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Propositional Logics for Three

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ABSTRACT. Semantics of propositional logic can be formulated in terms of 2-player games of perfect information. In the present paper the question is posed what would a generalization of propositional logic to a 3-player setting look like. Two formulations of such a '3-player propositional logic' are given, denoted PL^3_0 and PL^3. An overview of some metalogical properties of these logics is provided.

Semantics of classical propositional logic is typically given by laying down recursive rules which compute the truth-values of complex formulas from valuations that specify the truth-values of propositional atoms. Alternatively, the very same truth-conditions can be captured by defining semantics in terms of games of perfect information between two players (say *Eloise*, *Abelard*), with the property that a formula φ is true (false) under a valuation V in the usual sense if and only if there is a winning strategy for *Eloise* (*Abelard*) in the associated game $G(\varphi, V)$.[1] Conjunction and disjunction are interpreted by choices (between conjuncts and disjuncts) made by the two players, while negation is interpreted in terms of role switch (from 'Verifier' to 'Falsifier' and *vice versa*).

Classical propositional logic can, then, be seen as a logic for choice and role switching in a 2-player setting. In the present paper we wish to ask what 'logics for choice and role switching' would look like in multi-player settings.[2] In particular, we take first steps in exploring how to generalize propositional logic to the case where there are three players. The main goal of this paper is conceptual: to see how such a generalization can be carried out. The success of the proposed generalizations can, of course, only be assessed by reference to their formal properties, which is why we take up various related technical issues. The framework of the

[1] The original definition stems from Hintikka's [10]. Game-theoretic ideas were systematically applied in 20th century logic before Hintikka, in dialogical logic (starting with Lorenzen's [11, 12]). However, dialogic deals primarily with uninterpreted formulas and proof theory, while Hintikka's approach, influenced by Henkin's [9], is model-theoretic.

[2] Just before sending off the final version of the paper to the editors, we became aware of two other publications on multi-player logic. In [1, 2], Abramsky develops a compositional semantics of such logics in terms of multiple concurrent strategies (formalized as closure operators on certain concrete domains). It will be of obvious interest to further investigate this connection.

present paper can be approached from many perspectives: besides technical developments, one can analyze the use of game-theoretical notions for logical purposes and *vice versa*; also the philosophical significance of the emerging framework can be discussed. Given that the ground being covered is previously unexplored, we consider it as rewarding to ask questions on several fronts, and to follow in our discussion more than one lead from more than one viewpoint.

Two formulations of a '3-player propositional logic' will be presented, denoted PL^3_0 and PL^3. (Whether these are *stricto sensu* logics or not, it turns out that they can be studied as if they were.) Concerning these logics, we ask: Which are the 'semantic attributes' corresponding to the truth-values *true* and *false*? In which form, if any, do analogues of the *law of excluded middle*, *law of double negation*, *negation normal form*, or *conjunctive* and *disjunctive normal forms* emerge? What is the computational complexity of determining the semantic attribute of a formula relative to a valuation in the 3-player setting? Regarding one of the formulations, PL^3, some remarks are further made concerning the existence of a tableau-based proof system; 32 related decision problems are furthermore solved. Interestingly, many properties that fall together in classical propositional logic — typically due to the determinacy of the corresponding games — turn out to be distinguished in the multi-player setting.

1 Propositional logic or PL^2

Throughout the present paper, **prop** will be a countable set of propositional atoms. Formulas of *propositional logic* are generated by the grammar

$$\varphi ::= p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \neg\varphi,$$

with $p \in \mathbf{prop}$. Propositional logic will be denoted by PL^2. The basic semantic notion of PL^2 is, of course, that of a valuation. *Valuations* for PL^2 are functions $V : \mathbf{prop} \longrightarrow \{true, false\}$; they provide a distribution of truth-values over the propositional atoms considered. Well-known recursive clauses compute the truth-values of complex formulas, relative to a valuation V, from the truth-values that V gives to the atoms. We write $V \models \varphi$ to indicate that φ is true under the valuation V, whereby $V \not\models \varphi$ will indicate that φ is false under V. The truth-values *true* and *false* will occasionally be referred to as *semantic attributes* of propositional formulas.

There is an alternative way to define the semantics of PL^2, employing tools from game theory. Let us associate, with every formula φ and valuation V, a 2-player game $G(\varphi, V)$ of perfect information (between *Eloise* and *Abelard*). The following game rules determine the set of all (partial) *plays* of game $G(\varphi, V)$. The players are associated roles. In the beginning, *Eloise* occupies the role of 'Verifier' and *Abelard* that of 'Falsifier'; in the course of a play, the roles may get switched.

- If $\varphi := p$, a play of the game has come to an end. If $V(p) = true$, the player whose current role is 'Verifier' wins and the one whose current role is 'Falsifier' loses; otherwise 'Falsifier' wins and 'Verifier' loses.
- If $\varphi := (\psi \vee \chi)$, then 'Verifier' chooses a disjunct $\theta \in \{\psi, \chi\}$ and the play of the game continues as $G(\theta, V)$.
- If $\varphi := (\psi \wedge \chi)$, then 'Falsifier' chooses a conjunct $\theta \in \{\psi, \chi\}$ and the play of the game continues as $G(\theta, V)$.
- If $\varphi := \neg\psi$, the players switch their roles ('Verifier' assumes the role of 'Falsifier', and *vice versa*), and the play continues as $G(\psi, V)$.

If $\mathcal{P}$ is one of the players, a *strategy* for $\mathcal{P}$ is any function that specifies a move for $\mathcal{P}$ corresponding to each partial play at which it is $\mathcal{P}$'s turn to move, depending on the opponent's earlier moves. A strategy for $\mathcal{P}$ is *winning*, if it leads to a play won by $\mathcal{P}$ against any sequence of the opponent's moves. The usual propositional semantics is captured by the above games:

FACT 1. Let φ be a formula of PL^2 and V a valuation. Then: $V \models \varphi$ iff there is a winning strategy for *Eloise* in $G(\varphi, V)$; and $V \not\models \varphi$ iff there is a winning strategy for *Abelard* in $G(\varphi, V)$. □

2 From two to three — basic ideas

In n-player logic games, there will be n roles in addition to n players. This generalizes the 2-player case where there are two players (*Eloise*, *Abelard*) and two roles ('Verifier', 'Falsifier'). The roles are bijectively distributed to the players at each stage in a play of a game. Conceptually, players and their roles must be kept apart. In order to be able to say who is having which role at a given stage of a play, one cannot simply identify players with their roles. The same player may assume different roles during a play. Various semantically crucial notions will be defined by reference to the *initial role distribution*: the roles the players have when the playing of the game begins.

When generalizing PL^2 to multi-player settings, there are two mutually independent parameters that admit of variation: (1) the payoff function, and (2) the interpretation of negation symbols. Two generalizations of PL^2 to the 3-player setting will be presented, to be denoted PL^3_0 and PL^3. The former will be technically somewhat simpler to deal with. It will retain the binary character of the payoff function: each play is won by some players and lost by the others. Further, for each pair (i, j) of distinct roles there is a negation symbol $\sim_{ij}$, interpreted in terms of a transposition of the roles i and j. By contrast, in PL^3 payoffs are defined in terms of rankings of the players. In this respect it represents a more straightforward generalization of PL^2 than PL^3_0 does. On the other hand, the treatment of negations is more complicated: the syntax of PL^3 provides two negation symbols, which are interpreted by functions mapping role distributions to role distributions — instead of being interpreted simply by permutations of roles.

It should be noted that the parameters (1) and (2) could be instantiated in further ways. In particular, it might be of interest to combine the definition of payoffs as rankings with the interpretation of negations in terms of transpositions. The investigations into the 'logics' PL^3_0 and PL^3 presented in this paper are best viewed as case studies.

The most central question in the generalization of classical propositional logic is what happens to negation. Games defining the semantics of PL^2 are *determined*: in each game either *Eloise* or *Abelard* has a winning strategy.[3] Hence in PL^2, the truth of $\neg\varphi$ under V can be equivalently characterized in one of the two ways: (a) there is a winning strategy for 'Falsifier' in $G(\varphi, V)$; and (b) there is no winning strategy for 'Verifier' in $G(\varphi, V)$. In the former case negation is defined in terms of role shift, in the latter case by the absence of a winning strategy. In multi-player settings no analogous equivalence holds. Precisely because the two characterizations of negation in PL^2 — (a) and (b) — are equivalent, the classical framework as such does not dictate which characterization we should take as the model of our generalization. In connection with PL^3_0 and PL^3, we will continue to interpret negations in terms of changing roles. There is, admittedly, a rather strong pretheoretical tendency to construe negation in terms of 'complementation' or 'absence'. Therefore one might wish to think of the negations in the logics PL^3_0 and PL^3 as *contrarieties* rather than negations proper.

2.1 First formulation: logic PL^3_0

Syntax. Formulas of PL^3_0 are generated by the grammar

$$\varphi ::= p \mid (\varphi \vee_i \varphi) \mid {\sim_{ij}}\varphi,$$

where $i, j \in \{0, 1, 2\}$, $i \neq j$, and $p \in \mathbf{prop}$. Intuitively, there are three players. The numbers $0, 1, 2$ stand for the roles that the players may have. The connective $\vee_i$ is interpreted by the player whose current role is i. Note that the syntax of PL^3_0 involves 3 disjunction signs and 6 negation signs. Whenever no confusion threatens, brackets may be dropped.

Valuations. Valuations of the logic PL^3_0 assign to propositional atoms subsets of the set $\{0, 1, 2\}$ of all roles, viz. they are functions $V : \mathbf{prop} \longrightarrow Pow(\{0, 1, 2\})$. Intuitively, those players whose roles at the end of a play are in the set $V(p)$ *all win* the play, and the rest lose the play.

[3]This follows from the Gale-Stewart theorem [8]. The theorem saying that all two-player zero-sum perfect information games of finite length are determined, is often termed 'Zermelo's theorem'. However, the result is not due to Zermelo. For details, see [13].

If *arbitrary* subsets of $\{0,1,2\}$ are allowed as values of V, it may happen that all players win a play, or that no player does. Not to deviate from the 2-player setting in our generalization, we should ban $\emptyset$ and $\{0,1,2\}$ as possible values of a valuation function. In what follows, valuations V of PL_0^3 meeting the extra requirement that neither $\emptyset$ nor $\{0,1,2\}$ lies in the image of V, will be termed *restricted valuations*.[4] The counterpart in PL^2 of arbitrary PL_0^3-valuations would be the generalized valuations allowing any of the sets $\emptyset$, $\{true\}$, $\{false\}$ and $\{true, false\}$ as possible values. A 4-valued propositional logic evaluated precisely relative to such valuations was introduced by Belnap in [3]. (See also Dunn's article [7].)

Game rules. The games are played by three players: *Alice*, *Bob* and *Cecile* (in short: a, b, c). Relative to valuations $V : \mathbf{prop} \longrightarrow Pow(\{0,1,2\})$, the semantics of PL_0^3-formulas is specified by means of 3-player games $G(\varphi, V)$. To introduce these games, we first define something a bit more general, namely 3-player games $G(\varphi, V, \rho)$, where the extra input ρ is a bijection $\{0,1,2\} \longrightarrow \{a,b,c\}$, i.e., a distribution of roles to the players. If ρ is a role distribution, let ρ_{ij} be its transposition satisfying: $\rho_{ij}(i) = \rho(j)$ and $\rho_{ij}(j) = \rho(i)$ and $\rho_{ij}(k) = \rho(k)$ for $k \notin \{i, j\}$. With every formula φ of PL^3, valuation V and role distribution ρ, a 3-player game $G(\varphi, V, \rho)$ of perfect information between a, b and c is introduced. The game rules are these:

- If $\varphi \in \mathbf{prop}$, a play of the game has come to an end. Those players whose roles are in the set $V(\varphi)$ win the play, the others lose it. That is, a player $\mathcal{P}$ is one of the winners of the play iff $\rho^{-1}(\mathcal{P}) \in V(\varphi)$.[5]
- Let $i \in \{0,1,2\}$. If $\varphi = (\psi \vee_i \chi)$, then the player $\rho(i)$ chooses $\theta \in \{\psi, \chi\}$, and the play goes on as $G(\theta, V, \rho)$.
- Let $i, j \in \{0,1,2\}$ and $i \neq j$. If $\varphi = \sim_{ij}\psi$, the play continues as $G(\psi, V, \rho_{ij})$.

