# Markov-perfect Nash equilibria in models with a single capital stock

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# Markov–Perfect Nash Equilibria in Models With a Single Capital Stock

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**Abstract**

Many economic problems can be formulated as dynamic games in which strategically interacting agents choose actions that determine the current and future levels of a single capital stock. We study necessary conditions that allow us to characterise Markov perfect Nash equilibria for these games. These conditions result in an auxiliary system of ordinary differential equations that helps us to explore stability, continuity and differentiability of these equilibria. The techniques are used to derive detailed properties of Markov-perfect Nash equilibria for several games including the exploitation of a finite resource, the voluntary investment in a public capital stock, and the inter-temporal consumption of a reproductive asset.

**JEL classification** C73, D92, Q22

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## 1 Introduction

Many economic problems can be formulated as dynamic games in which strategically interacting agents choose actions that determine the current and future levels of a single capital stock. Consider, for example, a single stock of an exhaustible or reproductive resource that is simultaneously exploited by several agents that do not cooperate. Each agent chooses an extraction strategy to maximise the discounted stream of future utility. The actions taken by agents not only determine their levels of utility but also the level of the capital stock. Alternatively, look at the problem that agents voluntarily contribute to a single public stock of capital, like a park or a church. They choose their contributions (investments in the public stock of capital) to maximise the discounted stream of utility from consuming the public stock net of investment costs. Private investment builds up the public stock of capital that eventually can be consumed by all agents.

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Both examples have several things in common. First, the actions taken by agents determine the size of a single capital stock that fully describes the current state of the economic system. Second, if there is no mechanism that forces players to coordinate their actions, they will act strategically and play a non-cooperative game. Third, the equilibrium outcome will critically depend on the strategy spaces available to the agents.

We make use of these features and formulate a differential game in which agents act non-cooperatively and use Markov strategies. We provide a detailed analysis of Markov perfect Nash equilibria (MNPE) for this class of differential games with a single capital stock and discuss several economic examples that belong to this class.

In a differential game, strategically interacting agents try to maximise an inter-temporal objective function by choosing a strategy that results in an action at every point in time.[1] Collectively, these actions influence the state of the economic system and its time evolution.

There is a wide choice of possible strategies taken by the agents. They may choose a simple time profile of actions and precommit themselves to these fixed actions over the entire planning horizon: the players are then using open-loop strategies. Alternatively players might choose Markov strategies where they condition their actions on the current state of the system and react immediately every time the state variable changes. When agents use feedback or Markov strategies they are not required to precommit. Instead they play credible strategies if these are derived through backward induction.

To better understand the difference between open-loop and Markov strategies let us look at the following example of several agents strategically exploiting the same renewable resource, like for instance a stock of fish. If the fisheries use open-loop strategies they specify a time path of fishing effort at the beginning of the game and commit themselves to stick to these preannounced actions over the entire planning horizon. Alternatively, if they use Markov or feedback strategies they choose decision rules that determine current actions as a function of the current stock of the resource. Markov decision rules capture the strategic interactions present in a dynamic game. If a rival fishery makes a catch today that necessarily results in a lower level of the fish stock, the opponents react with actions that take this change in the stock into account. In that sense Markov strategies capture all the features of strategic interactions.

Assuming that agents use Markov strategy spaces we restrict our attention to the derivation of subgame perfect Nash equilibria. Within a differential game framework these strategies have the property that players choose a state dependent decision rule that for every subgame assigns an equilibrium action to the current state of the economic system. Finding Markov Nash equilibrium strategies of differential games, even if the game is of the linear-quadratic type, is a formidable analytical problem. For instance, to find a Markov-perfect Nash equilibrium in the general situation of $n$ players and $m$ state variables leads to the problem of determining solutions of a system of $n$ coupled nonlinear implicit $m$-dimensional partial differential equations (PDE). Only if the economic system can be described by a single state variable (a single capital stock) will the system of PDE's collapse to a system of ordinary differential equations that is much easier to deal with. Because of this, the paper focuses on the least complex situation $m = 1$. There are many economic problems that result in a dynamic game with a single capital stock.

Consider for instance $n$ agents non-cooperatively exploiting a single exhaustible or renewable resource. The resource stock is the single state variable and agents choose extraction

[1]For a general introduction to the theory of differential games we refer the reader to Dockner et al. (2000).

strategies to maximise the present value of utility over a given planning horizon. Markov perfect equilibria for these classes of games have been derived in many places, see for example Levhari and Mirman (1980), Sundaram (1989), Benhabib and Radner (1992), Clemhout and Wan (1994), Dutta and Sundaram (1993), Dockner and Sorger (1996), Rincón-Zapatero et al. (1998) and more recently Benchekroun (2003).

As a second class of models consider private investment in a public capital stock. The capital stock is the single state variable and agents choose investment strategies to maximise the present value of future net utility. Utility is derived from the available stock of public capital. This problem of private investment in a single public stock of capital has been formulated by Fershtman and Nitzan (1991) and Markov-perfect Nash equilibria have been studied by Wirl (1996) and Rowat (2007), and in a discrete time framework by Marx and Matthews (2000) and Dockner and Nishimura (2005).

Dynamic games with a single capital stock can also be applied to study transboundary pollution control. The emissions of two or more countries accumulate a single stock of pollution. Countries derive utility from consumption but production results in emissions that accumulate and generate costs. Markov equilibria for these types of games have been studied by van der Ploeg and de Zeeuw (1992), Dockner and Long (1994) and Dockner et al. (1996). Finally, environmental economists have recently started to explore equilibria in the shallow lake problem. This problem is structurally similar to the exploitation of a single renewable resource stock but with a non-concave production function. Recent papers dealing with the shallow lake problem include Brock and Starrett (2003), Mäler et al. (2003), and Wagener (2003). A numerical analysis of a Markov-perfect Nash equilibrium in the shallow lake problem has been given by Kossioris et al. (2008).

In this paper we formulate a class of differential games in which the actions of the agents influence a single capital stock, the state variable, and develop a solution methodology whose core is formed by necessary conditions that have to be satisfied by Markov strategies. If the economic system is described by a single state variable standard dynamic programming techniques to derive Markov-perfect Nash equilibria result in a system of ordinary differential equations in the value function. For a restricted class of specific functional forms of the primitives of the model this system of differential equations can be solved explicitly and the Markov-perfect Nash equilibrium can be derived analytically. Starting with the pioneering work by Case (1979), differential game theorists have modified this approach. Instead of working with the ordinary differential equations in the value functions, they derive a system of differential equations in shadow prices, that is, in the first derivatives of the value functions. Structurally this system is much simpler to work with, in particular when symmetric equilibrium strategies are analysed. For the shadow price system reduces then to a single quasi-linear[2] differential equation with non-constant state dependent coefficients, which for specific functional forms of the state equation and the objective functionals can be solved explicitly.

Using the shadow price system approach, Tsutsui and Mino (1990) derive non-linear Markov equilibria for a linear quadratic differential game. The same approach was used by Dockner and Long (1994) in a model of transboundary pollution control and by Wirl (1996) in a public goods investment problem. For a differential game with $m$ state variables and $n$ players, Rincón-Zapatero et al. (1998) give a general formulation of the shadow price system approach to derive Markov-perfect Nash equilibria. They differentiate the Bellman equations

[2] A differential equation is called quasi-linear if it is linear in the highest derivatives of the unknown function.

to arrive at a system of quasi-linear partial differential equations in the shadow prices and point out that if there is only a single state variable this system reduces to a linear system of ordinary differential equations. Using an example with exhaustible resources they derive analytical solutions for symmetric Markov-perfect Nash equilibria for constant elasticity utility functions. Finally, Kossioris et al. (2008) apply the shadow price system approach in the area of environmental economics.

This paper extends the shadow price system approach in a substantial way for $n$-player games with a single state variable. The $n$-dimensional system of ordinary quasi-linear non-autonomous differential equations in shadow prices is used to derive an auxiliary $(n+1)$-dimensional system of ordinary autonomous differential equations whose solution trajectories trace out graphs of the equilibrium strategies. The auxiliary system opens up the opportunity to use phase diagram analysis and study Markov-perfect Nash equilibria for games with general functional forms. Moreover, it can be used to gain important insights into the continuity and differentiability of Markovian strategies. We are able to derive necessary conditions for possible failures of differentiability and continuity of the Markov equilibrium strategies. Points where the Markov strategies are continuous but not differentiable can very conveniently be described by singularities of the *auxiliary* system. Moreover, we are also able to find non-continuous Markov-perfect Nash strategy equilibria.

In the important special case that all players have the same utility functionals, it is useful to focus on a *symmetric* equilibrium in which all agents play the same strategy. The symmetric equilibrium is the solution of an ordinary first order non-autonomous differential equation. It is possible to draw the phase diagram of the associated two-dimensional autonomous auxiliary system, which yields detailed information about the qualitative properties of the symmetric Markov equilibrium.

The article is organised as follows. Section 2 serves as an introduction to the central ideas of the paper, which are presented in the context of the Fershtman-Nitzan model of voluntary provisions to a public good. In section 3, necessary conditions are derived which have to hold at points where a Markov strategy is differentiable, or continuous, or discontinuous, respectively. Section 4 illustrates these conditions by determining Markov equilibria for different applications of differential games with a single capital stock: (i) extraction of exhaustible resources; (ii) management of an economical-ecological system, known as the *shallow lake system*; and (iii) exploitation of renewable resources. Section 5 concludes.

## 2 Example: voluntary provision of a public good

This section introduces our approach to finding Markov-perfect Nash equilibria for differential games with a single capital stock by applying it to a linear-quadratic differential game.

