$ will be less than the mean residual vacation time $\frac{1}{\gamma}$. If the tagged customer decides to stay till the end of the vacation, then his sojourn time after return of the server depends on the number of customers (still) in front of him. Define π as the probability that the tagged customer stays in the system till the end of the vacation period and define π^* as the probability that the tagged customer and a customer, who was already present at his arrival, both stay in the system. Then $\pi^* E(L_0)\frac{1}{\mu} + \pi\frac{1}{\mu}$ is his mean sojourn time, from the moment the server returns from vacation. Hence,

$$E(S) = p_1\left(E(L_1)\frac{1}{\mu} + \frac{1}{\mu}\right) + p_0\left(E(V^*) + \pi^* E(L_0)\frac{1}{\mu} + \pi\frac{1}{\mu}\right), \tag{4.1}$$

and Little's law yields

$$E(L) = \lambda E(S), \tag{4.2}$$

where the unconditional $E(L)$ is related to the conditional ones as

$$E(L) = p_0 E(L_0) + p_1 E(L_1). \tag{4.3}$$

Also, if we would act as if the customers arriving during a vacation are waiting in a "vacation area" and transferred to the queue as soon as the server returns, then application of Little's law to the vacation area yields

$$E(L_0) = \lambda E(V^*). \tag{4.4}$$

Analogous to (3.4), conservation of work gives the relation

$$p_1 = (\lambda p_0 \pi + \lambda p_1)\frac{1}{\mu}, \tag{4.5}$$

and clearly,

$$p_0 + p_1 = 1. \tag{4.6}$$

Now we need additional relations for the quantities π, π^* and $E(V^*)$. By conditioning on the next event after the arrival of the tagged customer, whether it is the end of the vacation (with probability $\frac{\gamma}{\zeta+\gamma}$) or an opportunity of abandonment (with probability $\frac{\zeta}{\zeta+\gamma}$), we have that

$$\pi = \frac{\gamma}{\zeta+\gamma} \times 1 + \frac{\zeta}{\zeta+\gamma} \times q\pi.$$

Hence

$$\pi = \frac{\gamma}{\zeta p + \gamma}. \tag{4.7}$$

Along the same lines,

$$\pi^* = \frac{\gamma}{\zeta + \gamma} \times 1 + \frac{\zeta}{\zeta + \gamma} \times q^2 \pi^*,$$

so

$$\pi^* = \frac{\gamma}{\zeta(1 - q^2) + \gamma}. \tag{4.8}$$

Finally, again conditioning on the next event after the arrival of the tagged customer, we obtain

$$E(V^*) = \frac{1}{\zeta + \gamma} + \frac{\gamma}{\zeta + \gamma} \cdot 0 + \frac{\zeta}{\zeta + \gamma} \cdot qE(V^*),$$

yielding

$$E(V^*) = \frac{1}{\zeta p + \gamma}. \tag{4.9}$$

This completes the formulation of the mean value relations. By solving (4.1)-(4.9) we get the following result.

Theorem 4.1 *The mean sojourn time is given by*

$$E(S) = \frac{1}{1 - \rho + \rho\pi} \left(\frac{1}{\zeta p + \gamma} + (q^2 - 1) \frac{\zeta \pi}{\zeta(1 - q^2) + \gamma} \cdot \frac{\rho}{\gamma} \right) + \frac{\pi}{1 - \rho + \rho\pi} \times \frac{1}{\mu(1 - \rho)} \tag{4.10}$$

and the fraction of time the server is inactive and active, respectively,

$$p_0 = \frac{1 - \rho}{1 - \rho + \rho\pi}, \quad p_1 = \frac{\rho\pi}{1 - \rho + \rho\pi}. \tag{4.11}$$

4.2 Equilibrium distribution

The set of balance equations for this model is given as follows:

$$\begin{aligned}
(\lambda + \zeta)\pi(0,0) &= \mu p \pi(1,1) + \zeta \sum_{j=0}^{\infty} \binom{j}{j} p^j q^0 \pi(j,0) && (4.12)\\
(\lambda + \gamma + \zeta)\pi(n,0) &= \lambda\pi(n-1,0) + \zeta \sum_{j=n}^{\infty} \binom{j}{j-n} p^{j-n} q^n \pi(j,0), \quad n \geq 1 && (4.13)\\
(\lambda + \mu)\pi(1,1) &= \gamma\pi(1,0) + \mu\pi(2,1) && (4.14)\\
(\lambda + \mu)\pi(n,1) &= \gamma\pi(n,0) + \lambda\pi(n-1,1) + \mu\pi(n+1,1), \quad n \geq 2. && (4.15)
\end{aligned}$$

Note that in the balance equations (4.12) and (4.13) we included the pseudo-transitions $(n,0) \to (n,0)$ with rates $\zeta\binom{n}{n}p^{n-n}q^n = \zeta q^n$, which correspond to epochs in the Poisson abandonment process where all customers remain in the system, i.e., no abandonments occur. This simplifies the writing of the balance equations.

In Theorem 4.2 of this subsection we will determine the equilibrium probability $\pi(0,0)$ and the equilibrium PGF $\Pi_0(z)$ in the form of infinite series of finite products. These series can be expressed compactly in terms of q-hypergeometric series (also known as basic hypergeometric series). Moreover, we will see that the theory of q-hypergeometric series easily yields interesting results for some limiting regimes.