By stipulation the game $G(\varphi, V)$ equals the game $G(\varphi, V, \rho_0)$, where ρ_0 is specified by putting $\rho_0(0) = a$, $\rho_0(1) = b$, and $\rho_0(2) = c$. This particular role distribution ρ_0 will be referred to as the 'standard initial role distribution'.

Strategies. Any sequence of moves made according to the game rules is a *partial play*. A *play* is a partial play at which no player is to move. If $\mathcal{P}$ is any of the players, a *strategy* for $\mathcal{P}$ is any function providing a choice for $\mathcal{P}$ at any partial play at which it is his or her turn to move; the choice may depend on the moves made by $\mathcal{P}$'s opponents earlier in the course of the relevant partial play. A strategy for $\mathcal{P}$ is *winning*, if against any sequence of moves by his or her opponents, it leads to a play won by $\mathcal{P}$.

[4] If $f : A \longrightarrow B$ is a function, its *image*, denoted $Im(f)$, is the set $\{f(a) : a \in A\}$.

[5] Were payoffs taken to be numbers, the games being defined would *not* in general be constant-sum games. If winning corresponds to 1 and losing to 0, the sum of the players' payoffs may be any integer m with $0 \leq m \leq 3$.

Semantic attributes. For each set S of players, exactly the members of S might have a winning strategy in a game $G(\psi, V)$. Each subset S of $\{a, b, c\}$ constitutes a *semantic attribute* such that φ has by definition the attribute S relative to V, if the set of players having a winning strategy in $G(\varphi, V)$ is S. There are, then, 8 semantic attributes. We will write $|\varphi, V| = S$ to indicate that the semantic attribute of φ relative to V is S.

REMARK 2. Semantic attributes were just defined in terms of players. Arguably a more intrinsic definition would be in terms of roles. However, throughout this paper we will think of formulas as evaluated starting with the standard initial role distribution, assigning to *Alice* the role 0, to *Bob* the role 1, and to *Cecile* the role 2. We could indeed leave the semantics neutral with respect to the initial role distribution, and let semantic attributes to be sets of roles rather than sets of players. The players corresponding to these attributes would then vary with the particular initial role distribution. We stay with a fixed initial role distribution for clarity of exposition. □

Observe that directly by the semantics, for any formula φ, any valuation V, and any distinct roles i and j, we have: $|{\sim_{ij}}\varphi, V| = |{\sim_{ji}}\varphi, V|$. Therefore the logic PL_0^3 involves 'really' only three negations.

Let us think of the 8 attributes. There is a one-one correspondence between sets of roles and sets of players via the standard initial role distribution ρ_0. Given a set S of players, we write $\rho_0^{-1}(S)$ for the set $\{\rho_0^{-1}(\mathcal{P}) : \mathcal{P} \in S\}$. Similarly, given a set R of roles, we write $\rho_0(R)$ for $\{\rho_0(i) : i \in R\}$. Observe that trivially, each attribute can appear as an attribute of an atomic formula. If S is a set of players and $V(p) = \rho_0^{-1}(S)$, then $|p, V| = S$. Next, note that even if $\emptyset$ were not allowed in the image of a valuation function — as when considering only restricted valuations — *still* there would be formulas having $\emptyset$ as their semantic attribute relative to some valuation. To see this, let $V(p) = \{1\}$, $V(q) = \{2\}$, and consider determining the value $|p \vee_0 q, V|$. Now no player has a winning strategy in $G(p \vee_0 q, V)$. For, no matter which disjunct *Alice* chooses, she herself loses. On the other hand, choosing 'left' she will prevent *Cecile* from winning, and choosing 'right' she will prevent *Bob* from winning. Thus $|p \vee_0 q, V| = \emptyset$.

By contrast, no similar fact holds for the other extreme. If V is a valuation whose image does not involve the full set $\{0, 1, 2\}$, no formula can have $\{a, b, c\}$ as its semantic value relative to V. For, suppose that $\{0, 1, 2\} \notin Im(V)$, but still all players have a winning strategy in $G(\varphi, V)$. Let f, g and h be winning strategies of *Alice*, *Bob* resp. *Cecile*. These strategies determine a certain play of $G(\varphi, V)$. Let p be the atom reached at the end of the play. Since the play is determined by the three winning strategies, it is won by all players, i.e., $V(p) = \{0, 1, 2\}$. This is a contradiction.

It is noteworthy that when arbitrary valuations are employed, the very same 8 semantic attributes are available for all formulas, both atomic and complex. But if restricted valuations are used — valuations whose image excludes both $\emptyset$ and the full set $\{0,1,2\}$ — then there are 6 semantic attributes available for atomic formulas, but 7 attributes for complex formulas.

Every formula $\varphi \in PL_0^3$ determines a map, call it $|\varphi|$, from restricted valuations to semantic attributes, namely $|\varphi| : V \mapsto S$, with $S = |\varphi, V|$. On the other hand, as noted above, any semantic attribute except $\{a,b,c\}$ is realizable by a formula of PL^3 under restricted valuations. Let **prop** be a finite set of propositional atoms, and f a map from restricted valuations on **prop** to realizable semantic attributes. An important systematic question related to PL_0^3 then is whether there always is a formula φ of PL_0^3 such that $f = |\varphi|$. This issue of functional completeness is left as an open question.

Case of n players. It would be straightforward to generalize PL_0^3 to the case of an arbitrary finite number n of players. This would involve having in the syntax n disjunction symbols and $n \cdot (n-1)$ negation symbols. The semantics would require n roles in addition to the n players. The disjunction symbol $\vee_i$ would be interpreted by the player having the role i, and the negation symbol $\sim_{ij}$ by the transposition of the roles i and j. Semantic attributes would simply be subsets of the set of all players. Hence there would be 2^n distinct semantic attributes.

2.2 Second formulation: logic PL^3

In PL_0^3, certain features of the 2-player framework were preserved that admit of generalization. For one thing, payoffs in games for PL_0^3 are simply *win* and *loss* — the relevant difference with respect to PL^2 is just that several players may receive a given payoff. For another thing, the negations of PL_0^3 are interpreted by means of transpositions — just like the negation of PL^2. One might consider interpreting negation symbols of a 3-player logic by permutations of the three roles, not in general merely switching two roles at a time. Such permutations are arbitrary bijections of type $\{0,1,2\} \longrightarrow \{0,1,2\}$. However, in PL^3 we will go one step further, and interpret negation symbols by bijections taking role distributions as arguments, and yielding role distributions as values: bijections of type $\mathbf{P} \longrightarrow \mathbf{P}$, where $\mathbf{P}$ is the set of all role distributions (the set of all bijections in the set $\{a,b,c\}^{\{0,1,2\}}$). In PL^3, negations are hence interpreted by *permutations of role distributions* rather than by permutations of the set of roles. Such a 'higher-order' interpretation of negation symbols has obvious drawbacks. There are $6! = 720$ such bijections, so which ones should we consider? Should all these functions be expressible by means of those we introduce explicitly? In connection with PL^3 a more modest approach is adopted: we introduce 2 negation symbols, and simply content ourselves with being able to express the 6 permutations of the set of roles in terms of these 2 negation symbols — interpreted by means of permutations of role distributions.

Syntax. Formulas of PL^3 are generated by the following grammar:

$$\varphi ::= p \mid (\varphi \vee_i \varphi) \mid \neg\varphi \mid \sim\varphi,$$

with $i \in \{0,1,2\}$ and $p \in$ **prop**. Again, there are intuitively three players, each of whom occupies one of the roles $0,1,2$. And as before, the connective $\vee_i$ is interpreted by the player whose current role is i. To the two negation signs, $\neg$ and $\sim$, two permutations of role distributions will correspond.

Valuations. Valuations of the logic PL^3 are functions from propositional atoms to role distributions, i.e., functions of type **prop** $\longrightarrow$ **P**. Let us write $V(p) = (r, r', r'')$ to indicate that $V(p)(r) = a$, $V(p)(r') = b$, and $V(p)(r'') = c$. Intuitively, the value $V(p)$ serves to rank the players in a linear order with respect to the propositional atom p. If $V(p) = (r, r', r'')$, this means that relative to the atom p, *Alice* has the role r, *Bob* the role r', and *Cecile* the role r''. The numerical values of r, r' and r'' then determine a ranking among the three players. Such a numerical value is termed the *rank* of the player. The best rank is 0, the next best 1, and the worst 2. If, e.g., $V(p) = (2,0,1)$, then relative to p *Alice* gets the worst rank, *Bob* the best rank and *Cecile* the next best rank.[6]

Negations. Negation of PL^2 turns truths into falsehoods and *vice versa*; game-theoretically this negation is interpreted by a transposition acting on a pair of roles. transpositions are a (representative) special case of permutations of a finite set. The generalization to be considered next involves interpreting the two negations of PL^3 by permutations of *role distributions*. The 'higher order' permutations to be considered are $\pi_\neg$ and $\pi_\sim$:

$\pi_\neg(0,1,2) = (1,2,0)$	$\pi_\neg(0,2,1) = (1,0,2)$
$\pi_\neg(1,2,0) = (2,0,1)$	$\pi_\neg(1,0,2) = (2,1,0)$
$\pi_\neg(2,0,1) = (0,1,2)$	$\pi_\neg(2,1,0) = (0,2,1)$

$\pi_\sim(0,1,2) = (0,2,1)$	$\pi_\sim(0,2,1) = (0,1,2)$
$\pi_\sim(1,2,0) = (1,0,2)$	$\pi_\sim(1,0,2) = (1,2,0)$
$\pi_\sim(2,0,1) = (2,1,0)$	$\pi_\sim(2,1,0) = (2,0,1)$

FIGURE 1

The two negations $\neg$ and $\sim$ will, then, correspond to two different ways in which distributions of roles to the players are changed: in one case according to $\pi_\neg$, in the other according to $\pi_\sim$. It can be noted that $\pi_\neg$ is definable as a permutation of roles, mapping the role 0 to 1, the role 1 to 2, and the role 2 to 0. By contrast, $\pi_\sim$ cannot be defined simply by reference to roles, but is genuinely a permutation of entire role distributions. As a matter of fact, all permutations of roles can now be defined by means of compositions of these two permutations of role distributions. (Certainly not all of the 720 permutations of role distributions are definable in

[6] For games with rankings as payoffs, see [4, 5].

terms of $\pi_{\neg}$ and $\pi_{\sim}$, but then again, this is not posed as a *desideratum* in our case study of PL^3.)

Game rules. With every formula φ of PL^3, valuation V : **prop** $\longrightarrow$ **P** and role distribution $\rho : \{0, 1, 2\} \longrightarrow \{a, b, c\}$, a 3-player game $G(\varphi, V, \rho)$ of perfect information between *Alice*, *Bob* and *Cecile* is introduced.[7] The game rules are as follows:

- If $\varphi \in$ **prop**, a play of the game has come to an end. If $V(\varphi) = (r, r', r'')$, then r determines the payoff for *Alice*, r' for *Bob*, and r'' for *Cecile*. Role 0 yields payoff $\mathfrak{g}$, role 1 payoff $\mathfrak{s}$, and role 2 payoff $\mathfrak{b}$. (Mnemonics: $\mathfrak{g}$ for 'gold', $\mathfrak{s}$ for 'silver' and $\mathfrak{b}$ for 'bronze'.)
- Let $i \in \{0, 1, 2\}$. If $\varphi = (\psi \vee_i \chi)$, then the player $\rho(i)$ chooses $\theta \in \{\psi, \chi\}$, and the play goes on as $G(\theta, V, \rho)$.
- If $\varphi = \neg\psi$, then the play continues as $G(\psi, V, \rho')$ with $\rho' = \pi_{\neg}(\rho)$.
- If $\varphi = \sim\psi$, then the play continues as $G(\psi, V, \rho')$ with $\rho' = \pi_{\sim}(\rho)$.

We stipulate that the game $G(\varphi, V)$ equals the game $G(\varphi, V, \rho_0)$, where ρ_0 is the standard initial role distribution, defined as in *Subsection* 2.1.