The model that will serve as example is the analysis of private investment in a public capital stock. This game was first treated by Fershtman and Nitzan (1991). They assumed that each agent derives quadratic utility from the consumption of the public capital stock. Investment in the stock, however, is costly and results in quadratic adjustment costs. Fershtman and Nitzan solved both the open-loop game and the game with Markov strategies and found that the dynamic free rider problem is more severe when agents use linear Markov strategies. Wirl (1996) challenged these results and studied the identical linear quadratic game but solved it for non-linear Markov equilibria. He found that if the discount rate is small enough non-linear

Markov strategies can support equilibrium outcomes that are close to the efficient provision of the public capital. Finally, Rowat (2007) derived explicit analytic expressions for the non-linear Markov equilibria. We shall use this example to introduce our methodology.

### 2.1 The game

There are $n$ players; player $i$ voluntarily invests in the public capital stock at the rate $u_i$. The single public capital stock evolves according to

$$\dot{x} = \sum_{j=1}^{n} u_j - \delta x;$$

here $\delta > 0$ is the constant depreciation rate. Following Fershtman and Nitzan we assume that player $i$'s utility functional is given by

$$\mathcal{J}_i = \int_0^{\infty} \left( ax - \frac{b}{2}x^2 - \frac{1}{2}u_i^2 \right) \mathrm{e}^{-\rho t} \, \mathrm{d}t,$$

where $a, b > 0$ are positive parameters. Note that compared to the formulation of Wirl (1996), one parameter has been scaled away. The corresponding present value Pontryagin function[3] becomes

$$\mathcal{P}_i(x, p_i, u) = \mathcal{P}_i(x, p_i, u_1, \cdots, u_n) = ax - \frac{b}{2}x^2 - \frac{1}{2}u_i^2 + p_i \left( \sum_{j=1}^{n} u_j - \delta x \right).$$

Contributions to the public good are assumed to be nonnegative: $u_i \geq 0$. The function $u_i \mapsto \mathcal{P}_i(x, p_i, u)$ is maximised at

$$u_i = v_i(p_i) = \begin{cases} p_i & p_i \geq 0, \\ 0 & p_i < 0. \end{cases}$$

Introduce $\hat{\mathbf{u}}_i = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n)$. The present value Hamilton function $H_i$ of player $i$ reads then as

$$H_i(x, p_i, \hat{\mathbf{u}}_i) = \begin{cases} ax - \frac{b}{2}x^2 + \frac{1}{2}p_i^2 + p_i \left( \sum_{j \neq i} u_j - \delta x \right), & \text{if } p_i \geq 0, \\ ax - \frac{b}{2}x^2 + p_i \left( \sum_{j \neq i} u_j - \delta x \right), & \text{otherwise.} \end{cases}$$

We now restrict our attention to the symmetric case, for which all players use the same strategy. If $V$ is the common value function of the players, using $p_i(x) = V'(x)$ we obtain, following Fershtman and Nitzan (1991), the Hamilton-Jacobi equation for $V$. If $V' \geq 0$, this equation reads as

$$\rho V = ax - \frac{b}{2}x^2 + \frac{2n-1}{2}(V')^2 - \delta x V'. \tag{1}$$

[3] Also called Hamilton, pre-Hamilton or unmaximised Hamilton function.

Fershtman and Nitzan obtained a solution to this equation by the well-known method of substituting $V(x) = c_0 + c_1 x + c_2 x^2$ and comparing coefficients of $x$.

Wirl (1996) pointed out that due to the fact that the Hamilton-Jacobi equation (1) has no initial conditions, there may be actually more solutions to this equation; he applied the shadow price system approach to derive these equilibria. This approach consists of two steps: (i) differentiating the Hamilton-Jacobi equation once with respect to $x$, to get

$$\rho V' = a - bx + (2n-1)V'V'' - \delta V' - \delta x V''.$$

and (ii) solving for $V''$ by substituting $V'' = p' = u'$ (if $V' \geq 0$) to arrive at the shadow price system,

$$[(2n-1)p - \delta x]p' = (\rho + \delta)p - a + bx. \tag{2}$$

Note that while equation (1) was an implicit nonlinear first order ordinary differential equation in $V$, equation (2) is an explicit quasi-linear equation in $p$ with non-constant coefficients. Rowat (2007) derives an explicit solution for this quasi-linear differential equation by carefully considering the singularity where $(2n-1)p - \delta x = 0$ holds. We do not repeat his approach here but refer to his paper instead.

The idea of our approach is to reformulate the shadow price system and to derive an auxiliary system from (2). In the region $p \geq 0$ this system is given by

$$\frac{\mathrm{d}p}{\mathrm{d}s} = (\rho + \delta)p - a + bx, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = (2n-1)p - \delta x.$$

Note that the derivatives are taken with respect to a parameter $s$ which has no *a priori* economic interpretation; the point of the auxiliary system is that a solution $(x(s), p(s))$ traces out the graph of a solution $p = p(x)$ of equation (2), as $\frac{\mathrm{d}p}{\mathrm{d}s} = \frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s}$ and consequently

$$\frac{\mathrm{d}p}{\mathrm{d}x}(x) = \frac{\frac{\mathrm{d}p}{\mathrm{d}s}}{\frac{\mathrm{d}x}{\mathrm{d}s}} = \frac{(\rho+\delta)p - a + bx}{(2n-1)p - \delta x}, \tag{3}$$

which corresponds to equation (2). Some phase curves of the auxiliary system are shown in figure 1. There, solutions of the auxiliary system are represented by drawn curves. They can, at least locally, be interpreted as the graphs of possible symmetric feedback strategies $u(x) = p(x)$.

## 2.2 Geometrical analysis of feedback equilibria

In this subsection, the solutions of the auxiliary system are analysed geometrically by investigating the phase diagram given in figure 1. For this, introduce the state dynamics

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,p) = \begin{cases} np - \delta x, & p \geq 0, \\ \quad - \delta x, & p < 0. \end{cases} \tag{4}$$

**Excluded trajectories.** In figure 1, note the region in the lower right hand corner. Any solution of the auxiliary system intersects (under the dynamics of the state equation) the line $l_2$ : $(2n-1)p - \delta x = 0$. As this is also, not coincidentally, the curve of minima of the value function for given $x$, the strategies cannot 'jump away' from $l_2$ (see subsection 3.4 for conditions on

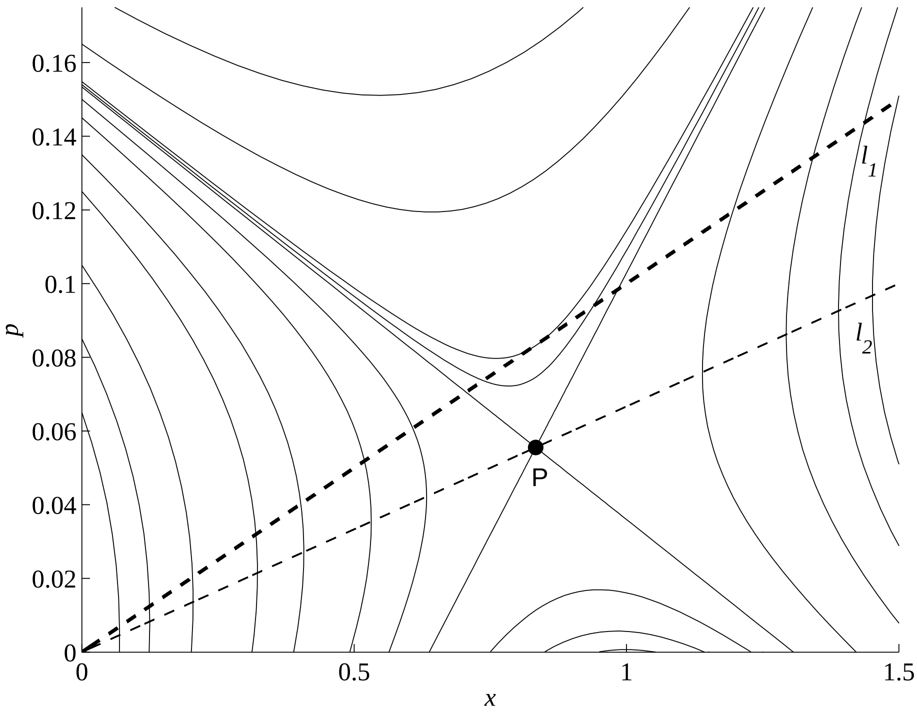



Figure 1: *Solutions of the auxiliary system (drawn curves) as well as the line of equilibria* $l_1$ : $\mathrm{d}x/\,\mathrm{d}t = 0$ *(thickly dashed line) and the isocline* $l_2$ : $\mathrm{d}x/\,\mathrm{d}s = 0$ *of the auxiliary system (thinly dashed line). Parameters:* $a = 0.1$, $b = 0.1$, $\delta = 0.2$, $\rho = 0.1$.

non-continuous strategy equilbria). Neither can the strategies stay there, for none of the points on $l_2$, excluding the origin, is an equilibrium of the state equation (4). Finally, continuation along solutions of the auxiliary system is impossible as well, as the trajectories bend back, they cannot represent univalent functions of the state variable any more.