There exists a rich theory for the class of q-hypergeometric series and their q-calculus, which enables fast calculations and simplifications. In the queueing theory literature there exist only few papers where this theory has been applied (see e.g. Ismail (1985), Kemp (1998, 2005)). Therefore, for the sake of completeness, we below summarize the basic definitions about q-hypergeometric series (for details see the reference book of Gasper and Rahman (2004), Chapters 1-3 and Appendices I-III).

The q-hypergeometric series are series of the form $\sum_{n=0}^{\infty} c_n$ where $c_0 = 1$ and $\frac{c_{n+1}}{c_n}$ is a rational function of q^n for a deformation parameter $|q| < 1$. They were initially introduced by Heine who developed their basic theory, following Gauss' fundamental paper on hypergeometric series. Observing that the ratio $\frac{c_{n+1}}{c_n}$, being rational in q^n, can be written in the form

$$\frac{c_{n+1}}{c_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}(-q^n)^{1+s-r}z,$$

we have that every such series assumes the form

$$ {}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{array}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \qquad (4.16)$$

where $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$, $n \geq 1$. In the definition of a q-series through (4.16) it is assumed that $b_i \neq q^{-m}$ for $m = 0, 1, \ldots$ and $i = 1, 2, \ldots, s$. This is the standard ${}_r\phi_s$ notation for q-series. If $0 < |q| < 1$, the ${}_r\phi_s$ series converge absolutely for all z when $r \leq s$ and for $|z| < 1$ when $r = s+1$. We use the abbreviation $(a_1, a_2, \ldots, a_r; q)_n$ to denote the product $(a_1;q)_n(a_2;q)_n \ldots (a_r;q)_n$. The quantity $(a;q)_n$ is referred to as the q-shifted factorial. We also define $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ and use the abbreviation $(a_1, a_2, \ldots, a_r; q)_\infty$ to denote the product $(a_1;q)_\infty(a_2;q)_\infty \ldots (a_r;q)_\infty$. A q-calculus has been developed that parallels the theory of hypergeometric functions.

We are now in position to state the main result of this section.

Theorem 4.2 *Provided $\rho < 1$, the equilibrium state probability of an empty system $\pi(0,0)$ is given by*

$$\begin{aligned} \pi(0,0) &= \frac{A}{\zeta}\sum_{j=0}^{\infty}\prod_{k=0}^{j}\frac{\zeta}{\gamma+\zeta+\lambda q^k} \\ &= \frac{A}{\gamma+\zeta}\frac{(q;q)_\infty}{(-\frac{\lambda}{\gamma+\zeta}, \frac{\zeta}{\gamma+\zeta}; q)_\infty}\, {}_2\phi_1\left(-\frac{\lambda}{\gamma+\zeta}, \frac{\zeta}{\gamma+\zeta}; 0; q, q\right). \end{aligned} \qquad (4.17)$$

The partial PGFs $\Pi_0(z)$ and $\Pi_1(z)$ are given by

$$\begin{aligned} \Pi_0(z) &= \frac{A}{\zeta}\sum_{j=0}^{\infty}\prod_{k=0}^{j}\frac{\zeta}{\gamma+\zeta+\lambda q^k(1-z)} \\ &= \frac{A}{\gamma+\zeta}\frac{(q;q)_\infty}{(-\frac{\lambda(1-z)}{\gamma+\zeta}, \frac{\zeta}{\gamma+\zeta}; q)_\infty}\, {}_2\phi_1\left(-\frac{\lambda(1-z)}{\gamma+\zeta}, \frac{\zeta}{\gamma+\zeta}; 0; q, q\right) \qquad (4.18) \\ \Pi_1(z) &= -\frac{Az}{\lambda z+\mu z-\lambda z^2-\mu} + \frac{\gamma z}{\lambda z+\mu z-\lambda z^2-\mu}\Pi_0(z), \qquad (4.19) \end{aligned}$$

where

$$A = \frac{\gamma(\mu-\lambda)(\gamma+\zeta(1-q))}{\mu\gamma+(\mu-\lambda)\zeta(1-q)}. \tag{4.20}$$

The convergence of the series is absolute in $\{z \in \mathbb{C} : |z| \le 1\}$ *and uniform in every compact subset of* $\{z \in \mathbb{C} : |z| < 1\}$.

Proof. Multiplying both sides of equations (4.12) and (4.13) by z^0 and z^n, respectively, and summing them for all $n = 0, 1, 2, \dots$ we obtain

$$(\lambda+\gamma+\zeta)\Pi_0(z) - \gamma\pi(0,0) = \mu\pi(1,1) + \lambda z \Pi_0(z) + \zeta \sum_{n=0}^{\infty}\sum_{j=n}^{\infty}\binom{j}{n} p^{j-n} q^n \pi(j,0) z^n \tag{4.21}$$

or

$$(\lambda+\gamma+\zeta-\lambda z)\Pi_0(z) = \gamma\pi(0,0) + \mu\pi(1,1) + \zeta\Pi_0(1-q+qz), \tag{4.22}$$

which leads to

$$\Pi_0(z) \quad = \quad \frac{\gamma\pi(0,0)+\mu\pi(1,1)}{\gamma+\zeta+\lambda(1-z)} + \frac{\zeta}{\gamma+\zeta+\lambda(1-z)}\Pi_0(1-q+qz). \tag{4.23}$$