Strategies. The notions of (partial) play and strategy are defined as with PL_0^3. Since in PL^3 a play has more than two possible outcomes (there are $3! = 6$ possible outcomes), unqualified talk of winning strategies would not make sense. This fact motivates the following definitions: a $\mathfrak{g}$-*strategy* for player $\mathcal{P}$ is a strategy for $\mathcal{P}$ leading to the payoff $\mathfrak{g}$ for $\mathcal{P}$, against any sequence of moves by $\mathcal{P}$'s opponents; and an $\mathfrak{s}$-*strategy* for $\mathcal{P}$ is a strategy for $\mathcal{P}$ which is not a $\mathfrak{g}$-strategy, and which leads *at least* to the payoff $\mathfrak{s}$ for $\mathcal{P}$, against any sequence of moves by $\mathcal{P}$'s opponents. If *Alice*, say, has an $\mathfrak{s}$-strategy, she *cannot* use it to gain $\mathfrak{g}$ against *all* sequences of moves by *Bob* and *Cecile*. Yet, if she follows her $\mathfrak{s}$-strategy, she may obtain $\mathfrak{g}$ against *some* moves by them, and whenever she does not, she gains the payoff $\mathfrak{s}$.

Semantic attributes. When extending the semantics from atomic to complex formulas, there are the following questions to consider: *Is there a* $\mathfrak{g}$*-strategy for one of the players? Is there an* $\mathfrak{s}$*-strategy for (at least) one of the players?* It can happen that no player has a $\mathfrak{g}$-strategy, and it can happen that exactly one has. Further, it can happen that no player has an $\mathfrak{s}$-strategy, and it can happen that one player has or that two players have an $\mathfrak{s}$-strategy. If no player has a y-strategy (for $y \in \{\mathfrak{g}, \mathfrak{s}\}$) in a game, the game is said to be *non-determined* with respect to having a y-strategy; if more than one player has a y-strategy (for $y = \mathfrak{s}$), the game is *over-determined* with respect to having a y-strategy. Schematically, we are interested

[7]Unlike games for PL_0^3, games for PL^3 would indeed be constant-sum games, should we define payoffs as numbers.

in global properties of games $G(\varphi, V)$ represented by the following 16 pairs ('?' stands for non-determinacy, '$!_{\mathcal{P},\mathcal{P}'}$' for overdeterminacy due to players' $\mathcal{P}$ and $\mathcal{P}'$ both having an $\mathfrak{s}$-strategy):

(a,b)	(a,c)	$(a,?)$		
(b,a)	(b,c)	$(b,?)$		$(?,!_{a,b})$
(c,a)	(c,b)	$(c,?)$		$(?,!_{a,c})$
$(?,a)$	$(?,b)$	$(?,c)$	$(?,?)$	$(?,!_{b,c})$

FIGURE 2

The first member x of a pair (x, y) indicates whether one of the players has a $\mathfrak{g}$-strategy, and if one of them has, it also indicates who does. The second member y indicates whether some of the players have an $\mathfrak{s}$-strategy, and if at least one of them has, it indicates which one does or which ones do.

We distinguish between 16 semantic attributes of a formula: each pair P listed in Figure 2 constitutes a *semantic attribute* such that φ has by definition the attribute P relative to a valuation V, if the players' status in terms of having or lacking a $\mathfrak{g}$-strategy *resp.* an $\mathfrak{s}$-strategy in game $G(\varphi, V)$ is as specified by the pair P. Hence for instance the attribute (a, b) corresponds to *Alice*'s having a $\mathfrak{g}$-strategy and *Bob*'s (and only *Bob*'s) having an $\mathfrak{s}$-strategy; while $(?, !_{b,c})$ corresponds to no player's having a $\mathfrak{g}$-strategy and both of the players' *Bob* and *Cecile* having an $\mathfrak{s}$-strategy. We write $|\varphi, V| = P$ to indicate that the semantic attribute of φ relative to V is P.

Like with PL^3_0, also in connection with PL^3 the semantics might be left neutral with respect to the initial role distribution, and we could define the semantic attributes by reference to roles rather than players. However, we believe it to serve clarity of exposition to refer to players and stay with the standard initial role distribution (cf. Remark 2).

REMARK 3. Semantic attributes in logics PL^2, PL^3_0 and PL^3 can be viewed from a unifying perspective using the game-theoretic notion of *security level* (guaranteed minimum payoff);[8] for this notion see e.g. [5]. In each case semantic attributes can be considered as maps representing the security levels of the players. With PL^2, the only possible maps are $\{(Eloise, 1), (Abelard, 0)\}$ and $\{(Eloise, 0), (Abelard, 1)\}$ — since the corresponding games are determined. Semantic attributes S of PL^3_0 give rise to maps f_S of type $\{a, b, c\} \longrightarrow \{0, 1\}$, where $f_S(\mathcal{P}) = 1$ iff $\mathcal{P} \in S$. Finally, in PL^3 semantic attributes P induce maps $f_P : \{a, b, c\} \longrightarrow \{\mathfrak{g}, \mathfrak{s}, \mathfrak{b}\}$, where the value $f_P(\mathcal{P})$ indicates the optimal minimum rank that player $\mathcal{P}$ can guarantee by a suitable choice of strategy. We leave systematic investigation of the use of security levels and related game-theoretic notions in connection with game-theoretically defined logics for future research. □

[8] We are indebted to an anonymous referee for this observation.

Let us now check in detail that all the 16 pairs of Figure 2 can indeed occur as semantic attributes of a formula. Let us begin by considering $\mathfrak{g}$-strategies. The number of players having a $\mathfrak{g}$-strategy in a game $G(\varphi, V)$ can be zero or one. (Evidently there are no games where more than one player has a $\mathfrak{g}$-strategy.)

EXAMPLE 4 (No one has a $\mathfrak{g}$-strategy). Put $V(p) = (2, 0, 1)$, $V(q) = (0, 2, 1)$, and $V(r) = (1, 2, 0)$; and let $\varphi := (p \vee_0 (q \vee_1 r))$. No player has a $\mathfrak{g}$-strategy in $G(\varphi, V)$: *Alice* gets second position if *Bob* chooses 'right' for $\vee_1$ (*Alice* having first herself chosen 'right' for $\vee_0$). *Cecile* gets second position if *Alice* chooses 'left' for $\vee_0$. And *Bob* gets only the last position if *Alice* chooses 'right' for $\vee_0$, no matter what *Bob* himself chooses for $\vee_1$. □

EXAMPLE 5 (Exactly one player has a $\mathfrak{g}$-strategy). Let $V(p) = (0, 1, 2)$. Then trivially *Alice* has a $\mathfrak{g}$-strategy in game $G(p, V)$. □

Let us proceed to think of $\mathfrak{s}$-strategies. The number of players having an $\mathfrak{s}$-strategy in a game $G(\varphi, V)$ can be zero, one or two. (Two players can have an $\mathfrak{s}$-strategy only if no player has a $\mathfrak{g}$-strategy.)

EXAMPLE 6 (No one has a $\mathfrak{g}$-strategy, nor an $\mathfrak{s}$-strategy). Consider the formula $\varphi := ((p \vee_1 q) \vee_0 (r \vee_2 s))$. Let $V(p) = (0, 1, 2)$, $V(q) = (2, 1, 0)$, $V(r) = (0, 2, 1)$, $V(s) = (2, 0, 1)$. In the left 0-disjunct *Bob* (i.e., the player responsible for moving) becomes second no matter which 1-disjunct he chooses, but he can decide whether *Alice* becomes first or third; and in the right 0-disjunct *Cecile* (i.e., the player responsible for moving) becomes second no matter which 2-disjunct she chooses, but she can decide whether *Alice* becomes first or third. It is *Alice* who decides whether the play proceeds to the left or to the right 0-disjunct. Hence in $G(\varphi, V)$ no player has a $\mathfrak{g}$-strategy. Actually no player even has an $\mathfrak{s}$-strategy. For, by what noted above, *Alice* cannot exclude the possibility that she becomes third. And jointly *Alice* and *Bob* can guarantee that *Cecile* becomes third, and likewise jointly *Alice* and *Cecile* can guarantee that *Bob* becomes third. □

EXAMPLE 7 (Someone has a $\mathfrak{g}$-strategy; no one has an $\mathfrak{s}$-strategy). Consider the formula $\varphi := (p \vee_0 q)$, and the valuation V with $V(p) = (0, 1, 2)$ and $V(q) = (0, 2, 1)$. *Alice* has a $\mathfrak{g}$-strategy in $G(\varphi, V)$, but neither *Bob* nor *Cecile* can guarantee second position: it is up to *Alice* to decide. □

Trivially there are games where some player has a $\mathfrak{g}$-strategy and another one has an $\mathfrak{s}$-strategy. E.g., in the game of Example 5, *Alice* has a $\mathfrak{g}$-strategy and *Bob* an $\mathfrak{s}$-strategy. Let us look at games where no one has a $\mathfrak{g}$-strategy.

EXAMPLE 8 (No one has a $\mathfrak{g}$-strategy; exactly one player has an $\mathfrak{s}$-startegy). Consider the formula $\varphi := (p \vee_0 (q \vee_1 r))$, and let $V(p) = (2, 1, 0)$, $V(q) = (2, 0, 1)$, $V(r) = (0, 1, 2)$. Clearly no player has a $\mathfrak{g}$-strategy in $G(\varphi, V)$: *Alice* cannot force that all plays of the game end with the atom r; *Bob* cannot force that all plays end with q; and *Cecile* cannot force that all plays end with p. On the other hand, *Bob* has an $\mathfrak{s}$-strategy: it consists of doing nothing if *Alice* picks out the left 0-disjunct, and choosing either of the 1-disjuncts if *Alice* picks out the right 0-disjunct. By contrast, *Cecile* cannot guarantee becoming second (since *Alice* can choose 'right' for $\vee_0$ and *Bob* can continue by picking out 'right' for $\vee_1$). And *Alice* cannot exclude ending up third (for if she chooses 'right' for $\vee_0$, *Bob* can choose 'left' for $\vee_1$). Hence *Bob* alone has an $\mathfrak{s}$-strategy in $G(\varphi, V)$. □

EXAMPLE 9 (No one has a $\mathfrak{g}$-strategy; two players have an $\mathfrak{s}$-strategy). A case in point is the formula $\varphi := (p \vee_0 q)$ and the valuation V such that $V(p) = (2, 0, 1)$ and $V(q) = (2, 1, 0)$. No player has a $\mathfrak{g}$-strategy in $G(\varphi, V)$; but both *Bob* and *Cecile* have a way of guaranteeing (without any personal effort, for that matter) that they receive at least the payoff $\mathfrak{s}$. By contrast, *Alice* ends inevitably last no matter how the play goes. □

The following fact holds trivially:

FACT 10. (**a**) It cannot happen that each of the three players has either a $\mathfrak{g}$-strategy or an $\mathfrak{s}$-strategy. (**b**) It cannot happen that one of the players has a $\mathfrak{g}$-strategy, and both remaining players have an $\mathfrak{s}$-strategy. □

The above considerations show that indeed each of the 16 pairs listed in Figure 2 determines a possible semantic attribute of a PL^3-formula.

Like in the case of PL^3_0, also in connection with PL^3 the question of functional completeness naturally arises. Every PL^3-formula φ determines a map — let us call it $|\varphi|$ — from valuations to semantic attributes, namely $|\varphi| : V \mapsto P$, where $P = |\varphi, V|$. Let, then, **prop** be a finite set of propositional atoms, and f any map from valuations on **prop** to the 16 semantic attributes. The question then is whether there always is a formula φ of PL^3 such that $f = |\varphi|$. We leave this as an open question.

Case of n players. Could PL^3 be naturally generalized to the case of n players? Insofar as the payoff function is considered, the answer is affirmative. There would be $n!$ possible rankings of the n players. The semantic attribute of a formula relative to a valuation would be defined by specifying for each player his or her optimal minimum rank in the correlated game. It would be straightforward to define the semantics of disjunction symbols. By contrast, it is less evident how to effect a generalization with respect to negations. If $\mathbf{P}_n$ is the set of all distributions of the n roles to the n players, the requirement would be to choose a minimal set of bijections of type $\mathbf{P}_n \longrightarrow \mathbf{P}_n$ which would suffice for defining all bijections of type $n \longrightarrow n$. The fact that the relevant set of permutations of role distributions is not uniquely determined can be seen as diminishing the theoretical interest of PL^3. Another theoretically problematic feature of PL^3 and its generalizations is that if we are willing to climb to the level of permutations of role distributions, why should not all permutations of role distributions be studied within the logic? In the n-player case there are $(n!)!$ of them.