Solutions for which there exists a constant $S > 0$ such that $p(s) < 0$ for $s \geq S$, that is, trajectories of the auxiliary system that pass through the line $p = 0$ at $x_0 = x(s)$, are also excluded. They correspond to strategies that satisfy $u(x) = 0$ for $0 \leq x \leq x_0$. In that region, the costate equation of a player would take the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = (\rho + \delta)p - a + bx;$$

note that the derivative here is with respect to time, as we are considering the optimisation problem of a single player. As $u(x) = 0$ implies that $x(t)$ tends to 0, it follows that $p \propto -\mathrm{e}^{(\rho+\delta)t}$, violating the transversality condition

$$\lim_{t \to \infty} p\,\mathrm{e}^{-\rho t} \geq 0$$

at $(x, p) = (0, 0)$.

In the lower left hand corner of the figure, points that start at the line $l_2$ move away from it under state dynamics and either hit the exceptional line $p = 0$ or the line of steady states $l_1$. The former solutions have to be excluded, as they lead to trajectories violating the transversality condition; the latter solutions of the auxiliary system represent feedback strategies that are not defined for all states.

Also, solutions along which $x$ increases indefinitely over time are to be excluded. This cannot always be done on the basis of a transversality argument, but we have to invoke a global argument. For $x(t)$ sufficiently large, both terms $ax - \frac{1}{2}bx^2$ and $-\frac{1}{2}u^2$ of the utility integrand are negative. Therefore along an indefinitely increasing solution it is always better for a player to play the deviating strategy $u = 0$. We end up with a substantially reduced set of strategies, shown in figure 2.

**Corner points.** The feedback strategy that is formed by the upper two invariant manifolds of the steady state $P$ of the auxiliary system is of the 'kink' or 'corner' type mentioned below in subsection 3.3. Note that the corner point is on the line $l_2$, as predicted. Also the globally defined strategy, thickly drawn in figure 2, has a corner: it is located at the point where the invariant manifold of $P$ intersects the horizontal axis. This corner is however of a different kind, as it represents a control constraint that becomes active.

**Stability of equilibria.** Consider the line $l_1 = \{(x, p) : f(x, p) = 0\}$ of equilibria of the state dynamics (the broken thickly drawn line in the figure): the quantity $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive above $l_1$, and negative below. From the figure, it is readily apparent that points on $l_1$ close to the origin (lower left hand corner) are stable, while points on $l_1$ in the upper right hand corner are unstable. Hence there is a point on $l_1$ where equilibria change from stable to unstable; it is the unique point $(x_{**}, p_{**})$ where a solution curve of the auxiliary system touches the line $l_1$.

Let $p(x)$ be a feedback solution. The stock then evolves according to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, p(x)) = np(x) - \delta x. \tag{5}$$

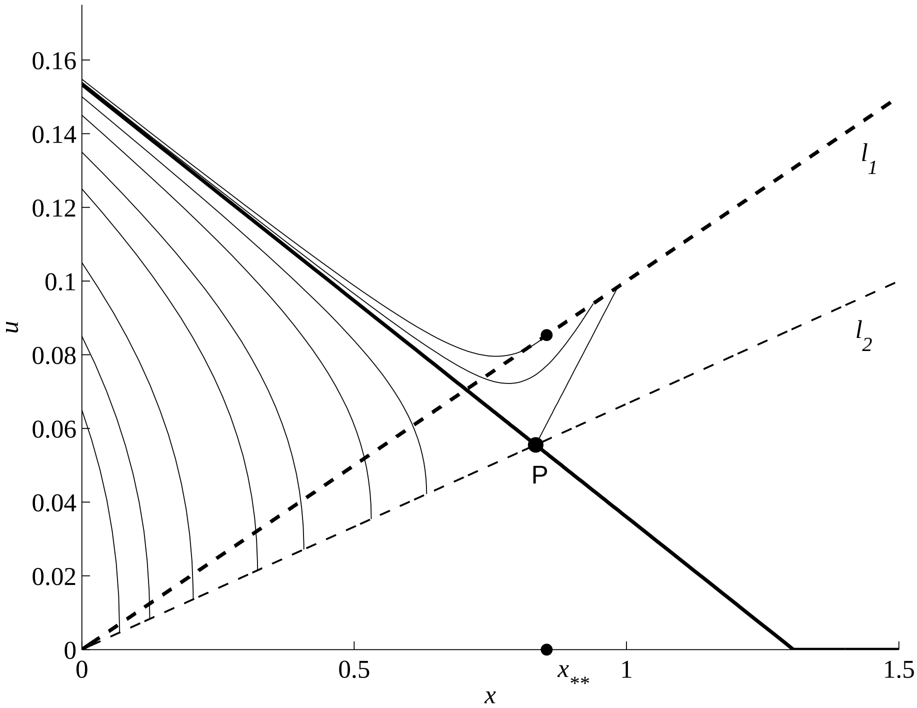



Figure 2: *Locally (thin curves) and globally (thick curve) defined Markov-perfect Nash equilibrium strategies, together with the lines* $l_1$ *and* $l_2$*. Also indicated is the supremum* $\bar{x}_*$ *of the state values that can be stabilised by a locally defined Markov-perfect Nash equilibrium strategy. Parameters as in figure 1.*

Let $x_*$ be a steady state equilibrium of this equation; then we have that $p_* = p(x_*) = (\delta/n)x_*$ and $(x_*, p_*) \in l_1$. This equilibrium is stable if

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, p(x)) \Big|_{x=x_*} = n\frac{\mathrm{d}p}{\mathrm{d}x}(x_*) - \delta < 0.$$

This stability condition holds, using (3), when

$$\frac{\mathrm{d}p}{\mathrm{d}x}(x_*) = \frac{(\rho+\delta)p_* - a + bx_*}{(2n-1)p_* - \delta x_*} = \frac{(\rho+\delta)\frac{\delta}{n}x_* - a + bx_*}{(2n-1)\frac{\delta}{n}x_* - \delta x_*} < \frac{\delta}{n}$$

is satisfied. This condition can be simplified to read as

$$x_* < \frac{a}{b + \frac{\delta\rho}{n} + \frac{\delta^2}{n^2}} = x_{**}.$$

In other words, the value $x_{**}$ is the supremum of the stock values that can be stabilised by a local Markov-perfect Nash equilibrium strategy. By this we mean the following.

**Definition.** *A state $x_* \in X$ can be* stabilised by a local Markov-perfect Nash equilibrium strategy*, if there is an open interval $I \subset X$ containing $x_*$ and a Markov-perfect Nash equilibrium strategy $p : I \to \mathbb{R}$ such that $x_*$ is a stable steady state under the stock evolution dynamics.*

In the present situation we have that for every $x_* < x_{**}$, there is a Markov-perfect Nash equilibrium strategy $p$ such that $x_*$ is a stable steady state under the dynamics (5).

The maximal utility stream that is extracted from the public good, that is, the maximum of $ax - \frac{1}{2}bx^2$, is obtained at $x_m = b/a$. Note that as the number $n$ of players tends to infinity, the value $x_{**}$, and with it the region of stock values that can be stabilised, increases towards $x_m$. This was to be expected: as the adjustment costs are convex, it is better in terms of average costs per player that they are distributed over more players.

**Optimal strategies.** From figure 1 we can also draw conclusions to which strategies maximise the pay–off for the players, if the initial state $x_0 = x(0)$ of the system is given; we obtain from equation (1) that

$$\rho V = ax - \frac{b}{2}x^2 + \frac{2n-1}{2}p^2 - \delta xp = G(x, p). \tag{6}$$

In figure 3 we added the level curves of $G$ to the strategies of figure 2 as dotted lines. Note that $G$, and hence $V$, is large if for instance $p$ is large.

Consider first the case that $x_0 = 0$. Then

$$\rho V(0) = G(0, p) = \frac{2n-1}{2}p^2,$$

and we see that the highest payoff is attained if $p$ is chosen as large as is feasible; from figure 3 we infer that this corresponds to the strategy that ends at the semi-stable steady state $x = x_{**}$.

In general, for fixed $x$, the function $p \mapsto G(x, p)$ is convex and takes its minimum if $(x, p) \in l_2$. It follows that to maximise payoff for all players, the initial value of $p$ has to be taken as large as is feasible for $x \le x_P$. Beyond that point, the solutions with maximal $p$ have to be compared with the globally defined strategy. For $x$ sufficiently large, there is only a single candidate, which is necessarily optimal.

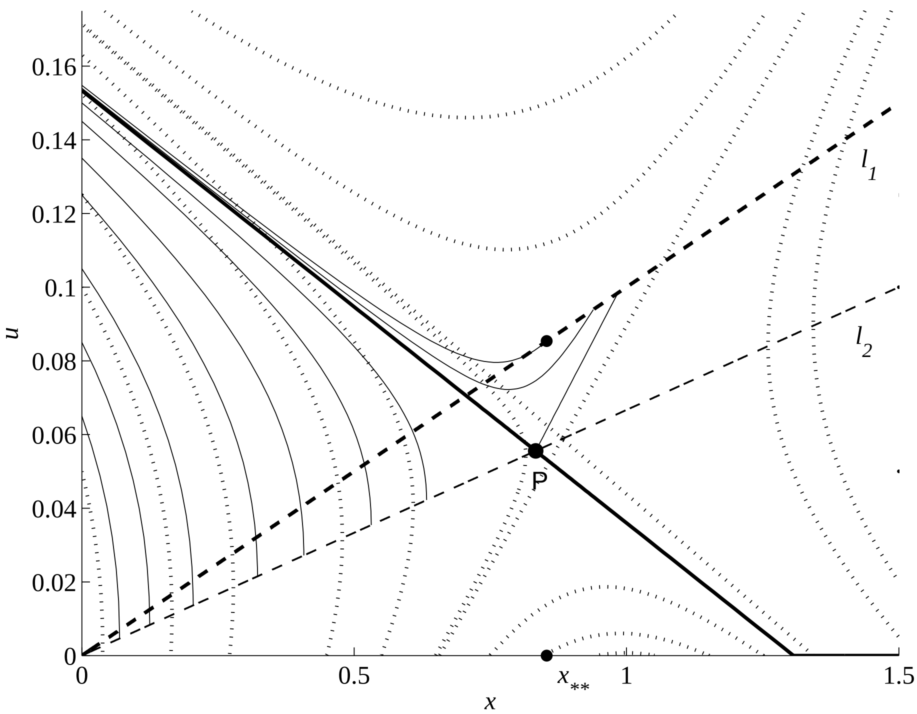



Figure 3: *As figure 2, but with the level sets of G added (dotted curves), from which the payoff of the strategies can be determined.*

### 2.3 General formulation of a symmetric game

It is clarifying to reformulate the derivation of the auxiliary equations in terms of the *game Hamiltonian* $G$, introduced in equation (6).