Furthermore, by multiplying both sides of equations (4.14) and (4.15) by z and z^n, respectively, and summing them for all $n = 1, 2, 3, \dots$ we obtain after some rearrangements that

$$\Pi_1(z) \quad = \quad -\frac{(\gamma\pi(0,0)+\mu\pi(1,1))z}{\lambda z+\mu z-\lambda z^2-\mu} + \frac{\gamma z}{\lambda z+\mu z-\lambda z^2-\mu}\Pi_0(z). \tag{4.24}$$

By iterating equation (4.23) and setting

$$A = \gamma\pi(0,0) + \mu\pi(1,1) \tag{4.25}$$

we obtain

$$\Pi_0(z) = \frac{A}{\zeta}\sum_{j=0}^{n}\prod_{k=0}^{j}\frac{\zeta}{\gamma+\zeta+\lambda q^k(1-z)} + \Pi_0(1-q^{n+1}+q^{n+1}z)\prod_{k=0}^{n}\frac{\zeta}{\gamma+\zeta+\lambda q^k(1-z)}, \ n \ge 0. \tag{4.26}$$

By letting $n \to \infty$ we obtain

$$\Pi_0(z) \quad = \quad \frac{A}{\zeta}\sum_{j=0}^{\infty}\prod_{k=0}^{j}\frac{\zeta}{\gamma+\zeta+\lambda q^k(1-z)}, \tag{4.27}$$

which is expressed as a q-hypergeometric series in the form (4.18). This shows also that the infinite series does converge. We set $z = 0$ in (4.27), yielding

$$\pi(0,0) \quad = \quad \frac{A}{\zeta}\sum_{j=0}^{\infty}\prod_{k=0}^{j}\frac{\zeta}{\gamma+\zeta+\lambda q^k}, \tag{4.28}$$

which can be put in the form (4.17). We set $z = 1$ in (4.27), which leads to

$$\Pi_0(1) = \frac{A}{\zeta}\sum_{j=0}^{\infty}\prod_{k=0}^{j}\frac{\zeta}{\gamma+\zeta} = \frac{A}{\gamma}. \tag{4.29}$$

Note also that by (4.24), (4.25) and (4.29) we obtain

$$\Pi_1(1) \quad = \quad \frac{\gamma}{\mu-\lambda}\Pi_0'(1). \tag{4.30}$$

To obtain $\Pi_0'(1)$ multiply (4.23) by $\gamma+\zeta+\lambda(1-z)$, differentiate and take $z \to 1$. We then have

$$\Pi_0'(1) = \frac{\lambda A}{\gamma(\gamma+\zeta(1-q))}. \tag{4.31}$$

Equations (4.30) and (4.31) yield

$$\Pi_1(1) \quad = \quad \frac{\lambda A}{(\mu-\lambda)(\gamma+\zeta(1-q))}. \tag{4.32}$$

We have now expressed the various quantities of interest and the PGFs $\Pi_0(z)$ and $\Pi_1(z)$ in terms of the parameters of the model and the parameter A. Using (4.29) and (4.32) and the normalization equation we obtain (4.20) which concludes the proof. ∎

Remark 4.1 By differentiating twice (4.23) and once (4.24) and taking $z \to 1$ we obtain, after some long calculations, the mean number of customers in system and by Little's law also the mean sojourn time of a customer. This coincides with (4.10). However, the mean value approach gives the result much more easily.

4.3 Limiting regimes

To emphasize the dependence on the parameters of the model in the rest of this section, we will denote $\pi(n,i)$, $\Pi_0(z)$ and $\Pi_1(z)$ by $\pi(n,i;\lambda,\mu,\zeta,p,\gamma)$, $\Pi_0(z;\lambda,\mu,\zeta,p,\gamma)$ and $\Pi_1(z;\lambda,\mu,\zeta,p,\gamma)$ respectively. Note that ζp can be thought of as the effective abandonment rate per customer. Indeed the overall abandonment time of a customer is a geometric sum of exponentially distributed random variables with rate ζ; hence it is also exponentially distributed with parameter ζp. Under this perspective, if we have two models with the same parameters λ, μ and γ that differ only in ζ and p, but with $\zeta p = \zeta^*$ fixed, we can think that the models have identical arrival rates, service rates, effective abandonment rates per customer and vacation rates and differ only in the 'level of synchronization' p. Indeed, the case $p \to 0^+$ corresponds to no synchronization since the customers abandon almost singly the system. On the contrary, the case $p \to 1^-$ corresponds to full synchronization since almost all present customers abandon simultaneously.

We are interested in studying the equilibrium behavior of the system for the case where λ, μ, ζ^* and γ are kept fixed in the two limiting cases $p \to 0^+$ ($q \to 1^-$) and $p \to 1^-$ ($q \to 0^+$). For the limiting case of no synchronization we introduce

$$\pi^{(1)}(0,0) = \lim_{q\to 1^-} \pi(0,0;\lambda,\mu,\frac{\zeta^*}{1-q},1-q,\gamma), \tag{4.33}$$

$$\Pi_i^{(1)}(z) = \lim_{q\to 1^-} \Pi_i(z;\lambda,\mu,\frac{\zeta^*}{1-q},1-q,\gamma), \quad i=0,1, \tag{4.34}$$

and for the limiting case of full synchronization,

$$\pi^{(2)}(0,0) = \lim_{q\to 0^+} \pi(0,0;\lambda,\mu,\frac{\zeta^*}{1-q},1-q,\gamma), \tag{4.35}$$

$$\Pi_i^{(2)}(z) = \lim_{q\to 0^+} \Pi_i(z;\lambda,\mu,\frac{\zeta^*}{1-q},1-q,\gamma), \quad i=0,1. \tag{4.36}$$

The corresponding results for (4.33)-(4.36) are presented in Theorems 4.3 and 4.4.