3 Basic features of PL^3_0 or PL^3

Let L be one of the logics PL^3_0 or PL^3. Valuations for L will be called *L-valuations*. The possible semantic attributes of L-formulas will be called *L-attributes*. In what follows, L-formulas φ and ψ are said to be *logically equivalent*, in symbols $\varphi \iff \psi$, if for all L-valuations V and L-attributes A, we have that $|\varphi, V| = A$ iff $|\psi, V| = A$. Formulas $\varphi, \psi \in L$ are *incompatible*, if there is no L-valuation V such that $|\varphi, V| = |\psi, V|$. Let us proceed to make some basic observations about the behavior of negations, literals, and the definability of semantic attributes in the logics PL^3_0 and PL^3.

3.1 Behavior of negations

In PL^2, the negation $\neg$ obeys the law of double negation: φ and $\neg\neg\varphi$ are logically equivalent, for any formula φ. Let us consider what types of laws hold for iterations of negation symbols in PL^3_0 and PL^3.

Let us begin with PL^3_0. Let $\varphi \in PL^3_0$ be arbitrary. We have already seen that whenever i, j are distinct roles, the formulas $\sim_{ij}\varphi$ and $\sim_{ji}\varphi$ are logically equivalent. Furthermore, it is easily observed that the following *laws of double negation* hold in PL^3_0: $\sim_{ij}\sim_{ij}\varphi \iff \varphi$.

What about the interaction of different negation signs? To begin with, it is readily checked that $\sim_{01}\sim_{02}\varphi \iff \sim_{02}\sim_{12}\varphi \iff \sim_{12}\sim_{01}\varphi$, and $\sim_{01}\sim_{12}\varphi \iff \sim_{02}\sim_{01}\varphi \iff \sim_{12}\sim_{02}\varphi$.

Obviously all six role distributions $\rho \in \{a, b, c\}^{\{0,1,2\}}$ can be expressed in terms of the three negations $\sim_{01}$, $\sim_{02}$, and $\sim_{12}$. That is, for any such ρ there is a string $\bar{n}$ of length at most 2, formed in the alphabet $\{\sim_{01}, \sim_{02}, \sim_{12}\}$, such that if $\pi_{\bar{n}} \in \{0, 1, 2\}^{\{0,1,2\}}$ is the permutation corresponding to the string of negations $\bar{n}$, then $\rho = (\rho_0 \circ \pi_{\bar{n}})$, where ρ_0 is the standard initial role distribution.[9] Identifying role distributions with the triples of roles they give to *Alice*, *Bob* and *Cecile* in this order, we may note that for the role distributions $(0, 1, 2)$, $(0, 2, 1)$, $(1, 0, 2)$, $(1, 2, 0)$, $(2, 1, 0)$ and $(2, 0, 1)$, the corresponding strings $\bar{n}$ are:[10] empty string; $\sim_{12}$; $\sim_{01}$; $\sim_{01}\sim_{02}$; $\sim_{02}$; and $\sim_{01}\sim_{12}$. It should be observed that from the viewpoint of expressive power, not all negations $\sim_{01}$, $\sim_{02}$ and $\sim_{12}$ are actually *needed* for expressing all role distributions. For instance $\sim_{12}$ is definable in terms of $\sim_{01}$ and $\sim_{02}$: namely, $\sim_{12}\varphi \iff \sim_{01}\sim_{02}\sim_{01}\varphi$, for any formula φ.

Negations of the logic PL^3 behave rather differently from the case of PL^3_0. Let $\varphi \in PL^3$ be arbitrary. It is straightforward to see that $\sim$ obeys the *law of double negation*, $\sim\sim\varphi \iff \varphi$, while $\neg$ obeys the *law of triple negation*: $\neg\neg\neg\varphi \iff \varphi$. Further, the relative order of the different negation symbols obeys the *law of*

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[9] If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions, the composite function $(g \circ f) : A \longrightarrow C$ satisfies: $(g \circ f)(a) = g(f(a))$, for any $a \in A$.

[10] Instead of the combination $\sim_{01}\sim_{02}$, we might just as well use either $\sim_{02}\sim_{12}$ or $\sim_{12}\sim_{01}$; and in place of $\sim_{01}\sim_{12}$ either $\sim_{02}\sim_{01}$ or $\sim_{12}\sim_{02}$ could be used.

order invariance: $\sim\neg\varphi \iff \neg\sim\varphi$. (That is, $\pi_\neg$ and $\pi_\sim$ commute.) By contrast, the following six formulas are pairwise incompatible: φ, $\neg\varphi$, $\neg\neg\varphi$, $\sim\varphi$, $\neg\sim\varphi$, $\neg\neg\sim\varphi$. Note that on the other hand, by the *law of order invariance* we have: $\sim\neg\neg\varphi \iff \neg\sim\neg\varphi \iff \neg\neg\sim\varphi$. Inspecting the definitions of the permutations $\pi_\neg$ and $\pi_\sim$ (given in Fig. 1), it is evident that the two negations $\neg$ and $\sim$ are *not* interdefinable.

3.2 Literals

In PL^2, there are for each propositional atom p precisely two mutually incompatible literals, namely p and $\neg p$. Let L be either of the logics PL^3_0 or PL^3. If $p \in \mathbf{prop}$, a formula ℓ of L is a *p-literal*, if ℓ is syntactically built from p using the negation symbols of L only. A set Λ of p-literals is *basic*, if (*i*) any two elements in Λ are pairwise incompatible, and (*ii*) every p-literal is logically equivalent to some element of Λ. (Recall the definition of incompatibility from the beginning of the present section.) Extending this terminology to PL^2, for instance sets $\{p, \neg p\}$ and $\{\neg\neg p, \neg p\}$ are both basic sets of p-literals, while the sets $\{\neg p\}$ and $\{p, \neg p, \neg\neg p\}$ are not. A formula of L is a *literal*, if it is a p-literal for some atom p. What interests us in connection with L is determining whether *some* basic set of p-literals exists, for any given atom p.

Consider PL^3_0 first. By observations made in *Subsection* 3.1, in order to find out whether PL^3_0 admits of basic sets of p-literals in the first place, it suffices to check whether $X := \{p, \sim_{01} p, \sim_{02} p, \sim_{12} p, \sim_{01}\sim_{02} p, \sim_{01}\sim_{12} p\}$ is such a set. Let S be either $\{a, b, c\}$ or $\varnothing$. Now if $V(p) = \rho_0^{-1}(S)$, then actually the semantic attribute of *all* formulas of the set X is S, relative to V. It follows that relative to arbitrary valuations, there simply exists no basic set of p-literals at all.

What about considering restricted valuations only — valuations whose value on an atom is neither $\varnothing$ nor $\{0, 1, 2\}$? Under this assumption, if $V(p) = R$ and $S = \rho_0(R)$, we can find distinct roles i, j such that $\{i, j\}$ coincides either with R or with $\{0, 1, 2\} \setminus R$. In both cases $|p, V| = |\sim_{ij} p, V| = S$. It follows that the p-literals p, $\sim_{01} p$, $\sim_{02} p$ and $\sim_{12} p$ are *not* pairwise incompatible in the sense that under no valuation no two of them would have the same semantic attribute. Actually, under any restricted valuation exactly two of them have the same attribute. Hence no basic set of p-literals exists, even when attention is confined to restricted valuations. With respect to literals, PL^3_0 behaves, then, very differently from PL^2.

By contrast, PL^3 comes closer to PL^2 in its behavior with respect to literals. The p-literals p, $\neg p$, $\neg\neg p$, $\sim p$, $\neg\sim p$ and $\neg\neg\sim p$ are indeed pairwise incompatible, and moreover any further p-literal is equivalent to one of them. Therefore these formulas form a basic set of p-literals in PL^3.

3.3 Capturing semantic attributes

In PL^2, both truth-values can be *captured* by a formula — or are *definable* — in the sense that there is a formula $\top$ of PL^2 whose truth-value is *true* under all valuations, and likewise there is a formula $\bot$ of PL^2 whose truth-value is *false* under all valuations. For instance $(p \vee \neg p)$ is such a formula $\top$, and $(p \wedge \neg p)$ such a formula $\bot$. Letting L be one of the logics PL^3_0 or PL^3, let us now consider the question whether all L-attributes can be similarly captured within L, that is, whether for each L-attribute A there is a formula φ_A of L such that $|\varphi_A, V| = A$, for all L-valuations V.

Let us begin by considering the logic PL^3_0. First we may notice that relative to arbitrary valuations such capturing is *not* possible in PL^3_0. This is a corollary to the following lemma:

LEMMA 11. *Let R be either $\{0, 1, 2\}$ or $\emptyset$; and let V be a valuation mapping all propositional atoms to the set R. Then all formulas of PL^3_0 receive, relative to V, the semantic attribute $S = \rho_0(R)$.*

Proof. By assumption the claim holds for atoms. Assuming inductively that the claim holds for formulas φ, ψ, it immediately follows that it also holds for formulas $(\varphi \vee_i \psi)$ and $\sim_{ij}\varphi$. ■

If $\{a, b, c\} \neq S \neq \emptyset$, then by Lemma 11 the semantic attribute S cannot be captured by any formula of PL^3_0. For, relative to a valuation as in the statement of the lemma, any formula will receive a semantic attribute other than S. What about turning attention to restricted valuations, then?

Consider the attribute $\{a\}$. Evidently under any restricted valuation V, *Alice* has a winning strategy in one of the games $G(p, V)$, $G(\sim_{01} p, V)$ and $G(\sim_{02} p, V)$. Now in order for a formula to have the semantic attribute $\{a\}$, it is required, not just that *Alice* has a winning strategy, but also that neither of the other two players has one. Actually the formula

$$(p \vee_0 \sim_{12} p) \;\vee_0\; (\sim_{01} p \vee_0 \sim_{01}\sim_{02} p) \;\vee_0\; (\sim_{02} p \vee_0 \sim_{01}\sim_{12} p)$$

captures the attribute $\{a\}$ relative to restricted valuations. To see this, let V be any such valuation. If $0 \in V(p)$, then already by her first choice *Alice* can guarantee that she will win: by choosing the leftmost disjunct. What is more, by her next choice she may prevent any of the other two players from winning. Namely, since V is a restricted valuation, either 1 or 2 falls outside of $V(p)$. Hence neither *Bob* nor *Cecile* has a winning strategy in both games $G(p, V)$ and $G(\sim_{12} p, V)$, and consequently neither of them has a winning strategy for $(p \vee_0 \sim_{12} p)$, since it is *Alice* who chooses the disjunct. Similarly, if $1 \in V(p)$, *Alice* may choose the middle disjunct, and if $2 \in V(p)$, the rightmost disjunct, being able to make sure by her remaining move that neither of the other players can also win the play.

The attributes $\{b\}$ and $\{c\}$ are captured similarly. On the other hand, by Lemma 12 the attributes $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ are *not* definable.

LEMMA 12. *Suppose $|V(p)| = 1$ for all propositional atoms p.*[11] *Then no formula φ of PL^3_0 has relative to V an attribute S, for any S with $|S| = 2$.*

Proof. Suppose V satisfies the premise of the lemma. Then the claim holds for atomic formulas. Go on to assume inductively that the claim holds for formulas φ, ψ, and let i, j be any two roles. Consider the formula $(\varphi \vee_i \psi)$. By inductive hypothesis, in both games $G(\varphi, V)$ and $G(\psi, V)$ at most one player has a winning strategy. Clearly the number of players with a winning strategy in game $G(\varphi \vee_i \psi, V)$ can be at most $\max\{n_\varphi, n_\psi\}$, where $n_\theta \leq 1$ is the number of players having a winning strategy in $G(\theta, V)$, with $\theta \in \{\varphi, \psi\}$. Further, given the inductive hypothesis, evidently at most one player has a winning strategy in $G(\sim_{ij}\varphi, V)$. ■

As witnessed by the fact that the singleton attributes S can be captured relative to restricted valuations, the 'mirror image' of Lemma 12 does *not* hold: it *can* very well happen that a formula has the semantic attribute S for a singleton S, even relative to a valuation V with $|V(p)| = 2$, for all atoms p. Generally, the number of players having a winning strategy for a complex formula can very well be smaller than the number of players having a winning strategy for its components.