Taking $p(x) = V'(x)$ and deriving (6) with respect to $x$ yields

$$\rho p(x) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial p} p'(x).$$

The shadow price system takes the form

$$\frac{\partial G}{\partial p} p'(x) = \rho p - \frac{\partial G}{\partial x},$$

and the auxiliary system takes the familiar form

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial G}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}s} = \rho p - \frac{\partial G}{\partial x}.$$

Mathematically speaking, these equations are the characteristic equations of the Hamilton-Jacobi equation (6). However, in crucial contrast to the "one-player" optimal control situation, the parameter $s$ is different from the time parameter $t$. In a way, it is this fact that allows the occurrence of "many" Nash equilibrium strategies.

## 3 General theory

In this section we derive the auxiliary system for general feedback Nash equilibria in a dynamic game with a single state variable. In this game, $n$ players choose Markov strategies, $u_i(x)$, to maximise an inter-temporal objective function. The strategies determine the level of a single capital stock, $x$, that is governed by the state dynamics. For this game we characterise Markov perfect Nash equilibria that are either differentiable, or continuous, or have at most a finite number of jump points.

Recall that such an equilibrium is by definition a vector–valued function $\mathbf{u} : X \to \mathbb{R}^n$, such that if the strategies of all players except player $i$ are given by the component functions $u_j(x)$, $j \neq i$, then $u_i(x)$ would be the optimal feedback control for player $i$ of the resulting optimal control problem.

### 3.1 The vector Hamilton-Jacobi equation.

Consider an $n$ player game, where each player has a payoff functional

$$\mathcal{J}_i = \int_0^\infty L_i(x, \mathbf{u})\, \mathrm{e}^{-\rho t}\, \mathrm{d}t. \tag{7}$$

Here $x \in X$, where $X$ is an open subinterval of the real line $\mathbb{R}$, and $\mathbf{u}$ denotes the vector $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$ of the strategies of the players. For known $\mathbf{u}$, the state equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \mathbf{u}) \tag{8}$$

determines the evolution of the system.

Introduce the notation $\hat{\mathbf{u}}_i = (u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n)$ for the vector of strategies of all players except player $i$. Given these strategies in feedback form $\hat{\mathbf{u}}_i = \hat{\mathbf{u}}_i(x)$, the Pontryagin function of player $i$ reads as

$$\mathcal{P}_i(x, p_i, \hat{\mathbf{u}}_i, u_i) = L_i(x, \mathbf{u}) + p_i f(x, \mathbf{u}).$$

This function is maximised at

$$u_i = v_i(x, p_i, \hat{\mathbf{u}}_i) \tag{9}$$

yielding the Hamilton function

$$H_i(x, p_i, \hat{\mathbf{u}}_i) = \mathcal{P}_i\left(x, p_i, \hat{\mathbf{u}}_i, v_i(x, p_i, \hat{\mathbf{u}}_i)\right).$$

The Hamilton-Jacobi equation for the value function of player $i$ reads then as

$$\rho V_i(x) = H_i(x, V_i'(x), \hat{\mathbf{u}}_i(x)).$$

Introduce the notations $\mathbf{v} = (v_1, \cdots, v_n)$ and

$$\frac{\partial \mathbf{v}}{\partial \hat{\mathbf{u}}} = \begin{pmatrix} 0 & \frac{\partial v_1}{\partial u_2} & \cdots & \cdots & \frac{\partial v_1}{\partial u_n} \\ \frac{\partial v_2}{\partial u_1} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & \frac{\partial v_{n-1}}{\partial u_n} \\ \frac{\partial v_n}{\partial u_1} & \cdots & \cdots & \frac{\partial v_n}{\partial u_{n-1}} & 0 \end{pmatrix},$$

etc. In order to eliminate the functions $u_i(x)$ from the problem, the system of equations

$$F_i(x, \mathbf{u}) = u_i - v_i(x, p_i, \hat{\mathbf{u}}_i) = 0, \qquad i = 1, \cdots, n \tag{10}$$

has to be solved for the $u_i$; in vector notation, this system reads as

$$\mathbf{F}(x, \mathbf{u}) = \mathbf{u} - \mathbf{v}(x, \mathbf{p}, \mathbf{u}) = 0.$$

A sufficient condition for the solvability of this system is that the matrix

$$\frac{\partial \mathbf{F}}{\partial \mathbf{u}} = I - \frac{\partial \mathbf{v}}{\partial \hat{\mathbf{u}}}$$

should be invertible. We make therefore the general assumption that this inversion is always possible, and find solutions $u_i = u_i(x, \mathbf{p})$.

Consequently, it may be assumed that the *game* Hamilton functions $G_i$, $i = 1, \cdots, n$ of the players can be written as

$$G_i = G_i(x, V_1', \cdots, V_n') = H_i(x, V_i', \hat{\mathbf{u}}_i(x, \mathbf{V}'(x))),$$

and that we are to solve the following vector Hamilton-Jacobi equation

$$\rho \mathbf{V}(x) = \mathbf{G}(x, \mathbf{V}'(x)) = \mathbf{H}(x, \mathbf{V}'(x), \mathbf{u}(x, \mathbf{V}'(x))), \tag{11}$$

where $\mathbf{H} = (H_1, \cdots, H_n)$, etc. Taking derivatives with respect to $x$, and substituting $\mathbf{p}(x) = \mathbf{V}'(x)$ yields

$$\rho\mathbf{p}(x) = \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial \mathbf{G}}{\partial \mathbf{p}}\mathbf{p}'(x). \tag{12}$$

Note that $\partial\mathbf{G}/\partial x$ is an $n$-dimensional vector, whereas $\partial\mathbf{G}/\partial\mathbf{p}$ is an $n \times n$ matrix. We obtain finally the equation

$$\frac{\partial \mathbf{G}}{\partial \mathbf{p}}\mathbf{p}'(x) = \rho\mathbf{p} - \frac{\partial \mathbf{G}}{\partial x}, \tag{13}$$

which in terms of the vector $\mathbf{H}$ of original Hamilton functions reads as

$$\left(\frac{\partial \mathbf{H}}{\partial \mathbf{p}} + \frac{\partial \mathbf{H}}{\partial \hat{\mathbf{u}}}\frac{\partial \mathbf{u}}{\partial \mathbf{p}}\right)\mathbf{p}'(x) = \rho\mathbf{p} - \frac{\partial \mathbf{H}}{\partial x} - \frac{\partial \mathbf{H}}{\partial \hat{\mathbf{u}}}\frac{\partial \mathbf{u}}{\partial x}. \tag{14}$$

Equation (13) or (14) is referred to as the the shadow price system. Due to the special structure of our class of games the shadow price system is a system of quasi-linear differential equations in $\mathbf{p}(x)$. Rincón-Zapatero et al. (1998) analyse this general system and demonstrate its applicability by considering specific examples. As already pointed out the shadow price system approach traces back to the analysis of Case (1979) who studied non-linear Markov equilibria for the sticky price model also analysed in detail by Tsutsui and Mino (1990). The shadow price system approach has also been applied by Dockner and Long (1994) and by Wirl (1996) to derive non-linear symmetric Markov-perfect Nash equilibria.

Note that if the relation $\mathbf{u} = \mathbf{u}(x, \mathbf{p})$ can be solved for $\mathbf{p}$, say by $\mathbf{p} = \mathbf{q}(x, \mathbf{u})$, then we can rewrite equation (14) in terms of the controls $\mathbf{u}$, which is often more convenient in applications.

Note further that in the important symmetric special case that all players are equal and play the same strategies, the vector $\mathbf{G}$ of game Hamilton functions can be replaced by the symmetric function

$$G(x, p) = \frac{1}{n}\sum_{i=1}^{n} G_i(x, p, \cdots, p) \tag{15}$$

and equations (11)-(13) now hold as scalar equations with $\mathbf{G}$ replaced by $G$.

## 3.2 Auxiliary system.

Recall the definition of the *adjoint* matrix $A^*$ of a given matrix $A$: it is the matrix whose elements are the cofactors of $A$, which are obtained by deleting the $i$'th row and $j$'th column of $A$ and taking the determinant of the remaining matrix. We have that $AA^* = (\det A)I$, where $I$ is the identity matrix; hence $A^{-1} = (\det A)^{-1}A^*$. Multiplying an equation of the form

$$Ax = b$$

from the left with $A^*$ yields

$$(\det A)x = A^*b.$$

Multiplying the shadow price system (13) from the left with the cofactor matrix $(\partial\mathbf{G}/\partial\mathbf{p})^*$ yields

$$\left(\det\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x, \mathbf{p})\right)\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}x}(x) = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x, \mathbf{p})\right)^*\left(\rho\mathbf{p}(x) - \frac{\partial \mathbf{G}}{\partial x}(x, \mathbf{p})\right).$$

The *auxiliary system* to equation (11) is now defined as

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \left(\dfrac{\partial \mathbf{G}}{\partial \mathbf{p}}(x,\mathbf{p})\right)^{*}\left(\rho\mathbf{p} - \dfrac{\partial \mathbf{G}}{\partial x}(x,\mathbf{p})\right), \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = \det \dfrac{\partial \mathbf{G}}{\partial \mathbf{p}}(x,\mathbf{p}), \end{cases} \tag{16}$$

where $s \in \mathbb{R}$ is some real parameter that has no immediate economic significance.