To obtain immediately the results, we will use some results of the q-theory, concerning the definite q-integral of a function on an interval $[0, a]$ which is defined by

$$\int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n.$$

As $q \to 1^-$ the q-analogues reduce to their standard counterparts. In particular we have the relationships:

$$\lim_{q \to 1^-} (a(1-q); q)_\infty = e^{-a}, \tag{4.37}$$

$$\lim_{q \to 1^-} \frac{(q^a s; q)_\infty}{(s; q)_\infty} = (1-s)^{-a}, \tag{4.38}$$

$$\lim_{q \to 1^-} \int_0^a f(t) d_q t = \int_0^a f(t) dt. \tag{4.39}$$

(see Gasper and Rahman (2004) Eq. (1.3.17), (1.3.19) and (1.11.6) respectively). Of particular importance is also the following transformation formula of q-hypergeometric series into q-integrals (see Gasper and Rahman (2004) p.26, 1.4(iii)):

$$\begin{aligned} {}_{r+1}\phi_r \left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix} ; q, q^z \right) &= \frac{(a_1, \ldots, a_{r+1}; q)_\infty}{(1-q)(q, b_1, \ldots, b_r; q)_\infty} \times \\ &\quad \times \int_0^1 s^{z-1} \frac{(qs, b_1 s, \ldots, b_r s; q)_\infty}{(a_1 s, \ldots, a_{r+1} s; q)_\infty} d_q s. \end{aligned} \tag{4.40}$$

Using these facts we can study the case of no synchronization (i.e., independent abandonments) that has been investigated by Altman and Yechiali (2006). The following theorem corresponds to their results for the $M/M/1$ type model (see their Sect. 2, in particular their equations (2.9), (2.8) and (2.3)).

Theorem 4.3 *In case $q \to 1^-$ and $\zeta(1-q) = \zeta^*$ fixed, we have*

$$\pi^{(1)}(0,0) = \frac{A^*}{\zeta^*} \int_0^1 (1-s)^{\frac{\gamma}{\zeta^*}-1} e^{-\frac{\lambda}{\zeta^*} s} ds \tag{4.41}$$

$$\Pi_0^{(1)}(z) = \frac{A^*}{\zeta^*} e^{\frac{\lambda}{\zeta^*} z} (1-z)^{-\frac{\gamma}{\zeta^*}} \int_z^1 (1-s)^{\frac{\gamma}{\zeta^*}-1} e^{-\frac{\lambda}{\zeta^*} s} ds \tag{4.42}$$

$$\Pi_1^{(1)}(z) = -\frac{A^* z}{\lambda z + \mu z - \lambda z^2 - \mu} + \frac{\gamma z}{\lambda z + \mu z - \lambda z^2 - \mu} \Pi_0^{(1)}(z), \tag{4.43}$$

where

$$A^* = \frac{\gamma(\mu - \lambda)(\gamma + \zeta^*)}{\mu\gamma + (\mu - \lambda)\zeta^*}. \tag{4.44}$$

Proof. Using (4.40) we express (4.18) as a q-integral and we obtain that

$$\begin{aligned} \Pi_0(z) &= \frac{A}{\gamma + \zeta} \frac{(q; q)_\infty}{(-\frac{\lambda(1-z)}{\gamma+\zeta}, \frac{\zeta}{\gamma+\zeta}; q)_\infty} {}_2\phi_1 \left(-\frac{\lambda(1-z)}{\gamma+\zeta}, \frac{\zeta}{\gamma+\zeta}; 0; q, q \right) \\ &= \frac{A}{(\gamma+\zeta)(1-q)} \int_0^1 \frac{(qs; q)_\infty}{(-\frac{\lambda(1-z)s}{\gamma+\zeta}, \frac{\zeta s}{\gamma+\zeta}; q)_\infty} d_q s. \end{aligned} \tag{4.45}$$

Taking the limit as $q \to 1^-$ and using (4.37)-(4.39) yields

$$\lim_{q\to 1^-} \Pi_0(z) = \frac{A^*}{\zeta^*}\int_0^1 (1-s)^{\frac{\gamma}{\zeta^*}-1} e^{-\frac{\lambda}{\zeta^*}(1-z)s} ds, \tag{4.46}$$

where $A^* = \lim_{q\to 1^-} A$. After a change of variable in (4.46) we arrive at (4.42) which is Yechiali and Altman (2006) equation (2.8). Equations (4.41) and (4.43) are now obvious by taking limits as $q \to 1^-$ in (4.17) and (4.19). ■

For the case of full synchronization we have the following theorem.