Let us, then, take a look at the logic PL^3. As regards capacity to capture semantic attributes, PL^3 turns out to differ from PL^3_0. Actually, each of the 16 semantic attributes of PL^3 can be captured.

First think of the attribute (a, b). Divide the p-literals into three 'cells': $\{p, \sim p\}$, $\{\neg p, \neg\sim p\}$, $\{\neg\neg p, \neg\neg\sim p\}$, and observe that under any valuation, there is a cell such that *Alice* receives the highest rank relative to both formulas of that cell. For one of the formulas in that cell, it is *Bob* who becomes second in the ranking, while for the other, *Cecile* becomes second. So the formula $(p \vee_1 \sim p) \vee_0 (\neg p \vee_1 \neg\sim p) \vee_0 (\neg\neg p \vee_1 \neg\neg\sim p)$ receives under any valuation the attribute (a, b). For, *Alice* may make sure that she becomes first, whereafter *Bob* can guarantee that he becomes second. The attributes P with $P \in \{(a, c), (b, a), (b, c), (c, a), (c, b)\}$ are captured similarly.

Let us, then, consider the case of $(a, ?)$. This attribute is simply captured by the formula $(p \vee_0 \neg p \vee_0 \neg\neg p \vee_0 \sim p \vee_0 \neg\sim p \vee_0 \neg\neg\sim p)$. Namely, under any valuation, there are two 0-disjuncts which *Alice* can choose so that she herself reaches the highest rank. In one of them *Bob* becomes second, while in the other *Cecile* becomes second. *Bob* and *Cecile* have no control over which 0-disjunct *Alice* chooses. The attributes $(b, ?)$ and $(c, ?)$ are captured similarly.

For the remaining attributes, we state the relevant formulas; the reader is invited to check in detail that indeed they fit the bill. Let $\Lambda := \{p, \neg p, \neg\neg p, \sim p, \neg\sim p,$

[11] If X is a set, $|X|$ stands for its cardinality.

$\neg\neg{\sim}p\}$, and define formulas C_1 and C_2 by setting:

$$C_1 := \bigvee_{\ell \in \Lambda}{}_0 (\ell \vee_1 \neg \ell) \ \text{ and } \ C_2 := \bigvee_{\ell \in \Lambda}{}_0 ({\sim}\ell \vee_1 \neg \ell).$$

The formula $(C_1 \vee_0 C_2)$ captures $(?, a)$; while $(?, b)$ and $(?, c)$ are captured similarly. Further, $(?, ?)$ is captured by the formula $(D_1 \vee_0 D_2)$, where

$$D_1 := \bigvee_{\ell \in \Lambda}{}_1 \ \ell \ \text{ and } \ D_2 := \bigvee_{\ell \in \Lambda}{}_2 \ \ell.$$

Finally, the attribute $(?, !_{a,c})$ is captured by

$$[(p \vee_2 {\sim}p) \vee_1 (\neg p \vee_2 \neg{\sim}p)] \ \vee_0 \ [(\neg p \vee_2 \neg{\sim}p) \vee_1 (\neg\neg p \vee_2 \neg\neg{\sim}p)] \\ \vee_0 \ [(\neg\neg p \vee_2 \neg\neg{\sim}p) \vee_1 (p \vee_2 {\sim}p)],$$

the attributes $(?, !_{a,b})$ and $(?, !_{b,c})$ being captured similarly.

3.4 The notion of consequence

In connection with PL^2, logical consequence is defined in terms of truth-preservation: ψ is a logical consequence of φ, if all valuations making φ true, make ψ true as well. The game-theoretic content of this condition is as follows: for every valuation V, if there is a winning strategy for *Eloise* in $G(\varphi, V)$, there also is one for her in $G(\psi, V)$. A corresponding consequence relation for *Abelard* is defined by the requirement that for all valuations V, whenever there is a winning strategy for *Abelard* in $G(\varphi, V)$, there is one for him in $G(\psi, V)$. The relation defined by this condition is the converse of the relation of logical consequence, and is characterized by falsity-preservation.

The notion of consequence relation is naturally generalized to the multi-player setting by introducing one consequence relation for *each player*. Let $\mathcal{P}$ be one of the three players. It is said that ψ is a *$\mathcal{P}$-consequence of φ in PL^3_0*, provided that all valuations V satisfy: if $\mathcal{P}$ has a winning strategy in $G(\varphi, V)$, then $\mathcal{P}$ also has one in $G(\psi, V)$. This condition can be otherwise expressed thus: for all valuations V, if $\mathcal{P} \in \mid \varphi, V \mid$, then $\mathcal{P} \in |\psi, V|$.

In games corresponding to PL^3, the payoffs are in terms of rankings. Accordingly, what is of interest to a given player $\mathcal{P}$ on the level of strategies is the optimal minimum rank that $\mathcal{P}$ can guarantee by a suitable choice of strategy. In *Section* 2 the relevant notions of strategy were conceptualized in terms of the notions of $\mathfrak{g}$-strategy and $\mathfrak{s}$-strategy. (Alternatively, we might speak of security levels $\mathfrak{g}$ and $\mathfrak{s}$, respectively, as noted in Remark 3.) Now ψ is said to be a *$\mathcal{P}$-consequence of φ in PL^3*, if for all valuations V, the optimal minimum rank that $\mathcal{P}$ can guarantee in $G(\psi, V)$ is the same or better than the optimal minimum rank that $\mathcal{P}$ can guarantee in $G(\varphi, V)$. In terms of semantic attributes, this condition means the following. Suppose that $|\varphi, V| = (x, y)$ and $|\psi, V| = (x', y')$. Then if $x = \mathcal{P}$, also

$x' = \mathcal{P}$, while if y marks $\mathcal{P}$ as a player with an $\mathfrak{s}$-strategy, then either $x' = \mathcal{P}$ or y' marks $\mathcal{P}$ as a player with an $\mathfrak{s}$-strategy.

From the viewpoint of PL^2 one might equally well think of a generalization where *each semantic attribute* would induce its own consequence relation. We believe, however, that the generalization in terms of players is the fruitful one. In particular, when consequence relations for PL_0^3 and PL^3 are defined as above, they seem to have good algebraic counterparts in terms of semi-lattices. Systematically studying the consequence relations is left for future research.

4 Further features

The logic PL^2 satisfies the *law of excluded middle*: for any valuation V and any formula φ, we have $V \models (\varphi \vee \neg\varphi)$. Furthermore, PL^2 is subject to the *semantic principle of bivalence*: relative to any valuation, any formula φ is either true or false.[12] Let us take a look whether the logics PL_0^3 and PL^3 admit of analogous logical laws and semantic principles. We also take up the question whether analogues to De Morgan's laws and distributive laws can be formulated in PL_0^3 and PL^3.

The following definitions will be needed subsequently. Let L be any of the logics PL^2, PL_0^3 or PL^3. If φ is a formula of L, let us agree to write $L(\varphi)$ for the class of L-formulas that can be formed from (any number of tokens of) the formula φ using the operators of L. Hence for instance $\sim_{01}(p \vee_0 q) \in L(p \vee_0 q)$, but $p \notin L(p \vee_0 q)$.

DEFINITION 13 (Uniform characterization; characterizability). If $\chi \in L(p)$, write f_χ for the syntactic transformation of type $L \longrightarrow L$ defined by the following condition: for all $\varphi \in L$,

$$f_\chi(\varphi) = \chi[p/\varphi],$$

where $\chi[p/\varphi]$ stands for the result of substituting everywhere φ for p in χ. Let, then, A_0 be a fixed L-attribute, and A an arbitrary L-attribute. It is said that a formula $\chi \in L(p)$ *uniformly characterizes* A in terms of A_0, if for all formulas $\psi \in L$ and L-valuations V,

$$|\psi, V| = A \text{ iff } |f_\chi(\psi), V| = A_0.$$

It is merely said that A *can be characterized* in terms A_0, if for all $\psi \in L$, there is $\chi_\psi \in L$ such that for all V: $|\psi, V| = A$ iff $|\chi_\psi, V| = A_0$. □

Whenever a formula $\chi \in L(p)$ uniformly characterizes A in terms of A_0, the attribute A can of course be characterized in terms of A_0, but the converse does not hold. Crucially, in connection with the stronger notion, the *form* of the formula

[12] As stressed notably by Dummett (see, e.g., [6]), logical laws must be distinguished from semantic principles. From the perspective of the present paper, what is crucial about logical laws is that they are expressed in terms of a designated truth-value (*true*).

$f_\chi(\psi)$ does *not* depend on the particular input ψ: the form of $f_\chi(\psi)$ is that of χ, modulo ψ being substituted everywhere for p. With the weaker notion, again, there is no guarantee of syntactic similarity between formulas χ_ψ corresponding to distinct formulas ψ.

4.1 Analogues of the law of excluded middle

In PL^2, the truth-value *true* is captured by the formula $(p \vee \neg p)$. As it happens, the atom p can be replaced by an arbitrary formula φ, and the result will continue to capture *true*. This fact alone shows that the law of excluded middle holds for PL^2. In the case of a logic subject to the semantic principle of bivalence, the law of excluded middle can be understood as providing at the object language level an exhaustive list of the semantic values that an arbitrary formula φ may have. Expressing the different truth-values on the object language level is possible in PL^2, since falsity of a formula can be expressed by truth of another formula. Let us consider whether similar phenomena occur in the 3-player setting.

Logic PL^3_0. A necessary condition for the existence of an analogue of the law of excluded middle in PL^3_0 would be that any semantic attribute S of any given formula χ could be expressed in terms of some *designated* attribute S_0 of some other formula ψ^χ_S, i.e., that for all valuations V, we had $|\chi, V| = S$ iff $|\psi^\chi_S, V| = S_0$. Let us take $\{a\}$ as the designated attribute. The choice is to some extent arbitrary. In particular, one might ask why the designated attribute should be a singleton rather than a pair. To this a possible reply would be that the former are better-behaved from the logic-internal viewpoint: singleton attributes are definable in PL^3_0, while two-element attributes are not (see Lemma 12). Let us now turn attention to propositional atoms, and show that indeed for each atom p there is a formula φ_p which lists all the six semantic attributes that p may have relative to *restricted valuations*. (By Lemma 11 we know such a result cannot hold with respect to arbitrary valuations.) We first prove a lemma.

LEMMA 14. *Let V be an arbitrary restricted valuation. Define formulas $\psi_{\{a,b\}}$, $\psi_{\{a,c\}}$ and $\psi_{\{b,c\}}$ as follows:*

$$\begin{aligned}
\psi_{\{a,b\}} &:= (p \vee_0 \sim_{12} p) \vee_2 (\sim_{01} p \vee_1 \sim_{12} p),\\
\psi_{\{a,c\}} &:= (p \vee_0 \sim_{12} p) \vee_1 (\sim_{02} p \vee_2 \sim_{12} p),\\
\psi_{\{b,c\}} &:= (\sim_{01} p \vee_1 \sim_{02} p) \vee_0 (\sim_{01} p \vee_2 \sim_{02} p).
\end{aligned}$$

For all sets $S \in \{\{a,b\},\{a,c\},\{b,c\}\}$: $|p, V| = S$ iff $|\psi_S, V| = \{a\}$.