In the symmetric case, the auxiliary system simplifies to

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}s} = \rho p - \dfrac{\partial G}{\partial x}(x,p), \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\partial G}{\partial p}(x,p). \end{cases} \tag{17}$$

Note again, though, that the parameter $s$ is not the time.

### 3.3 Corner points.

Equation (13) answers the question of when a continuous equilibrium Markov strategy $u(x)$ may fail to be differentiable at certain (isolated) points.

**Theorem 1.** *Assume that $\mathbf{p} = \mathbf{p}(x)$ is a Markov Nash equilibrium costate of the differential game with payoffs (7) and state equation (8). If $\mathbf{p}$ is continuous in a neighborhood $U$ of $x_0$, and if $\det \frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0, \mathbf{p}(x_0)) \neq 0$, then $\mathbf{p}$ is differentiable at $x_0$ and*

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}x}(x_0) = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0,\mathbf{p}(x_0))\right)^{-1}\left(\rho\mathbf{p}(x_0) - \frac{\partial \mathbf{G}}{\partial x}(x_0,\mathbf{p}(x_0))\right).$$

**Proof**

Under the hypotheses of the theorem, the matrix $\partial\mathbf{G}/\partial\mathbf{p}(x,\mathbf{p}(x))$ is invertible on a neighbourhood $\tilde{U} \subset U$, and on $\tilde{U}$ the function $\mathbf{p}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}x}(x) = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x,\mathbf{p}(x))\right)^{-1}\left(\rho\mathbf{p}(x) - \frac{\partial \mathbf{G}}{\partial x}(x,\mathbf{p}(x))\right).$$

Hence $\mathbf{p}$ is differentiable with the stated derivative. ∎

This result suggests immediately a necessary criterion for the occurrence of corner points in Markov-perfect Nash equilibrium strategies.

**Theorem 2.** *Under the same assumptions as theorem 1, if $\mathbf{p}$ is differentiable in $U\backslash\{x_0\}$ and $\det \frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0, \mathbf{p}(x_0)) = 0$, and if the limits*

$$\lim_{x\uparrow x_0} \mathbf{p}'(x) \qquad or \qquad \lim_{x\downarrow x_0} \mathbf{p}'(x)$$

*exist, then*

$$\left(\frac{\partial \mathbf{G}}{\partial \mathbf{p}}(x_0,\mathbf{p}(x_0))\right)^{*}\left(\rho\mathbf{p}(x_0) - \frac{\partial \mathbf{G}}{\partial x}(x_0,\mathbf{p}(x_0))\right) = 0.$$

*Consequently, a strategy vector $\mathbf{p}(x)$ can have a corner point at $x = x_0$ if and only if the point $(x_0, \mathbf{p}(x_0))$ is an equilibrium point of the auxiliary system.*

The proof of this result is immediate.

### 3.4 Jump points

Note that theorem 1 yields a necessary condition at points where the equilibrium costates are continuous. Here, we look for necessary conditions that have to hold if the equilibrium strategy has an isolated jump discontinuity.

Let $\mathbf{p}(x) = (p_1(x), \cdots, p_n(x))$ be the vector of costates, and let $\mathbf{u} = \mathbf{u}(x, \mathbf{p}(x))$ be the corresponding vector of feedback strategies. The result can then be stated as follows.

**Theorem 3.** *Assume the value function $V_i$ of player $i$ is continuous. If a strategy vector*

$$\mathbf{u}(x) = \big(u_1(x, \mathbf{p}(x)), \cdots, u_n(x, \mathbf{p}(x))\big)$$

*has a jump discontinuity at $x = x_0$, then necessarily*

$$\lim_{x \uparrow x_0} G_i(x, \mathbf{p}(x)) = \lim_{x \downarrow x_0} G_i(x, \mathbf{p}(x)).$$

Note that it is possible to give a priory conditions that ensure the continuity of the value function $V_i$.

**Proof**

This is a direct consequence of the equation

$$\rho V_i(x) = G_i(x, \mathbf{p}(x))$$

together with the continuity of $V_i$. ∎

**Remarks.**

1. In the symmetric case, if the game Hamilton function $G(x, p)$ is convex in $p$, then there are at most two solutions to the equation $G(x, p) = c$, and consequently for given $(x_0, p_0)$, there is only one candidate value of $p$ to "jump to".
2. If all value functions are continuous, and given a point $(x_0, \mathbf{p}_0)$, the possible values of $\mathbf{p}$ to "jump to" are solutions of the system of equations

   $$\mathbf{G}(x_0, \mathbf{p}) = \mathbf{G}(x_0, \mathbf{p}_0).$$

3. Let $\mathbf{u}(x)$ be defined for all $x < x_0$, and assume that there is only one possibility for a nonzero jump at $x_0$, to a continuous strategy $\tilde{\mathbf{u}}(x)$ defined for all $x \geq x_0$. Let

   $$\lim_{x \uparrow x_0} f(x, \mathbf{u}(x)) = A, \qquad \text{and} \qquad \lim_{x \downarrow x_0} f(x, \tilde{\mathbf{u}}(x)) = f(x_0, \tilde{\mathbf{u}}(x_0)) = B.$$

   If the time evolution $x(t)$ is continuous and piecewise differentiable, then it is not possible that simultaneously $A > 0$ and $B < 0$. For, suppose it were the case. Then if $x(t_0) = x_0$, necessarily $x(t) = x_0$ for all $t > t_0$. Hence $B$ should be equal to 0, contradicting the assumption.

∎

# 4 Applications

The class of differential games introduced in the preceding section is fairly general and allows us to study Markov equilibria for a variety of different examples. Here we apply the techniques of the auxiliary system to three alternative models that have been dealt with in the literature: (i) the exploitation of an exhaustible resource (Eswaran and Lewis 1984, Clemhout and Wan 1994, Rincón-Zapatero et al. 1998), (ii) the shallow lake problem (Mäler et al. 2003, Brock and Starrett 2003, Wagener 2003) and (iii) the exploitation of a reproductive asset (Benhabib and Radner 1992, Dockner and Sorger 1996, Benchekroun 2003).

## 4.1 Exploitation of exhaustible assets.

Let $x$ be the stock of some exhaustible resource, and let $u_i$ be player $i$'s rate of exploitation of this resource. Assume that the objective function of this player is given by

$$\mathcal{J}_i = \int_0^\infty L_i(u_i)\,\mathrm{e}^{-\rho t}\,\mathrm{d}t,$$

where $L_i(u_i)$ is the instantaneous utility that exhibits positive and decreasing marginal utility. The stock dynamics of the resource is described by

$$\dot{x} = -\sum_{i=1}^{n} u_i.$$

The function $\mathcal{P}_i$ is given by

$$\mathcal{P}_i = L_i(u_i) - p_i \sum_{i=1}^{n} u_i, \tag{18}$$

yielding $p_i = L_i'(u_i) \equiv \frac{\mathrm{d}L_i(u_i)}{\mathrm{d}u_i}$ if $u_i > 0$, and $p_i = 0$ if $u_i = 0$. This equation is solved by $u_i = v_i(p_i)$. Substitution into (18) yields

$$G_i(\mathbf{p}) = L_i(v_i(p_i)) - p_i \sum_{j=1}^{n} v_j(p_j)$$

Restricting the analysis to symmetric agents for which $L_1 = \cdots = L_n = L$ and symmetric equilibria, we obtain the game Hamiltonian

$$G(p) = L(v(p)) - npv(p).$$

Equation (13) then takes the form

$$\left(L'v' - nv - npv'\right)p'(x) = \rho p.$$

Using that $p = L'(v(p))$ and $v'(p) = 1/L''(v(p))$, and taking out a factor $v$, transforms this to

$$\left(n + (n-1)\frac{L'}{L''v}\right) vp'(x) = -\rho p$$

Switching to controls rather than costates, taking into account the relation

$$p'(x) = L''(u(x))u'(x)$$

yields finally

$$(n + (1-n)E(u))\, u\frac{\mathrm{d}u}{\mathrm{d}x} = \rho E(u)u, \tag{19}$$

where $E(u)$ measures the inverse of relative risk aversion, i.e.,

$$E(u) = -\frac{L'(u)}{L''(u)u}.$$

Using the shadow price system (19) we are able to derive the auxiliary system. It is given by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} &= (n + (1-n)E(u))\, u, \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} &= \rho E(u)u. \end{cases} \tag{20}$$

In the first specific example of this model we will demonstrate the use of the shadow prices instead of the auxiliary system. This is motivated by the fact that under the assumption of a constant elasticity utility function the shadow price system is fully integrable.

The shadow price system (19) reduces to a simple ordinary differential equation when either preferences with constant relative risk aversion (CRRA) or with constant absolute risk aversion (CARA) together with symmetric equilibria are assumed (see Rincón-Zapatero et al. 1998).