Theorem 4.4 *In case $q \to 0^+$ and $\zeta(1-q) = \zeta^*$ fixed, we have*

$$\pi^{(2)}(0,0) = \frac{A^*(\gamma+\zeta^*)}{\gamma(\gamma+\zeta^*+\lambda)} \tag{4.47}$$

$$\pi^{(2)}(n,0) = \left(\frac{\lambda}{\gamma+\zeta^*+\lambda}\right)^n \pi^{(2)}(0,0), \quad n \geq 1 \tag{4.48}$$

$$\pi^{(2)}(n,1) = \begin{cases} \frac{A^*}{\gamma+\zeta^*+\lambda-\mu}\left[\left(\frac{\lambda}{\mu}\right)^n - \left(\frac{\lambda}{\gamma+\zeta^*+\lambda}\right)^n\right], & n \geq 1, \quad \mu \neq \gamma+\zeta^*+\lambda \\ n\frac{A^*}{\mu}\left(\frac{\lambda}{\mu}\right)^n, \quad n \geq 1, & \mu = \gamma+\zeta^*+\lambda \end{cases} \tag{4.49}$$

where

$$A^* = \frac{\gamma(\mu-\lambda)(\gamma+\zeta^*)}{\mu\gamma+(\mu-\lambda)\zeta^*}. \tag{4.50}$$

Proof. We take the limit as $q \to 0^+$ in (4.18). This yields

$$\Pi_0^{(2)}(z) = \frac{A^*(\gamma+\zeta^*)}{\gamma(\gamma+\zeta^*+\lambda(1-z))}, \tag{4.51}$$

where A^* is given by (4.50). By expanding (4.51) in power series of z we obtain easily (4.47) and (4.48). Taking $q \to 0^+$ in (4.19) implies, after some simplifications, that

$$\Pi_1^{(2)}(z) = \frac{A^*\lambda z}{\mu(\gamma+\zeta^*+\lambda)(1-\frac{\lambda}{\mu}z)(1-\frac{\lambda}{\gamma+\zeta^*+\lambda}z)}. \tag{4.52}$$

By analyzing $\left((1-\frac{\lambda}{\mu}z)(1-\frac{\lambda}{\gamma+\zeta^*+\lambda}z)\right)^{-1}$ in partial fractions for the two cases $\mu \neq \gamma+\zeta^*+\lambda$ and $\mu = \gamma+\zeta^*+\lambda$, and expanding in power series of z we obtain (4.49). ■

5 Non-Markovian UAE model

We now assume the general framework introduced in Section 2, i.e., the service times and vacation times are both generally distributed. First we determine the mean number of customers in the system and then we focus on the PGF of this number.

5.1 Mean value analysis

We suppose that the system is in equilibrium and consider a tagged arriving customer. Then, with probability p_1, he finds the server providing service, in which case his sojourn time is equal to the residual service time of the customer in service plus the service times of all customers waiting in the queue plus his own service time. Hence, his mean sojourn time is $E(B_e) + (E(L_1) - 1)E(B) + E(B)$. On the other hand, if he finds the server on vacation, then he first has to wait for the vacation time to expire; the mean residual vacation time is $E(V_e)$. Then, with probability p, he will abandon and, with probability q, he will remain for service, in which case the mean number of customers that he will find in front of him is $qE(L_0)$. So, in this case, his mean sojourn time is $E(V_e) + p \cdot 0 + q \cdot (qE(L_0)E(B) + E(B))$. Hence,

$$E(S) = p_1 \left(E(B_e) + E(L_1)E(B)\right) + p_0 \left(E(V_e) + q^2 E(L_0)E(B) + qE(B)\right)\right). \tag{5.1}$$

Further, Little's law states that

$$p_0 E(L_0) + p_1 E(L_1) = \lambda E(S). \tag{5.2}$$

Since p_0 and p_1 only depend on the arrival rate, mean service time and q, these quantities again satisfy (3.8). Finally, $E(L_0)$ is equal to the number of Poisson (λ) arrivals during the age of the vacation time. Since the age of the vacation time is in distribution the same as the residual vacation time, we get

$$E(L_0) = \lambda E(V_e). \tag{5.3}$$

Solution of (5.1)-(5.3) yields:

Theorem 5.1 *The mean sojourn time is given by*

$$E(S) = \frac{1}{1-\rho p}\left(E(V_e) + (q^2-1)\rho E(V_e)\right) + \frac{q}{1-\rho p}\left(\frac{\rho E(B_e)}{1-\rho} + E(B)\right).$$

5.2 Equilibrium distribution

The aim of this section is to determine the PGF of the number of customers in the system. By conditioning on the state of the server, we obtain

$$E(z^L) = p_0 E(z^{L_0}) + p_1 E(z^{L_1}),$$

where p_0 and p_1 are given by (3.8). The number of customers during a vacation, L_0, are exactly the ones who arrived during the age of the vacation, and the age is in distribution the same as the residual vacation. Hence, by conditioning on $V_e = t$, the number of arrivals is Poisson with parameter λt, and thus we get

$$\begin{aligned} E(z^{L_0}) &= \int_0^\infty e^{-\lambda t(1-z)} dV_e(t) \\ &= \tilde{V}_e(\lambda(1-z)). \end{aligned} \tag{5.4}$$