Proof. Let us check the case $S := \{a,b\}$; the other cases can be proven similarly. Let V be a restricted valuation. Suppose first that $|\psi_{\{a,b\}}, V| = \{a\}$. Hence *Alice* has a winning strategy in both games $G(\sim_{01} p, V)$ and $G(\sim_{12} p, V)$. Therefore $0, 1 \in V(p)$, and since V is restricted, in fact $V(p) = \{0,1\}$. So $|p, V| = \{a,b\}$. Conversely, suppose $|p, V| = \{a,b\}$. Thus *Alice* has a winning strategy in all games

$G(\sim_{01} p, V)$, $G(\sim_{12} p, V)$ and $G(p, V)$, and therefore in $G(\psi_{\{a,b\}}, V)$. What is more, *Bob* does not have a winning strategy in the game corresponding to $\psi_{\{a,b\}}$ (since *Cecile* may choose 'left' after which *Alice* may choose 'right'); and *Cecile* has no winning strategy either, because if she chooses 'left', *Alice* may choose 'left' to prevent *Cecile* from winning; and if she chooses 'right', *Bob* may choose 'left', with the consequence that *Cecile* loses. Hence indeed $|\psi_{\{a,b\}}, V| = \{a\}$. ■

THEOREM 15 (Atomic law of excluded seventh). *Let $S_1, \ldots, S_6$ be a list of all subsets of $\{a, b, c\}$ except $\{a, b, c\}$ and $\emptyset$. There is a formula $\varphi_p := (\psi_{S_1} \vee_0 \ldots \vee_0 \psi_{S_6})$ of PL^3_0 with p as its only atom, such that for all restricted valuations V,*

$$|\varphi_p, V| = \{a\}, \text{ and } |\psi_{S_i}, V| = \{a\} \text{ iff } |p, V| = S_i \ (1 \leq i \leq 6).$$

Proof. Let $\psi_{\{a\}} := p$, $\psi_{\{b\}} := \sim_{01} p$, and $\psi_{\{c\}} := \sim_{02} p$. Further, let $\psi_{\{a,b\}}$, $\psi_{\{a,c\}}$ and $\psi_{\{b,c\}}$ be as in the statement of Lemma 14. Then we may take φ_p to be the formula $(\psi_{\{a\}} \vee_0 \psi_{\{b\}} \vee_0 \psi_{\{c\}} \vee_0 \psi_{\{a,b\}} \vee_0 \psi_{\{a,c\}} \vee_0 \psi_{\{b,c\}})$. ■

The next thing to ask is whether the atomic law of excluded seventh can be generalized so as to apply to arbitrary formulas. Let us first formulate the question more precisely. To begin with, note that the law of excluded middle of PL^2 has the following general format: for any formula φ of PL^2, the formula $(f_p(\varphi) \vee f_{\neg p}(\varphi))$ is true under any valuation. Observe that p serves to uniformly characterize the truth-value *true* in terms of *true*, while $\neg p$ uniformly characterizes *false* in terms of *true*. (Recall the notion of uniform characterizability from Definition 13.) Looking for an analogue to the law of excluded middle in PL^3_0, we must first of all ask whether each of the relevant semantic attributes of PL^3_0 can be uniformly characterized in terms of a fixed attribute. But which *are* the relevant attributes?

As already noted at the end of *Subsection* 2.1, relative to restricted valuations the number of possible semantic attributes increases from 6 to 7 when considering arbitrary formulas instead of literals: there are complex formulas for which no player has a winning strategy, i.e., which have the attribute $\emptyset$ relative to suitable restricted valuations. It is not difficult to check that each of the $PL^3_0(p)$-formulas $\psi_{\{a\}}$, $\psi_{\{b\}}$, $\psi_{\{c\}}$, $\psi_{\{a,b\}}$, $\psi_{\{a,c\}}$, $\psi_{\{b,c\}}$ referred to in the proof of Theorem 15 uniformly characterizes the corresponding semantic attribute in terms of the attribute $\{a\}$. But is there a formula uniformly characterizing the attribute $\emptyset$ in terms of $\{a\}$? That is, can we find a formula χ such that $|\psi, V| = \emptyset$ iff $|f_\chi(\psi), V| = \{a\}$? By the argument given for Lemma 11, it is immediate that the answer is negative: if the attribute of ψ is $\emptyset$ relative to a valuation V, so will be the attribute of any formula of the class $PL^3_0(\psi)$.

We may conclude that a natural generalization of the law of excluded middle, applicable to arbitrary formulas, is *not* possible for PL^3_0. As a semantic principle we still have the principle of *7-valence*: any formula of PL^3_0 has, relative to restricted valuations, one of seven semantic attributes.[13]

So the attribute $\varnothing$ cannot be *uniformly* characterized in terms of $\{a\}$. Might it still not be possible to simply characterize $\varnothing$ in terms of $\{a\}$? We leave settling this issue for future research. The following example however shows that at least for some formulas ψ, a suitable formula χ_ψ can be found such that for all restricted valuations V: $|\psi, V| = \varnothing$ iff $|\chi_\psi, V| = \{a\}$.

EXAMPLE 16. Consider the formula $(p \vee_0 q)$, letting V be a restricted valuation. Evidently $|p \vee_0 q, V| = \varnothing$ iff: $[V(p) = \{1\}$ and $V(q) = \{2\}]$ or $[V(p) = \{2\}$ and $V(q) = \{1\}]$. Let, then, φ be the following formula:

$$(\sim_{01} p \vee_1 \sim_{02} q) \vee_0 (\sim_{01} q \vee_1 \sim_{02} p).$$

It is easy to check that $|\varphi, V| = \{a\}$ iff $|p \vee_0 q, V| = \varnothing$. □

Logic PL^3**.** Let us turn to PL^3. Does PL^3 have an analogue of the law of excluded middle? Again, an analogue is found when attention is restricted to propositional atoms. Let us take (a, b) as the designated attribute. (It has a clearly better claim on being a designated attribute than any attribute involving non-determinacy or overdeterminacy.) Let P_i denote the i-th pair in the list (a, b), (a, c), (b, a), (b, c), (c, a), (c, b); and let ψ_{P_i} stand for the i-th formula in the list p, $\sim p$, $\neg\neg\sim p$, $\neg\neg p$, $\neg p$, $\neg\sim p$. It is straightforward to check that $|\psi_{P_i}, V| = (a, b)$ iff $|p, V| = P_i$, for any valuation V $(1 \le i \le 6)$.

THEOREM 17 (Atomic law of excluded seventh). *There is a formula* $\varphi_p := (\psi_{P_1} \vee_1 \psi_{P_2}) \vee_0 (\psi_{P_3} \vee_1 \psi_{P_4}) \vee_0 (\psi_{P_5} \vee_1 \psi_{P_6})$ *of* PL^3 *with* p *as its only atom, such that for all valuations* V*,*

$$|\varphi_p, V| = (a, b), \text{ and } |\psi_{P_i}, V| = (a, b) \text{ iff } |p, V| = P_i \ (1 \le i \le 6).$$

Proof. Put $\varphi_p := (p \vee_1 \sim p) \vee_0 (\neg\neg\sim p \vee_1 \neg\neg p) \vee_0 (\neg p \vee_1 \neg\sim p)$. ■

When formulating the atomic law of excluded seventh for PL^3, two types of disjunction symbol are needed — unlike in the cases of PL^2 or PL^3_0. This is due to the 'combinatorial' nature of payoffs in games correlated with PL^3-formulas: not only must we express that one of the players has a $\mathfrak{g}$-strategy, but we also must express that another player has an $\mathfrak{s}$-strategy.

As in the case of PL^3_0, also in connection with PL^3 the next thing to ask is whether a variant of the atomic law of excluded seventh can be formulated which is applicable to arbitrary formulas. Recall that in PL^3 there are 16 semantic attributes

[13] Dummett [6, p. xix] remarks: "[W]hile acceptance of the semantic principle normally entails acceptance of the corresponding logical law, the converse does not hold." It may be of some interest to observe that by these criteria, PL^3_0 offers an instance of abnormality: while the semantic principle of 7-valence holds, the corresponding logical law, the law of excluded eighth, does not. (The latter cannot even be formulated.)

to consider: 10 attributes in addition to the 6 possible attributes P_i ($1 \leq i \leq 6$) of atomic formulas. It is not difficult to check that the above $PL^3(p)$-formula ψ_{P_i} uniformly characterizes the semantic attribute P_i in terms of the attribute (a, b) (with $1 \leq i \leq 6$). To see what happens with the further attributes, let us first prove a result that can be compared to Lemma 11, the case $R := \emptyset$. It should be noted, however, that whereas that lemma applies to all formulas of PL^3_0, the present result only pertains to complex PL^3-formulas capable of non-determinacy.

LEMMA 18. *Let* (x, y) *be any attribute such that* $\{x, y\} \cap \{?\} \neq \emptyset$. *Suppose* ψ *is a formula of* PL^3 *and* V *is a valuation such that* $|\psi, V| = (x, y)$. *Then for all formulas* $\chi \in PL^3(\psi)$, *we have that there is a pair* (x', y') *such that* $|\chi, V| = (x', y')$, *where:* ($x' = ?$, *if* $x = ?$) *and* ($y' = ?$, *if* $y = ?$).

Proof. By the premise of the lemma the claim holds for ψ. Assume inductively that it holds for $\theta, \theta' \in PL^3(\psi)$. No switching of roles can yield any of the players a z-strategy (with $z \in \{\mathfrak{g}, \mathfrak{s}\}$) in $G(\neg\theta, V)$ or in $G(\sim \theta, V)$, if no player has a z-strategy in $G(\theta, V)$. Similarly no player can have a z-strategy in $G(\theta \vee_i \theta', V)$ without having one either in $G(\theta, V)$ or in $G(\theta', V)$. ■

It follows from Lemma 18 that whenever at least one of x, y equals ?, no formula can uniformly characterize the attribute (x, y) in terms of the attribute (a, b). Consequently *only* the attributes P_i ($1 \leq i \leq 6$) can be uniformly characterized. Therefore we may conclude that a natural generalization of the law of excluded middle, applicable to arbitrary formulas, is *not* possible for PL^3 — just as we saw that it is not possible for PL^3_0. On the other hand, PL^3 is subject to the semantic principle of *16-valence*.[14]

Not all semantic attributes can, then, be uniformly characterized in terms of the attribute (a, b). Settling the issue whether all semantic attributes of PL^3 could, however, be characterized in the weak sense in terms of the attribute (a, b) is left for another occasion. Let us conclude the present considerations by looking at a particular semantic attribute involving non-determinacy, namely $(?, a)$. We may observe that actually there are formulas ψ for which another formula χ_ψ can be found such that $|\psi, V| = (?, a)$ iff $|\chi_\psi, V| = (a, b)$, for all valuations V.

EXAMPLE 19. Consider the formula $(p \vee_0 q)$, letting V be an arbitrary valuation. Let, then, φ be the following formula:

$$(\neg\neg{\sim}p \ \vee_2 \ \neg q) \ \vee_0 \ (\neg\neg{\sim}q \ \vee_2 \ \neg p).$$

Now $|\varphi, V| = (a, b)$ iff $|p \vee_0 q, V| = (?, a)$. Namely, $|\varphi, V| = (a, b)$ iff: either $|\neg\neg{\sim}p, V| = |\neg q, V| = (a, b)$, or $|\neg\neg{\sim}q, V| = |\neg p, V| = (a, b)$. This condition, again, is equivalent to requiring that either $V(p) = (1, 0, 2)$ and $V(q) = (2, 0, 1)$,

[14]Hence also PL^3 is 'abnormal', in the way PL^3_0 was observed to be in footnote 13.

or else $V(q) = (1, 0, 2)$ and $V(p) = (2, 0, 1)$, which is simply equivalent to the condition $|p \vee_0 q, V| = (?, a)$. □

We have now seen that in the logics PL^3_0 and PL^3, those semantic attributes that are possible attributes of *propositional atoms* can be uniformly characterized in terms of the fixed attribute $\{a\}$ *resp.* (a, b). By contrast, in both cases all attributes that only emerge in connection with complex formulas defy *uniform* characterization. This fact is perhaps made understandable by the negative character of those attributes: each of them involves the failure of all players to have a strategy of a certain type. The particular reason *why* in each relevant case this failure takes place is *not* a straightforward matter of logical form. The 'explanation' of the failure is sensitive to the syntax of the formula in a less robust way, in a way that precludes the possibility of uniform characterizability.

An important closure property that can be formulated in terms of a designated attribute A_0 is closure under 'complementary negation'. If L is one of the logics PL^3_0 or PL^3, it may be asked whether for every formula φ of L there is a formula $neg(\varphi)$ of L such that every valuation V satisfies: $|neg(\varphi), V| = A_0$ iff $|\varphi, V| \neq A_0$. We conjecture that for neither logic PL^3_0 or PL^3 can an attribute A_0 be so chosen as to admit of this property; settling the issue is left for another occasion.

4.2 Normal forms

De Morgan's laws for PL^2 enable transforming formulas into *negation normal form*, and distributive laws allow putting formulas, already in negation normal form, further into *disjunctive* and *conjunctive normal forms*. All these normal forms admit of analogues in the 3-player setting.