Let us consider the case of constant relative risk aversion first. Here the utility functions of the agents are given by

$$L_i(u_i) = \frac{u_i^{1-\eta}}{1-\eta}$$

and the inverse of relative risk aversion is given by

$$E(u) = \frac{1}{\eta}.$$

Equation (19) becomes

$$\left(\frac{n-1}{\eta} - n\right) u\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\rho}{\eta}u,$$

which is solved by either $u(x) = 0$, or, if $n > 1/(1-\eta)$, by the linear function

$$u(x) = Ax, \quad \text{where } A = \frac{\rho}{(1-\eta)n - 1}.$$

This is the symmetric Nash equilibrium strategy for all players; note that in equilibrium, the rate of extraction $A$ is proportional to the discount rate, and it decreases with the number of players. In the limiting case when agents have logarithmic utility, $L_i(u_i) = \ln u_i$ and the elasticity becomes $E(u) = 1$, extraction rates are independent of the number of firms exploiting the resource. They are given by $u(x) = \rho x$.

In case of constant absolute risk aversion, consumer preferences are given by

$$L_i(u_i) = -e^{-\alpha u_i}$$

where $\alpha > 0$ is the constant absolute risk aversion. In this case we get

$$E(u) = \frac{1}{\alpha u},$$

and equation (19) becomes

$$\left(n + (1-n)\frac{1}{\alpha u}\right) u \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\rho}{\alpha}.$$

Under the assumptions that $u(0) = 0$ this equation can be integrated to yield equilibrium extraction rates equal to

$$u(x) = \frac{(n-1) + \sqrt{(1-n)^2 + 2\rho n \alpha x}}{n\alpha}.$$

The integration is especially clear when performed using the auxiliary system (20), which in the present case is given by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} &= nu + \dfrac{1-n}{\alpha} \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} &= \dfrac{\rho}{\alpha}. \end{cases}$$

This can be easily integrated to yield

$$u(s) = u_0 + \frac{\rho}{\alpha}s, \qquad x(s) = x_0 + \left(nu_0 + \frac{1-n}{\alpha}\right)s + \frac{n\rho}{2\alpha}s^2.$$

It follows that the solution curves of the auxiliary system are a family of parabolas. For $u_0 = 0$ and $x_0 = 0$, we find $s = (\alpha/\rho)u$ and

$$x = \frac{1-n}{\rho}u + \frac{n\alpha}{2\rho}u^2.$$

Solving this equation for $u$ results in the explicit solution given above. These equilibrium strategies are decreasing both in the number of firms exploiting the resource and in the level of absolute risk aversion.

### 4.2 Shallow lake.

Consider the following environmental problem. There are $n$ players (countries, communes, farmers) sharing a shallow lake. Each player has revenues from farming, for which artificial fertiliser is used. The use of fertiliser has two opposing effects: more fertiliser means better harvests and hence higher revenues from farming. On the other hand fertiliser is washed from the fields by rainfall and eventually accumulates a stock of phosphorus in the shallow lake. The higher the level of phosphorus the higher are the costs (for fresh water, decreased income from tourism) to the player. Since the level of the stock of phosphorus is the result of activities of all players sharing the lake, the resulting problem can best be described by a differential game. The shallow lake system has been investigated in detail by Dechert and Brock (2000), Mäler et al. (2003), Wagener (2003); we refer to these papers for background information.

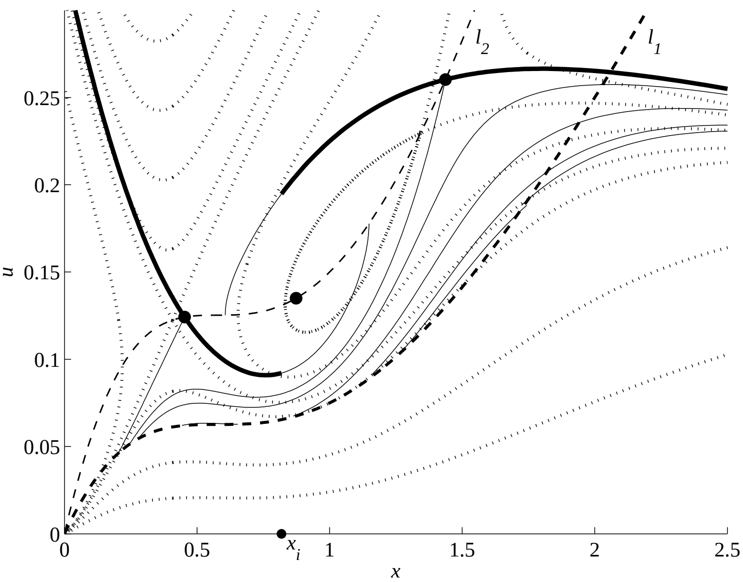



Figure 4: *Solutions of the auxiliary system (solid) and level curves of the value function (dotted) in the symmetric two player case of the shallow lake game. The highlighted curve is a globally defined non-continuous Nash feedback strategy, jumping at the state $x_i$.*

Let the stock variable $x$ represent the amount of phosphorus in a shallow lake and let $u_i$ be the amount of fertiliser used by farmer $i$. Assuming a concave technology to produce farming output and quadratic costs coming from the stock $x$, player $i$ maximises intertemporal utility

$$\mathcal{J}_i = \int_0^\infty (\log u_i - cx^2)\,\mathrm{e}^{-\rho t}\,\mathrm{d}t.$$

The level of phosphorus is assumed to evolve according to the following state equation:

$$\dot{x} = f(x,u) = \sum_{i=1}^n u_i - bx + \frac{x^2}{x^2+1};$$

where we have a constant rate of self-purification (sedimentation, outflow) and the nonlinear term $x^2/(x^2+1)$ is the result of biological effects in the lake.

For this differential game the function $\mathcal{P}_i$ is given by

$$\mathcal{P}_i = \log u_i - cx^2 + p_i \left( \sum_{j=1}^n u_j - bx + \frac{x^2}{x^2+1} \right).$$

Maximising over $u_i$ yields that $u_i = -1/p_i$. Restricting again our attention to symmetric strategies, we find on setting $p_j = p$ for all $j = 1, \cdots, n$ that

$$G(x,p) = -\log(-p) - cx^2 - n + p\left(-bx + \frac{x^2}{x^2+1}\right).$$

The auxiliary system now reads as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = & \dfrac{\partial G}{\partial p} & = -\dfrac{1}{p} - bx + \dfrac{x^2}{x^2+1}, \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} = \rho p - & \dfrac{\partial G}{\partial x} & = (\rho+b)p + 2cx - \dfrac{2px}{(x^2+1)^2}, \end{cases}$$

or, in terms of controls, as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} & = u - bx + \dfrac{x^2}{x^2+1}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} & = -(\rho+b)u + 2cu^2x + \dfrac{2ux}{(x^2+1)^2}. \end{cases}$$

Solutions to the auxiliary system are given in figure 4. The most important feature of the solution set is that there is a globally defined non-continuous Nash feedback strategy, indicated by a thick line in the figure. Indeed, it has been known for some time that the Hamilton-Jacobi equation of some economic optimal control problems may have jumps in the policy function, see Skiba (1978), and for the shallow lake model Mäler et al. (2003) and Wagener (2003). Since the game Hamilton-Jacobi equation for the case of two or more players is identical to that of the one player case, the same jump occurs. Note that the feedback Nash strategies that are parametrised by parts of the stable and unstable manifolds of one of the saddle points of the auxiliary system are continuous, but not continuously differentiable everywhere.

Finally notice that the auxiliary system does not depend on the number of agents, and therefore coincides with the state–control system of the shallow lake optimal control problem. In practical terms, this means that figure 4 can be used to analyse the situation for any number of players. The only difference is in the symmetric time dynamics

$$\dot{x} = nu - bx + \frac{x^2}{1+x^2}.$$

Increasing the number of players $n$ leads to a decrease of the isocline $\dot{x} = 0$. In particular, though this will not be demonstrated here, for large values of $n$ no states in the 'clean' region can be stabilised by a locally defined feedback Nash equilibrium strategy.

### 4.3 Exploitation of reproductive assets.

As the final example consider the problem where $n$ agents strategically exploit a single reproductive asset, like fish or other species (see Dockner and Sorger 1996). The reproduction of the stock $x$ occurs at rate $h(x)$, whereas player $i$ extracts the stock at rate $u_i$. Hence, the state dynamics is given by

$$\dot{x} = h(x) - \sum_{i=1}^{n} u_i. \tag{21}$$

Let us assume that the instantaneous utility that agent $i$ derives from the consumption of the stock is given by

$$L_i(u_1, \cdots, u_n)$$

so that his utility functional becomes

$$\mathcal{J}_i = \int_0^\infty L_i(u_1, \cdots, u_n)\, \mathrm{e}^{-\rho t}\, \mathrm{d}t.$$

This functional covers several cases. The fish catch can be sold in an imperfect output market. In that case the price of the fish depends on the total quantity produced by all fisheries and therefore the function $L_i(u)$ depends on the extraction rates of all players. In case of a duopoly market with linear demand this problem was studied in Benchekroun (2003).

Alternatively, the function $L_i(u)$ can only depend on the exploitation rate of player $i$. In this case each agents maximises the present value of future utility derived from consuming the fish. This formulation was dealt with in Dockner and Sorger (1996).