To find the PGF of L_1 we first need the number of customers in the system just after the end of a vacation; denote this number by L_e. The number of arrivals during a vacation of length t, who decide to stay at the end of the vacation, is Poisson with parameter $q\lambda t$. Hence, the PGF of the number of customers in the system, just after the end of the vacation, is

$$E(z^{L_e}) = \tilde{V}(q\lambda(1-z)). \tag{5.5}$$

We now proceed as in Fuhrmann (1984). Define the primary customers to be the ones just after the start of the busy period and the secondary customers to be the ones who arrive during the busy period. Further, we change the service discipline in non-preemptive LCFS; this does not affect the number of customers in the system. So, after servicing a primary customer, the server will serve any secondary customer until there is none present. So each primary customer generates a standard M/G/1 busy period, at the end of which the server either begins servicing the next primary customer or, if the system is empty, takes a vacation. Let Q_p be the number of primary customers waiting for service in the queue (so excluding the one possibly in service). If we remove server vacations from the time axis and glue together the service periods, then we readily obtain from the renewal reward theorem (see, e.g., Ross (2003)) that the fraction of time the queue contains n primary customers is equal to

$$P(Q_p = n) = \frac{P(L_e > n)}{E(L_e)}\ , \quad n \geq 0.$$

Hence,

$$E(z^{Q_p}) = \frac{1}{E(L_e)} \sum_{n=0}^{\infty} P(L_e > n) z^n = \frac{1 - E(z^{L_e})}{E(L_e)(1-z)} = \tilde{V}_e(q\lambda(1-z)), \tag{5.6}$$

where the last equality follows from (5.5). Let $L_{M/G/1}$ denote the number of customers in the M/G/1 with arrival rate λ and service time distribution $B(t)$, so, according to the Pollaczek-Khinchin formula,

$$E(z^{L_{M/G/1}}) = (1-\rho)\frac{(1-z)\tilde{B}(\lambda(1-z))}{\tilde{B}(\lambda(1-z)) - z}\ . \tag{5.7}$$

Since $L_1 = L_{M/G/1} + Q_b$, where $L_{M/G/1}$ and Q_b are independent, we obtain, by (5.6) and (5.7),

$$E(z^{L_1}) = E(z^{L_{M/G/1}})E(z^{Q_b}) = (1-\rho)\frac{(1-z)\tilde{B}(\lambda(1-z))}{\tilde{B}(\lambda(1-z)) - z}\tilde{V}_e(q\lambda(1-z)).$$

The results are summarized in the following theorem.

Theorem 5.2 *The PFG of the number of customers in the system is given by*

$$E(z^L) = \frac{1-\rho}{1-\rho p}\tilde{V}_e(\lambda(1-z)) + \frac{\rho q(1-\rho)}{1-\rho p}\ \frac{(1-z)\tilde{B}(\lambda(1-z))}{\tilde{B}(\lambda(1-z)) - z}\tilde{V}_e(q\lambda(1-z)).$$

6 Non-Markovian MAE model

In this section we consider the MAE model with generally distributed service times and vacation times. Below we first determine the mean number of customers in the system and then the PGF of this number.

6.1 Mean value analysis

We again consider a tagged arriving customer. If he finds the server providing service, then his mean sojourn time is $E(B_e) + E(L_1)E(B)$. Otherwise, he finds the server on vacation, in which case his mean sojourn time is equal to the mean time spent in the system till the end of the vacation, $E(V^*)$, plus the mean sojourn time from the end of the vacation, $\pi^* E(L_0)E(B) + \pi E(B)$. Hence,

$$E(S) = p_1\left(E(B_e) + E(L_1)E(B)\right) + p_0\left(E(V^*) + \pi^* E(L_0)E(B) + \pi E(B)\right). \tag{6.1}$$

Little's law states that

$$p_0 E(L_0) + p_1 E(L_1) = \lambda E(S) \tag{6.2}$$

and, when applied to the "vacation area",

$$E(L_0) = \lambda E(V^*) \tag{6.3}$$

Since p_0 and p_1 only depend on the arrival rate, mean service time and π, these quantities again satisfy (4.11). To derive additional relations for π, π^* and $E(V^*)$, we observe that, if the tagged customer arrives during a vacation, then the time till abandonment is exponential with rate ζp. By denoting the time till abandonment by A, we can write

$$V^* = \min(V_e, A) \quad \text{and} \quad \pi = P(V_e < A).$$

Hence, by conditioning on the length of V_e,

$$\begin{aligned} \pi &= \int_0^\infty P(t < A) dV_e(t) \\ &= \int_0^\infty e^{-\zeta p t} dV_e(t) \\ &= \tilde{V}_e(\zeta p), \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} E(V^*) &= E(V_e + A) - E(\max(V_e, A)) \\ &= E(V_e) + \frac{1}{\zeta p} - \left(E(V_e) + \pi \frac{1}{\zeta p} \right) \\ &= (1 - \pi) \frac{1}{\zeta p}. \end{aligned} \tag{6.5}$$

To compute π^* we condition on the length of V_e and on the number of abandonment epochs during V_e,

$$\begin{aligned} \pi^* &= \int_0^\infty \sum_{n=0}^\infty e^{-\zeta t} \frac{(\zeta t)^n}{n!} (q^2)^n dP(V_e \le t) \\ &= \int_0^\infty e^{-\zeta t} e^{\zeta t q^2} dV_e(t) \\ &= \int_0^\infty e^{-\zeta (1-q^2) t} dV_e(t) \\ &= \tilde{V}_e(\zeta(1 - q^2)). \end{aligned} \tag{6.6}$$

By solving (6.1)-(6.6) we finally get:

Theorem 6.1 *The mean sojourn time is equal to*

$$E(S) = \frac{1}{1 - \rho + \rho\pi} \left(E(V^*) + (\pi^* - 1)\rho E(V^*) \right) + \frac{\pi}{1 - \rho + \rho\pi} \left(\frac{\rho E(B_e)}{1 - \rho} + E(B) \right),$$

where π, π^* and $E(V^*)$ are given by (6.4), (6.6) and (6.5), respectively.