Let us first consider the logic PL^3_0. If i, j are distinct roles and ψ, χ any formulas, the following De Morgan's laws hold:

$$\begin{array}{ll} \sim_{ij}(\psi \vee_i \chi) \Longleftrightarrow (\sim_{ij}\psi \vee_j \sim_{ij}\chi) & \\ \sim_{ij}(\psi \vee_j \chi) \Longleftrightarrow (\sim_{ij}\psi \vee_i \sim_{ij}\chi) & \\ \sim_{ij}(\psi \vee_k \chi) \Longleftrightarrow (\sim_{ij}\psi \vee_k \sim_{ij}\chi) & \text{if } i \neq k \neq j \end{array}$$

By successive applications of these laws, together with the laws of double negation, any formula can be brought to an equivalent form where negation symbols only appear in literals, and the literals furthermore contain at most two negation signs. Such a formula is said to be in *negation normal form*, and can be formed from literals of the forms p, $\sim_{01} p$, $\sim_{02} p$, $\sim_{12} p$, $\sim_{01}\sim_{02} p$, $\sim_{01}\sim_{12} p$ using binary connectives $\vee_0$, $\vee_1$ and $\vee_2$ only.

Consider, then, the logic PL^3. As already noted in *Subsection* 2.2, the map $\pi_\neg$ is in effect a permutation of roles. For any argument $(r, r', r'') \in \mathbf{P}$, it does the same: *replaces* 0 *by* 1, 1 *by* 2, and 2 *by* 0. Define an operation $\ominus$ by putting $0 \ominus 1 = 2$, $1 \ominus 1 = 0$, and $2 \ominus 1 = 1$. The rule for driving $\neg$ deeper is this: if i is any role, then $\neg(\varphi \vee_i \psi) \Longleftrightarrow (\neg\varphi \ \vee_{i\ominus 1} \ \neg\psi)$.

The behavior of $\sim$ is trickier: $\pi_\sim$ is genuinely a permutation of role distributions, as opposed to a permutation of roles. The rule for driving the negation $\sim$ deeper is sensitive to the relative location of the occurrence of $\sim$ within the larger formula considered. To describe the effect of the map $\pi_\sim$, we must take into account *who* is having which role, or to be more precise, we must know which role *Alice* is having. What $\pi_\sim$ does is this: it keeps *Alice*'s role intact, no matter which role she has, while the roles of *Bob* and *Cecile* are interchanged. Since $\sim$ precisely does not affect *Alice*'s role, her role relative to a subformula token $\sim\psi$ only depends on the number of occurrences of the negation $\neg$ to which $\sim\psi$ is subordinate in the relevant larger formula χ.

Let us write $n[\chi,\theta]$ for the number of occurrences of $\neg$ to which a given subformula token θ is subordinate in a formula χ. Define, then, $a[\chi,\theta]$ as the unique number $m \in \{0,1,2\}$ such that

$$n[\chi,\theta] \equiv m \pmod 3.$$

It is immediate that (relative to the standard initial role distribution) *Alice*'s role at a subformula token θ in χ is $a[\chi,\theta]$.

If i is a role and $\sim(\varphi \vee_i \psi)$ is a subformula token in χ, put

$$i^* := \begin{cases} i, \text{ if } i = a[\chi,(\varphi \vee_i \psi)] \\ j, \text{ with } i \neq j \neq a[\chi,(\varphi \vee_i \psi)], \text{ otherwise} \end{cases}$$

If it is, e.g., *Bob* who has the role i at $(\varphi \vee_i \psi)$, then i^* is *Bob*'s role at the negated subformula token $\sim(\varphi \vee_i \psi)$. The following rule allows pushing $\sim$ deeper in a formula: if $\sim(\varphi \vee_i \psi)$ is a subformula token in a formula χ, it may be replaced in χ by the formula $(\sim\varphi \vee_{i^*} \sim\psi)$, and the resulting formula will be logically equivalent to χ. Observe that since the result of pushing an occurrence of $\sim$ deeper in a formula χ is sensitive to the relative location of this very occurrence within χ, the rule under consideration actually must be formulated for subformula tokens relative to a given larger formula.

By applying the given rules for driving the negation signs deeper, together with the laws of double negation, triple negation and order invariance, one can produce out of any PL^3-formula a logically equivalent PL^3-formula in which the negation signs $\neg$ and $\sim$ appear on the atomic level only, and which can be built from literals of the forms p, $\neg p$, $\neg\neg p$, $\sim p$, $\neg\sim p$, $\neg\neg\sim p$ using only the binary connectives $\vee_0$, $\vee_1$ and $\vee_2$. Such a formula of PL^3 is said to be in *negation normal form*.

Let L be either of the logics PL^3_0 or PL^3. Having seen that L admits of a negation normal form, let us proceed to ask whether it allows for analogues of disjunctive and conjunctive normal forms of PL^2. To begin with, straightforward distribution laws hold in L for any distinct roles i and j:

$$\varphi \vee_i (\psi \vee_j \chi) \Longleftrightarrow (\varphi \vee_i \psi) \vee_j (\varphi \vee_i \chi).$$

Provided that (i, j, k) is a triple of pairwise distinct roles, a formula is said to be in (i, j, k) *normal form*, if it has the form

$$\bigvee_i^{x\in I} \bigvee_j^{y\in J_x} \bigvee_k^{z\in K_{xy}} \theta_{xyz},$$

where I, the J_x and the K_{xy} all are finite sets of natural numbers, and θ_{def} is a literal of L, for any assignment of suitable values d, e, f to the variables x, y, z, respectively. For all six permutations (i, j, k) of the set $\{0, 1, 2\}$ and any formula φ of L, there is a logically equivalent formula $\psi_\varphi \in L$ which is in (i, j, k) normal form. This follows immediately from the distribution laws together with the De Morgan's laws.

5 Computing semantic attributes

Model-checking PL^2 can be done in linear time: there is an algorithm which, given a formula φ and valuation V as inputs, decides whether the relation $V \models \varphi$ holds or not; and the number of computation steps the algorithm uses — computation time — is bounded by a linear function of the size of φ. (The size of a formula is the number of tokens of symbols it contains.) It turns out that in this respect, no additional difficulties arise from moving to the logics PL^3_0 and PL^3.

THEOREM 20. *Let L be either PL^3_0 or PL^3. There is an algorithm such that given any L-formula φ and any L-valuation V, the algorithm computes the attribute $|\varphi, V|$ in computation time linear in the size of φ.*

Proof. First note that each L-formula φ has a uniquely determined *syntactic tree*. The root of the tree is the formula φ itself, and its leaves are propositional atoms. Any node $(\psi \vee_i \chi)$ has two immediate successors, namely ψ and χ; and any node $\sim_{ij} \psi$ (case $L := PL^3_0$) *resp.* any node $\neg\psi$ or $\sim\psi$ (case $L := PL^3$) has ψ as its unique immediate successor. Fix, then, a formula $\varphi \in L$ and an L-valuation V. The attribute $|\varphi, V|$ may be determined by labeling the nodes of the syntactic tree of φ. First label each leaf p with the attribute $V(p)$. Then work the way towards the root of the tree by computing the semantic attribute of each complex node from its simpler parts: the attributes of ψ and χ determine in a straightforward way the attribute of $(\psi \vee_i \chi)$, and the attribute of ψ determines the attribute of the result of applying any of the relevant negations to ψ. In this way the formula φ receives a label; this label is the attribute $|\varphi, V|$. Since the number of nodes cannot exceed the size of φ, the algorithm just described runs in time linear in the size of the input formula φ. ■

As pointed out in the proof of Theorem 20, it is straightforward to formulate recursive rules that determine the semantic attribute of an arbitrary formula of PL^3_0 or PL^3 from the attributes of its atomic components. That is to say, nothing prevents from defining the semantics of these 3-player logics by laying down a set of recursive semantic clauses — in perfect analogy to the usual recursive rules defining the semantics of PL^2. It should, however, be noted that in the case of PL^3_0 and PL^3, the motivation of such recursive rules comes from their relation to the game-theoretically defined semantics.

6 Issues of satisfiability and validity[15]

Let us take a look at some possible further developments in the context of 3-player propositional logic. More specifically, let us see how various issues related to satisfiability and validity can be studied in connection with PL^3.

6.1 Tableaus

What would *proof theory* of PL^3 look like? Let us restrict attention to formulas in negation normal form. A proof system for such formulas can be defined in terms of a set of tableau rules.[16] Our tableaus will deal with 'signed formulas'. There are two basic signs T and F as usual, but they appear relativized both to a player (a, b, or c) and to a feature (g for 'having a $\mathfrak{g}$-strategy', s for 'having an $\mathfrak{s}$-strategy'): the 12 signs $T^g_a, T^g_b, T^g_c, F^g_a, F^g_b$, $F^g_c, T^s_a, T^s_b, T^s_c, F^s_a, F^s_b, F^s_c$ are introduced. A *signed formula* is any expression $S^y_x\varphi$ with $S \in \{T, F\}$, $y \in \{g, s\}$ and $x \in \{a, b, c\}$, where φ is a formula of PL^3. A set B of signed formulas is *realizable*, if there is a valuation V such that any signed formula $S^y_x\varphi \in B$ satisfies: player x has ($S := T$) *resp.* lacks ($S := F$) a y-strategy in game $G(\varphi, V)$. Then V is said to *realize* the set B. From the fact that V realizes B, it does *not* follow that all formulas φ with $S^y_x\varphi \in B$ (for some S, y, x) have the same semantic attribute relative to V. It only follows that the requirements induced by the signed formulas in B can all be simultaneously satisfied by one and the same valuation.

No single sign S^y_x will correspond to a semantic attribute of PL^3. On the other hand, the attributes of PL^3-formulas can be analyzed by reference to the players: an attribute is expressed by specifying for each player his or her optimal minimum rank in the relevant game. It is precisely for dealing with such an analysis that the 12 signs are introduced. An example may clarify the idea. A formula φ receives the attribute (a, b) under all valuations, if for all valuations V, it is impossible for a to fail having a $\mathfrak{g}$-strategy in $G(\varphi, V)$, for b to fail having an $\mathfrak{s}$-strategy in $G(\varphi, V)$, and for c to have an $\mathfrak{s}$-strategy in $G(\varphi, V)$. The idea will be that in order to check whether these three conditions are met, it suffices to see whether the tableaus for the following three signed formulas are 'closed': $F^g_a\varphi$, $F^s_b\varphi$, $T^s_c\varphi$. Note that there is no obvious way in which to dispense with the signs F^y_x in favor of the two negations of PL^3 and the signs T^y_x. The tableau for $F^g_a\varphi$, for instance, will have an 'open' branch iff there is a valuation V such that a fails to have a $\mathfrak{g}$-strategy in $G(\varphi, V)$. It is far from evident that there are $y \in \{g, s\}$ and $x \in \{a, b, c\}$ such that for all φ there

[15] *Sect.* 6 describes continuing work by one of the authors (TT), cf. [15].

[16] For a classical presentation of tableau systems, see [14].

is φ' satisfying: a tableau for $T^y_x\varphi'$ has an 'open' branch iff a tableau for $F^g_a\varphi$ has one. It is an open question whether PL^3 enjoys such a player-relative property of 'closure under complementation'; we conjecture that it does not.

There will be two types of tableau rules: those that allow extending a branch, and those that allow closing a branch, or schematically:

$$\begin{array}{|c|}\hline S^y_x\varphi \\ | \\ \mathfrak{B} \\ \hline\end{array} \qquad \begin{array}{|c|}\hline B \\ | \\ X \\ \hline\end{array}$$

In the left rule, we have $S \in \{T, F\}$, $y \in \{g, s\}$, $x \in \{a, b, c\}$, and $\mathfrak{B}$ is a set of sets of signed formulas. In the right rule, B is a set of signed formulas and X is a specific additional symbol. The rules of the latter kind are termed 'closing rules'. Rules of the former kind will be so chosen that they can be proven to satisfy the following two properties: (P1) The singleton set $\{S^y_x\varphi\}$ is realized by a valuation V *only if* one of the branches $B \in \mathfrak{B}$ is realized by V; and (P2) If some of the sets $B \in \mathfrak{B}$ is realized by a valuation V, then $\{S^y_x\varphi\}$ is realized by V. Each closing rule, in turn, will be chosen so that (Q1) the set B that triggers closing a branch is not realized by any valuation. Furthermore, the totality of all closing rules is so selected that (Q2) if B' is a set of signed formulas not realized by any valuation, it has a subset B to which some closing rule can be applied.