### 4.3.1 Oligopoly.

Let us start with the duopoly model studied by Benchekroun (2003); actually we shall treat the general oligopoly model, as this involves no additional complexity. That is, there are $n$ agents and the utility (revenue) functions are given by

$$L_i(u) = \left( a - b \sum_{j=1}^{n} u_j \right) u_i.$$

Moreover, the production function is specified as

$$h(x) = \begin{cases} \delta x & \text{for} \quad x \le \frac{1}{2} \\ \delta(1-x) & \text{for} \quad x > \frac{1}{2}. \end{cases}$$

Note that the production function $h$ is piecewise linear and the utility function is quadratic: the problem can be seen as two linear-quadratic subproblems glued together. The analysis is restricted to the case that marginal product is large enough to satisfy

$$\delta > \max\left\{ \frac{(n^2+1)\rho}{2}, \frac{2(n^2+1)a}{(n+1)^2 b} \right\}.$$

The function $\mathcal{P}_i$ is given by

$$\mathcal{P}_i = a u_i - b u_i \sum_{j=1}^{n} u_j + p_i \left( h(x) - \sum_{j=1}^{n} u_j \right).$$

This yields for $i = 1, \cdots, n$ that

$$p_i = a - 2b u_i - b \sum_{j \ne i} u_j.$$

Solving this system of linear equations for the $u_i$ yields that

$$u_i = \frac{a - n p_i + \sum_{j \ne i} p_j}{(n+1)b}.$$

Restricting to the symmetric case reduces this to $u = (a-p)/(b(n+1))$. Substitution in $\mathcal{P}_i = \mathcal{P}$ and simplification yields the game Hamiltonian

$$G = \frac{(a - n^2 p)(a - p)}{(n + 1)^2 b} + ph(x).$$

The auxiliary system takes the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} & = h(x) - \dfrac{(n^2+1)}{(n+1)^2}\dfrac{a}{b} + \dfrac{2n^2}{(n+1)^2 b}p, \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} & = (\rho - h'(x))p \end{cases}$$

In state-control variables, it takes the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} & = h(x) + \dfrac{n-1}{n+1}\dfrac{a}{b} - \dfrac{2n^2}{n+1}u, \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} & = (\rho - h'(x))\left(u - \dfrac{a}{(n+1)b}\right). \end{cases} \tag{22}$$

Given the assumption on the production function, marginal product is piecewise constant, i.e. $h'(x) = \delta$ or $h'(x) = -\delta$. The steady states of the auxiliary system are then given as

$$(x_0, u_0) = \left(\frac{n^2+1}{(n+1)^2}\frac{a}{b\delta}, \frac{a}{(n+1)b}\right)$$

and

$$(x_0, u_0) = \left(1 - \frac{n^2+1}{(n+1)^2}\frac{a}{b\delta}, \frac{a}{(n+1)b}\right).$$

**Globally defined strategy.** Since the oligopoly game is of linear quadratic type, it makes sense to look for a linear Markov equilibrium. The linear equilibrium, however, cannot be applied for the entire state space. Whenever the stock level is large enough it is optimal for the firms to chose the steady state level $u_0 = a/((n+1)b)$ and stay at this level. Prior to reaching this upper steady state firms can choose linear Markov strategies. They can be derived making use of equation (22). Let us assume that in the appropriate state space range strategies are linear, that is $u(x) = \alpha x + \beta$, and that on that range $h(x) = \delta x$. Using the fact that for all $x$ the relationship

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha\frac{\mathrm{d}x}{\mathrm{d}s}$$

is satisfied, it follows that in order to determine $\alpha$ and $\beta$, we can substitute for the derivatives $\mathrm{d}u/\mathrm{d}s$ and $\mathrm{d}x/\mathrm{d}s$ the expressions given in equation (22). This yields on either side of the equality sign a linear function in $x$ whose coefficients should be equal. Equating the coefficients of $x$ yields the following equation for $\alpha$:

$$\alpha\left(\delta - \frac{2n^2}{n+1}\alpha\right) = \alpha(\rho - \delta).$$

We find that

$$\alpha = 0 \quad \text{or} \quad \alpha = \frac{n+1}{2n^2}(2\delta - \rho);$$

the linear strategies are then given by

$$u(x) = \frac{a}{(n+1)b} \tag{23}$$

and

$$u(x) = \frac{n+1}{2n^2}(2\delta - \rho)x - \frac{a}{b\delta}\,\frac{2\delta - (n^2+1)\rho}{2n^2(n+1)}. \tag{24}$$

It is now easily seen that the equilibrium strategy consists of three parts. For stock levels below

$$x_1 = \frac{a}{(n+1)^2 b\delta}\,\frac{2\delta - (n^2+1)\rho}{2\delta - \rho},$$

equilibrium production is zero. For levels above

$$x_2 = \frac{n^2+1}{(n+1)^2}\frac{a}{b\delta},$$

the optimal policy of the firms is to choose $u_0 = a/((n+1)b)$. For intermediate levels it is optimal to choose the linear Markov strategy given by (24). This equilibrium is illustrated in figure 5. Note that the conditions

$$\delta > \frac{(n^2+1)\rho}{2} \quad \text{and} \quad \delta > \frac{2(n^2+1)}{(n+1)^2}\frac{a}{b}$$

ensure that $0 < x_1 < x_2 < 1/2$.

**Locally defined strategies.** As can be seen in figure 5, the piecewise linear strategy is the only globally defined Markov perfect Nash equilibrium strategy. Indicated in the figure are also locally defined Markovian equilibrium strategies. We shall determine all states that can be stabilised by a locally defined Nash equilibria.

Note that a point $(x, u) = (x, u(x))$ corresponds to a steady state if $f(x) = h(x) - nu(x) = 0$. It corresponds to a stable steady state if moreover

$$f'(x) = h'(x) - nu'(x) < 0. \tag{25}$$

If the graph of $x \mapsto u(x)$ is parametrised as a trajectory $(x(s), u(s))$ of the auxiliary vector field, then $u'(x) = \frac{\mathrm{d}u}{\mathrm{d}s}/\frac{\mathrm{d}x}{\mathrm{d}s}$ along this trajectory. The stability condition (25) takes the form

$$h'(x) - n\frac{(\rho - h'(x))\left(u - \frac{1}{n+1}\frac{a}{b}\right)}{h(x) + \frac{n-1}{n+1}\frac{a}{b} - \frac{2n^2}{n+1}u} < 0. \tag{26}$$

In the region $0 < x < \frac{1}{2}$, where $h(x) = \delta x$ and $u = \delta x/n$, this inequality can be rewritten to

$$\delta + \frac{(\delta - \rho)(\delta x - \frac{n}{n+1}\frac{a}{b})}{\frac{n-1}{n+1}\left(\frac{a}{b} - \delta x\right)} < 0.$$

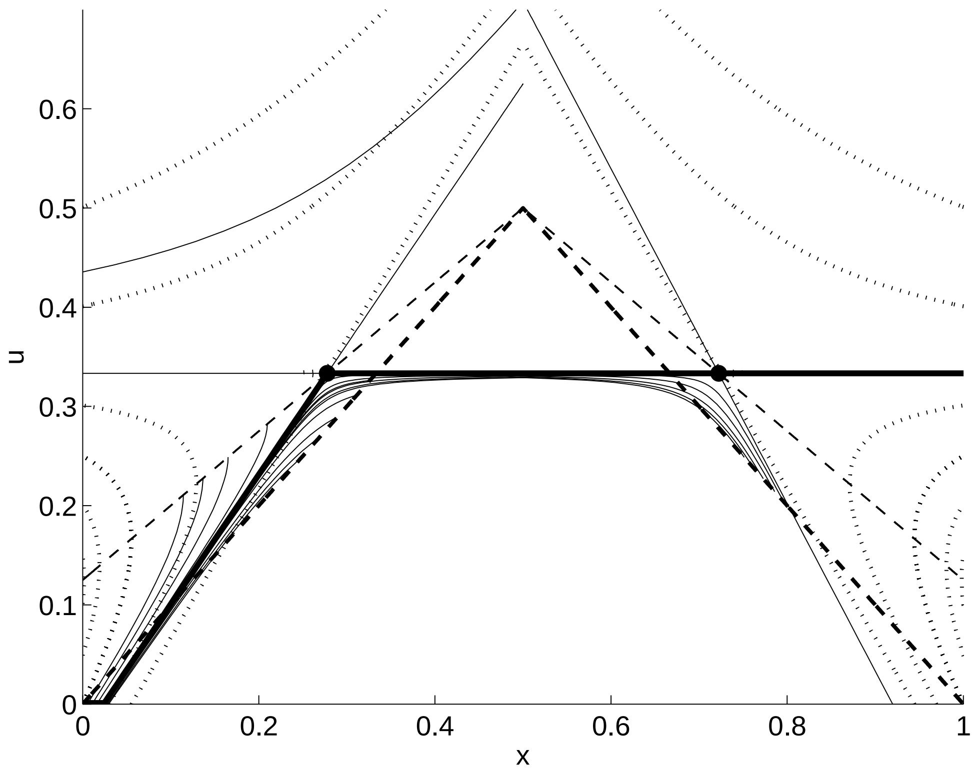



Figure 5: *Solutions of the auxiliary system (solid) and level curves of the value function (dotted) in the symmetric two player duopoly with production function* $h(x) = \delta \min\{x, 1-x\}$. *Parameters are* $a = b = 1$, $\delta = 2$, $\rho = 1/2$. *The highlighted curve is the piecewise linear solution described in the text.*

If $\frac{1}{2} < x < 1$, then $h(x) = \delta(1-x)$ and $u = \delta(1-x)/n$ and inequality (26) takes the form

$$-\delta - \frac{(\delta+\rho)(\delta(1-x) - \frac{n}{n+1}\frac{a}{b})}{\frac{n-1}{n+1}\left(\frac{a}{b} - \delta(1-x)\right)} < 0.$$

With a little straightforward algebra, we derive from this inequality the following result.