6.2 Equilibrium distribution

To determine the PGF of the number of customers in the system we can proceed along the same lines as in Section 5.2. In fact, we only need to find the PGFs of the number of customers during a vacation, L_0, and just after the end of a vacation, L_e. We start with the latter. Conditioning on the event that $V = t$, the number of abandonment epochs is Poisson with parameter ζt. Given the number of abandonment epochs is $n(>0)$, the event times $(s_1, s_2, \ldots, s_n)$ of these epochs will be distributed as the order statistics $(U_{1:n}, U_{2:n}, \ldots, U_{n:n})$ of a random sample $(U_1, U_2, \ldots, U_n)$ from the uniform distribution in $(0, t]$. The number of arrivals in each of the intervals $(0, s_1]$, $(s_1, s_2]$, …, $(s_{n-1}, s_n]$, $(s_n, t]$ are Poisson with parameters λs_1, $\lambda(s_2 - s_1)$, …, $\lambda(s_n - s_{n-1})$, $\lambda(t - s_n)$ respectively. Moreover, the individuals that arrive during these intervals with remain till time t with probabilities q^n, q^{n-1}, …, q, 1 respectively. Since the sum of Poisson random variables is again Poisson, we can conclude that the number of customers at the end of the vacation is Poisson with parameter

$$\begin{aligned}
\Lambda(t, n, s_1, \ldots, s_n) &= \lambda s_1 q^n + \lambda(s_2 - s_1)q^{n-1} + \cdots + \lambda(s_n - s_{n-1})q + \lambda(t - s_n) \\
&= -\lambda q^{n-1}(1-q)s_1 - \lambda q^{n-2}(1-q)s_2 - \cdots - \lambda(1-q)s_n + \lambda t,
\end{aligned}$$

valid for $n > 0$, and if $n = 0$, this number is Poisson with parameter λt. Hence,

$$E(z^{L_e}|V = t) = e^{-\zeta t}e^{-\lambda t(1-z)} + \sum_{n=1}^{\infty}\int_0^t\int_{s_1}^t\cdots\int_{s_{n-1}}^t e^{-\zeta t}\frac{(\zeta t)^n}{n!}e^{-\Lambda(t,n,s_1,\ldots,s_n)(1-z)}\frac{n!}{t^n}ds_n\cdots ds_1. \tag{6.7}$$

To put $E(z^{L_e}|V = t)$ in a more compact form we use the auxiliary identity

$$\begin{aligned}
I_n(t, n, \alpha_1, \alpha_2, \ldots, \alpha_n) &= \int_0^t\int_{s_1}^t\cdots\int_{s_{n-1}}^t e^{\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n}ds_n\cdots ds_2 ds_1 \\
&= \sum_{k=1}^{n+1}(-1)^{k+1}e^{t\sum_{i=k}^n \alpha_i}\frac{1}{\prod_{i=0}^{n-k}\sum_{j=k}^{k+i}\alpha_j \cdot \prod_{i=1}^{k-1}\sum_{j=k-i}^{k-1}\alpha_j},
\end{aligned} \tag{6.8}$$

which can be easily established by induction. In order to use (6.8) to simplify (6.7) we substitute

$$\alpha_j = \lambda(1-q)(1-z)q^{n-j}, j = 1, 2, \ldots, n \tag{6.9}$$

in (6.8), yielding (after some algebra)

$$I_n(t, n, \alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{e^{t\lambda(1-z)}}{[\lambda(1-z)]^n}\sum_{k=0}^{n}(-1)^k e^{-t\lambda(1-z)q^{n-k}} \cdot \frac{1}{(q;q)_k(q;q)_{n-k}q^{\binom{n-k}{2}+(n-k)k}}. \tag{6.10}$$

Using (6.7), (6.8) and (6.10) we obtain

$$\begin{aligned}
E(z^{L_e}|V = t) &= e^{-(\zeta+\lambda(1-z))t}\sum_{n=0}^{\infty}\zeta^n\sum_{k=0}^{n}\frac{e^{t\lambda(1-z)}}{(\lambda(1-z))^n}\frac{1}{(q;q)_k(q;q)_{n-k}}(-1)^k e^{-t\lambda(1-z)q^{n-k}}q^{-(n+k-1)(n-k)/2} \\
&= \frac{e^{-\zeta t}}{(-\frac{\zeta}{\lambda(1-z)};q)_\infty}\sum_{n=0}^{\infty}\frac{q^n e^{-t\lambda(1-z)q^n}}{(q;q)_n(-\frac{\lambda(1-z)q}{\zeta};q)_n}.
\end{aligned} \tag{6.11}$$