Familiarity with basic notions related to tableaus for PL^2 is assumed (maximal/closed/open branch, extending a branch); the reader may consult e.g. [14] for details. These notions are straightforwardly extended to the case of PL^3, cf. [15].

Tableau rules. Let us introduce a tableau system, to be denoted TS. Its tableau rules are divided into three groups: (1) 'g-rules', (2) 's-rules', and (3) closing rules. Here are the g-rules for the binary connective $\vee_0$:

$$\begin{array}{|cc|}\hline \multicolumn{2}{|c|}{T^g_a(\varphi \vee_0 \psi)} \\ / & \backslash \\ T^g_a\varphi & T^g_a\psi \\ \hline\end{array} \qquad \begin{array}{|c|}\hline T^g_b(\varphi \vee_0 \psi) \\ | \\ T^g_b\varphi \\ | \\ T^g_b\psi \\ \hline\end{array} \qquad \begin{array}{|c|}\hline T^g_c(\varphi \vee_0 \psi) \\ | \\ T^g_c\varphi \\ | \\ T^g_c\psi \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline F^g_a(\varphi \vee_0 \psi) \\ | \\ F^g_a\varphi \\ | \\ F^g_a\psi \\ \hline\end{array} \qquad \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{F^g_b(\varphi \vee_0 \psi)} \\ / & \backslash \\ F^g_b\varphi & F^g_b\psi \\ \hline\end{array} \qquad \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{F^g_c(\varphi \vee_0 \psi)} \\ / & \backslash \\ F^g_c\varphi & F^g_c\psi \\ \hline\end{array}$$

g-rules for $\vee_1$ ($\vee_2$) are entirely analogous: the only branching T-rule is the rule for T^g_b (*resp.* T^g_c), and the only non-branching F-rule is the one for F^g_b (*resp.* F^g_c). The s-rules for $\vee_0$ are as follows:

$$\begin{array}{c} T^s_a(\varphi \vee_0 \psi) \\ | \\ F^g_a\varphi \\ | \\ F^g_a\psi \\ / \quad \backslash \\ T^s_a\varphi \quad T^s_a\psi \end{array} \qquad \begin{array}{ccc} & T^s_b(\varphi \vee_0 \psi) & \\ / & | & \backslash \\ T^s_b\varphi & T^s_b\varphi & T^s_b\psi \\ | & | & | \\ T^g_b\psi & T^s_b\psi & T^g_b\varphi \end{array} \qquad \begin{array}{ccc} & T^s_c(\varphi \vee_0 \psi) & \\ / & | & \backslash \\ T^s_c\varphi & T^s_c\varphi & T^s_c\psi \\ | & | & | \\ T^g_c\psi & T^s_c\psi & T^g_c\varphi \end{array}$$

$$\begin{array}{ccc} & F^s_a(\varphi \vee_0 \psi) & \\ / & | & \backslash \\ T^g_a\varphi & F^s_a\varphi & T^g_a\psi \\ & | & \\ & F^s_a\psi & \\ & | & \\ & F^g_a\varphi & \\ & | & \\ & F^g_a\psi & \end{array} \qquad \begin{array}{ccc} & F^s_b(\varphi \vee_0 \psi) & \\ / & | & \backslash \\ F^g_b\varphi & T^g_b\varphi & F^g_b\psi \\ | & | & | \\ F^s_b\varphi & T^g_b\psi & F^s_b\psi \end{array} \qquad \begin{array}{ccc} & F^s_c(\varphi \vee_0 \psi) & \\ / & | & \backslash \\ F^g_c\varphi & T^g_c\varphi & F^g_c\psi \\ | & | & | \\ F^s_c\varphi & T^g_c\psi & F^s_c\psi \end{array}$$

s-rules for $\vee_1$ ($\vee_2$) are again completely analogous. Now it can be shown that both g-rules and s-rules satisfy the properties (P1) and (P2) mentioned in the beginning of the present subsection.

Within the confines of this article, we cannot formulate a full collection of closing rules with properties (Q1) and (Q2). Therefore we content us with an entirely 'non-constructive' proof to the effect that a suitable set of closing rules *exists*. Let ℓ be a fixed literal, and let $\{\ell_1, \ldots, \ell_6\}$ be a basic set of p-literals, for a fixed atom p. Define $B_\ell := \{S^y_x\ell : S \in \{T, F\}, y \in \{g, s\}$ and $x \in \{a, b, c\}\}$, and write B_p for the set of sets of signed formulas obtained by assigning one of the 12 possible signs to all members of some non-empty subset of $\{\ell_1, \ldots, \ell_6\}$. Then the size of the set $B_\ell \cup B_p$ is 4,095+4,826,808 = 4,830,903. *A fortiori*, the number of closing rules needed for meeting condition (Q2) cannot exceed 4,830,903. Hence there unavoidably exists a set of closing rules such that each rule individually satisfies (Q1), and they jointly satisfy (Q2). As a matter of fact — luckily enough! — we can do with only a handful of closing rules; for details, cf. [15].

We may, then, be convinced that a suitable set TS of tableau rules can be formulated. What are these rules applied to? As an *input* of a tableau we consider (finite) sets of signed formulas. If C is a set of signed formulas, a *tableau for* C is the result of applying the tableau rules of TS to formulas of the set C, and to the formulas thereby recursively generated, until all branches produced are maximal, i.e., cannot be further extended by the tableau rules. A tableau for a single signed formula $S^y_x\varphi$ is by definition a tableau for the set $\{S^y_x\varphi\}$. There are in general several tableaus for a given set of signed formulas. The differences between these tableaus are, however, immaterial for our purposes: If τ, τ' are both tableaus for

C, then τ is closed if and only if τ' is closed; and τ has an open maximal branch if and only if τ' has an open maximal branch.

6.2 Collection of sound and complete proof systems

If P is one of the 16 semantic attributes, a formula is *P-valid*, if it has the attribute P under all valuations. Dually, a formula is *P-satisfiable*, if it has the attribute P under some valuation. For every attribute P and every formula φ, there are sets $C^{VAL}_{P,\varphi}$ and $C^{SAT}_{P,\varphi}$ — each consisting of three signed formulas — with the following properties: φ is P-valid iff each signed formula in the set $C^{VAL}_{P,\varphi}$ individually has a closed tableau; and φ is P-satisfiable iff there is a closed tableau for the set $C^{SAT}_{P,\varphi}$. For instance, for the attribute $P_{(a,b)}$ the corresponding sets are $C^{VAL}_{(a,b),\varphi} := \{F^g_a\varphi, F^s_b\varphi, T^s_c\varphi\}$ and $C^{SAT}_{(a,b),\varphi} := \{T^g_a\varphi, T^s_b\varphi, F^s_c\varphi\}$.

A *P-proof* of a formula φ is by definition a triple of closed tableaus, one for each signed formula in the set $C^{VAL}_{P,\varphi}$. It will be said that our tableau system TS is *P-sound*, if actually every formula with a P-proof is P-valid. And TS is *P-complete*, if every P-valid formula has a P-proof.

THEOREM 21 (P-soundness, P-completeness). *For each of the 16 semantic attributes, P, TS is P-sound and P-complete.*

Proof. Let an attribute P and a formula φ be given. For P-soundness, it suffices to prove that if φ is *not* P-valid, then a tableau for one of the signed formulas in $C^{VAL}_{P,\varphi}$ is *not* closed. This can be done thanks to the properties ($P1$) and ($Q1$) of TS. For P-completeness, it is enough to show that if a tableau for one of the signed formulas in $C^{VAL}_{P,\varphi}$ is *not* closed, then φ is *not* P-valid. This, again, can be done by virtue of the properties ($P2$) and ($Q2$) of TS. For details, cf. [15]. ■

6.3 Decidability issues

If P is one of the 16 attributes, the *P-satisfiability problem* (or P-SAT) is by definition the problem of deciding whether a given formula of PL^3 is P-satisfiable. Similarly, the *P-validity problem* (or P-VAL) is the problem of deciding whether a given formula is P-valid. Actually, each of these 32 decision problems is decidable. What is more, the 16 satisfiability problems are decidable in **NP** and the 16 validity problems in **coNP**.

THEOREM 22. *Let P be any PL^3-attribute. Then P-VAL is in* **coNP** *and P-SAT is in* **NP**.

Proof. Consider the attribute (a, b); the claim can be similarly proven for the rest of the attributes. Let φ be any formula of PL^3. To decide whether φ is (a, b)-valid, first non-deterministically guess a valuation V. (The guess will yield a counterexample to the claim that φ is (a, b)-valid, if such a counterexample exists.) Then apply the polynomial time algorithm described in *Section* 5 to compute the attribute $|\varphi, V|$. If $|\varphi, V| = (a, b)$, then φ is (a, b)-valid, otherwise not. We have just described an algorithm that solves (a, b)-VAL and runs in **coNP**. Consider, then, (a, b)-SAT. Given a formula φ, non-deterministically guess a valuation V. (The guess will yield a witness to the claim that φ is (a, b)-satisfiable, if such a witness exists.) Then apply the polytime algorithm of *Section* 5 to determine whether $|\varphi, V| = (a, b)$. If yes, φ is (a, b)-satisfiable, otherwise not. The algorithm just described solves (a, b)-SAT and runs in **NP**. ■

It is possible to prove, at least for some semantic attributes, that the corresponding validity problem is **coNP**-*hard* and the corresponding satisfiability problem **NP**-*hard*.[17] In those cases, then, we have **NP**-complete and **coNP**-complete decision problems about the logic PL^3.

7 Concluding remarks

What has been accomplished in the present paper? Looking at the outcome, a minimalist answer would be that we have introduced two 'systems' of the general form $(\mathcal{L}, \mathcal{V}, A)$, where $\mathcal{L}$ is a set of 'formulas', $\mathcal{V}$ is a set of 'valuations', and $A : \mathcal{L} \times \mathcal{V} \longrightarrow \mathbb{N}$ is a function, mapping pairs of 'formulas' and 'valuations' to encodings of what we called 'semantic attributes'.

For each of the two systems, PL^3_0 and PL^3, the corresponding function A was, as a matter of fact, determined by reference to certain 3-player games. These games were obtained by generalizing in certain respects 2-player evaluation games of classical propositional logic, PL^2. Given this background, the two systems were thought of as 'logics' in an extended sense of the word. As evidenced by the body of the present paper, viewing PL^3_0 and PL^3 as logics is at least heuristically justified — virtually any question that can be asked of PL^2 can be formulated in connection with the two systems. Studying the systems PL^3_0 and PL^3 from a logical viewpoint may even be seen as throwing light on PL^2 itself. For, properties of PL^2 can be classified according to whether they do or do not survive the transition to the 3-player setting. The former properties can be regarded as 'robust'. Applying these criteria, for example model-checking in linear time is a robust property of PL^2 (*Sect.* 5), while uniform characterizability of semantic attributes is not (*Sect.* 4).

Various specific questions suggest themselves for future work. Algebraic perspective on multi-player logics may turn out to be of considerable interest. It is natural to study consequence relations proper to these logics in this connection. Further, multi-player logics offer novel ways of introducing game-theoretical notions into logical contexts — a case in point is the notion of security level, referred

[17] In [15] it is proven that whenever $P \in \{(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\}$, P-SAT is **NP**-hard and P-VAL is **coNP**-hard. The proof is based on showing that the satisfiability problem of PL^2-formulas in conjunctive normal form can be reduced in polynomial time to P-SAT, and that the validity problem of PL^2-formulas in disjunctive normal form has a polytime reduction to P-VAL.

to in Remark 3. Indeed, the framework of multi-player logic games calls for a systematic study of correspondences between theorems about multi-player logics and game-theoretic principles.

Many-valued logics and n-player logic games could be compared. In particular, one can ask for which many-valued logic PL_0^3 *resp.* PL^3 provides an alternative, game-theoretical semantics. (The precise formulation of this question is dependent on how consequence relations are defined for these 3-player logics.) Also, it might be of interest to study the variant of game-theoretical semantics for PL^2 with 4 possible payoffs: both players win, no player wins, and one of the two wins. The resulting logic could then be compared with Belnap's and Dunn's 4-valued logics [3, 7].

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