Hence, after unconditioning, we conclude that the PGF of L_e assumes the form

$$\begin{aligned}
E(z^{L_e}) &= \int_0^\infty E(z^{L_e}|V=t)dV(t) \\
&= \int_0^\infty \frac{e^{-\zeta t}}{(-\frac{\zeta}{\lambda(1-z)};q)_\infty} \sum_{n=0}^\infty \frac{q^n}{(q;q)_n} e^{-t\lambda(1-z)q^n} \frac{1}{(-\frac{\lambda(1-z)q}{\zeta};q)_n} dV(t) \\
&= \frac{1}{(-\frac{\zeta}{\lambda(1-z)};q)_\infty} \sum_{n=0}^\infty \frac{q^n \tilde{V}(\zeta+\lambda(1-z)q^n)}{(q;q)_n(-\frac{\lambda(1-z)q}{\zeta};q)_n}. \qquad (6.12)
\end{aligned}$$

Moreover,

$$E(L_e|V=t) = e^{-\zeta t}\lambda t + \sum_{n=1}^\infty \int_0^t \int_{s_1}^t \cdots \int_{s_{n-1}}^t e^{-\zeta t}\frac{(\zeta t)^n}{n!}\Lambda(t,n,s_1,\ldots,s_n)\frac{n!}{t^n}ds_n\cdots ds_1. \qquad (6.13)$$

To put $E(L_e|V=t)$ in a more compact form we use the auxiliary identity

$$\begin{aligned}
J_n(t,n,\alpha_1,\alpha_2,\ldots,\alpha_n) &= \int_0^t \int_{s_1}^t \cdots \int_{s_{n-1}}^t [\alpha_0+\alpha_1 s_1+\alpha_2 s_2+\cdots+\alpha_n s_n]\, ds_n\cdots ds_2 ds_1 \\
&= \alpha_0\frac{t^n}{n!}+\sum_{k=1}^n \alpha_k \frac{k\, t^{n+1}}{(n+1)!}, \qquad (6.14)
\end{aligned}$$

which can be easily established by induction. In order to use (6.14) to simplify (6.13) we substitute

$$\begin{aligned}
\alpha_0 &= \lambda t \\
\alpha_j &= -\lambda(1-q)q^{n-j}, j=1,2,\ldots,n
\end{aligned}$$

in (6.14), yielding

$$J_n(t,n,\alpha_1,\alpha_2,\ldots,\alpha_n) = \lambda\frac{t^{n+1}}{(n+1)!}\frac{1-q^{n+1}}{1-q}. \qquad (6.15)$$

Using (6.13) and (6.15) we obtain

$$E(L_e|V=t) = \frac{\lambda}{\zeta p}(1-e^{-\zeta p t}). \qquad (6.16)$$

Note that, after unconditioning, the mean value of L_e assumes the form

$$E(L_e) = \frac{\lambda}{\zeta p}(1-\tilde{V}(\zeta p)). \qquad (6.17)$$

To determine the PGF of the number of customers during a vacation, we can copy the approach above, where the vacation V should be replaced by its age V_e. This leads to

$$\begin{aligned}
E(z^{L_0}) &= \int_0^\infty E(z^{L_0}|V_e=t)dV_e(t) \\
&= \int_0^\infty \frac{e^{-\zeta t}}{(-\frac{\zeta}{\lambda(1-z)};q)_\infty} \sum_{n=0}^\infty \frac{q^n}{(q;q)_n} e^{-t\lambda(1-z)q^n} \frac{1}{(-\frac{\lambda(1-z)q}{\zeta};q)_n} dV_e(t) \\
&= \frac{1}{(-\frac{\zeta}{\lambda(1-z)};q)_\infty} \sum_{n=0}^\infty \frac{q^n \tilde{V}_e(\zeta+\lambda(1-z)q^n)}{(q;q)_n(-\frac{\lambda(1-z)q}{\zeta};q)_n}. \qquad (6.18)
\end{aligned}$$

Based on the PGFs of L_0 and L_e, we immediately obtain the following result.

Theorem 6.2 *The PFG of the number of customers in the system is given by*

$$E(z^L) = \frac{1-\rho}{1-\rho+\rho\pi}E(z^{L_0}) + \frac{\rho\pi(1-\rho)}{1-\rho+\rho\pi}\,\frac{\tilde{B}(\lambda(1-z))}{\tilde{B}(\lambda(1-z))-z}\,\frac{1-E(z^{L_e})}{E(L_e)},$$

where π*,* $E(z^{L_0})$*,* $E(z^{L_e})$ *and* $E(L_e)$ *are given by (6.4), (6.18), (6.12) and (6.17), respectively.*

7 Conclusion and possible extensions

In this paper we studied the abandonment phenomenon in queueing systems with vacations, where there exist a kind of synchronization for the abandonments. More specifically we analyzed two models with respect to the abandonment decisions of the customers (unique or multiple). We studied the stationary distributions for the number of customers in the system in continuous time for the Markovian models and we also consider some aspects of the analysis in the non-Markovian case. It would be interesting to consider extensions of this methodology for the study of other models with this type of binomial transitions.

A first direction is to carry out the analysis of the many variations of these models. For example, we can also consider the many-server case and the infinite-server case. We should then specify the way in which the servers take the vacations. The simplest case is the one where all servers take a vacation when the system becomes empty and all of them return as in the one-server case. This agrees to the many-server case with independent abandonments in the paper of Altman and Yechiali (2006). We can also consider the single-vacation case which is different from the multiple-vacation case described above in that the server takes just one vacation and then remains to the system even if there are not waiting customers.

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