**Theorem 4.** *If $\delta > (n+1)\rho/2$, then a state $x \in (0, \frac{1}{2})$ is stabilisable by a locally defined Markov perfect Nash equilibrium strategy, if*

$$0 < x < \frac{\delta - n\rho}{2\delta - (n+1)\rho}\frac{a}{b\delta} \quad \textit{or} \quad \frac{a}{b\delta} < x < \frac{1}{2}.$$

*The corresponding steady state values of $u$ satisfy*

$$0 < u < u_{\max} \stackrel{\text{def}}{=} \frac{\delta - n\rho}{2\delta - (n+1)\rho}\frac{a}{nb} \quad \textit{or} \quad \frac{a}{nb} < u < \frac{1}{2}.$$

*A state $x \in (\frac{1}{2}, 1)$ is stabilisable if*

$$1 - \frac{a}{b\delta} < x < 1 - \frac{\delta + n\rho}{2\delta + (n+1)\rho}\frac{a}{b\delta}.$$

*The corresponding steady state values of $u$ satisfy*

$$\frac{\delta + n\rho}{2\delta + (n+1)\rho}\frac{a}{nb} \stackrel{\text{def}}{=} u_{\min} < u < \frac{a}{nb}.$$

There is an interesting economic result that follows from these propositions. Recall that in the analogous static oligopoly game, where players act as if there is always a sufficient amount of stock available, the Cournot-Nash equilibrium strategy is given as

$$u_{\text{Nash}} = \frac{a}{(n+1)b},$$

whereas the collusive strategy is given as

$$u_{\text{collusive}} = \frac{a}{2nb}.$$

Note that the boundary points $u_{\max}$ and $u_{\min}$ of the intervals of stabilisable values of $u$, satisfy

$$u_{\text{collusive}} = u_{\max} + \frac{n-1}{4\delta - 2(n+1)\rho}\rho = u_{\max} + \mathsf{O}(\rho)$$

and

$$u_{\text{collusive}} = u_{\min} - \frac{n-1}{4\delta + 2(n+1)\rho}\rho = u_{\min} - \mathsf{O}(\rho)$$

as $\rho \to 0$. In other words:

**Theorem 5.** *In the dynamic oligopoly game with $n$ players, the collusive static exploitation rates can never be stabilised. However, they can be realised, up to a term of order $\rho$, as steady state exploitation rates of locally defined Markov perfect Nash equilibrium strategies.*

Actually, it is easy to show that this result is independent of the specifications of $h$ and the demand function.

### 4.3.2 Constant relative risk aversion.

We now proceed with the case in which each agent has a constant relative risk aversion utility function. Let $0 < \sigma < 1$ and specify the utility functional of player $i$ as

$$\mathcal{J}_i = \int_0^\infty \frac{u_i^{1-\sigma}}{1-\sigma} \mathrm{e}^{-\rho t} \, \mathrm{d}t.$$

The function $\mathcal{P}_i$ becomes

$$\mathcal{P}_i = \frac{u_i^{1-\sigma}}{1-\sigma} + p_i \left( h(x) - \sum_{j=1}^{n} u_j \right).$$

From $\partial \mathcal{P}_i / \partial u_i$ we obtain $p_i = u_i^{-\sigma}$ and $u_i = p_i^{-1/\sigma}$, and the game Hamilton functions read as

$$G_i = \frac{1}{1-\sigma} p_i^{(\sigma-1)/\sigma} + p_i \left( h(x) - \sum_{j=1}^{n} p_j^{-1/\sigma} \right)$$

In the symmetric case $p_1 = \cdots = p_n = p$, this simplifies to

$$G = \frac{1-n+n\sigma}{1-\sigma} p^{(\sigma-1)/\sigma} + p h(x),$$

and we obtain the auxiliary system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} & = \dfrac{n-1-n\sigma}{\sigma} p^{-1/\sigma} + h(x) \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} & = (\rho - h'(x)) p \end{cases}$$

Using the relation $u = p^{-1/\sigma}$, we find the form of the auxiliary system in state-control variables:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} & = \dfrac{n-1-n\sigma}{\sigma} u + h(x) \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} & = \dfrac{h'(x)-\rho}{\sigma} u \end{cases}$$

The case $\sigma = (1 - 1/n)$ is special, since then the system can be integrated analytically, yielding

$$u(x) = C h(x)^{n/(n-1)} \exp\left( -\frac{n\rho}{n-1} \int_{x_0}^{x} h(\xi)^{-1} \, \mathrm{d}\xi \right).$$

Compare equation (4) of Dockner and Sorger (1996).

**Stability of steady states.** As above, for a given symmetric Nash equilibrium strategy $u(x)$, the state dynamics are given as $f(x) = h(x) - nu(x)$. A state-control pair $(x, u)$, with $u = u(x)$, corresponds to a steady state for these dynamics if $f(x) = h(x) - nu = 0$, that is,

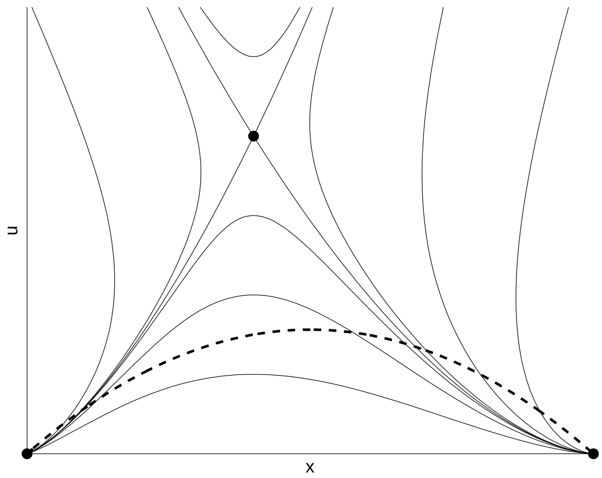



Figure 6: *Solutions of the auxiliary system (solid) and the isocline* $\dot{x} = h(x) - nu = 0$ *(dashed) in the symmetric two player case of the fishery model with production function* $h(x) = x(1-x)$ *and parameters* $\rho = 0.2$ *and* $\sigma = 0.8$.

if $u = h(x)/n$. The state is locally attracting if $f'(x) < 0$. We compute, using the relation $u = h(x)/n$:

$$f'(x) = h'(x) - nu'(x) = h'(x) - n\frac{\frac{\mathrm{d}u}{\mathrm{d}s}}{\frac{\mathrm{d}x}{\mathrm{d}s}}$$
$$= h'(x) - n\frac{\frac{h'(x)-\rho}{\sigma}u}{\frac{n-1-n\sigma}{\sigma}u + h(x)} = \frac{n\rho - h'(x)}{n-1}.$$

It follows that $(x, u)$ corresponds to an attracting steady state if

$$\rho < \frac{1}{n}h'(x),$$

and to an unstable state if the inequality sign is reversed. In particular, if $h'(x) < 0$, then the point $(x, h(x)/n)$ always corresponds to an unstable equilibrium for the state dynamics. Moreover, since the derivative $h'(x)$ is bounded from above, if $\rho > 1/n \cdot \max h'(x)$ then the state dynamics do not have stable equilibria in the interior of the state space.

Let us finally consider the "semi-stable" state $\bar{x}$ that satisfies

$$\rho = h'(\bar{x})/n;$$

note that this point is in the boundary of the set of all stabilisable states. Compare it to the optimal long term steady state $x_{\text{collusive}}$ of the collusive outcome, for which

$$\rho = h'(x_{\text{collusive}})$$

holds, the so-called "golden rule". Note that the strategic behaviour in the semi-stable state $\bar{x}$ can be described as each player behaving as if he had a private fish stock available with reproduction rate $h(x)/n$.

**Case study.** In the following, it will be assumed that $h$ is a twice differentiable strictly concave function which satisfies $h(0) = h(1) = 0$. Note that this implies that $h'(0) > 0$, $h'(1) < 0$, and that $h'$ is a strictly decreasing function on $(0, 1)$. In the illustrations the specification $h(x) = x(1 - x)$ will be used.

**Analysis of the auxiliary system.** We shall assume that $\rho \leq h'(0)$. Then there is a unique point $x_\rho \in [0, 1)$ such that $h'(x_\rho) = \rho$. The auxiliary system has then fixed points

$$(x, u) \in \left\{ (0,0), (1,0), \left( x_\rho, \frac{\sigma}{1 - n(1-\sigma)} h(x_\rho) \right) \right\}.$$

Note that the third equilibrium satisfies $u > 0$ only if $n < 1/(1 - \sigma)$. Solutions to the auxiliary system are sketched in figure 6.

**Lemma 1.** *No Markov perfect Nash equilibrium $x \mapsto u(x)$ is defined on an interval that does include the point $x = 1$.*

**Proof**
We linearise the auxiliary system at $(x, u) = (1, 0)$. Introducing linearising coordinates $\xi = x - 1$ and $\eta = u$, the linearised system reads as

$$\begin{cases} \dot{\xi} &= h'(1)\xi + \dfrac{n(1-\sigma) - 1}{\sigma} \eta \\ \dot{\eta} &= \dfrac{h'(1) - \rho}{\sigma} \eta. \end{cases}$$

The characteristic values $\lambda_1 = h'(1)$ and $\lambda_2 = h'(1) - \rho$ of this system satisfy $\lambda_2 < \lambda_1 < 0$. To $\lambda_1$ is associated the eigenvector $v_1 = (1, 0)$. It follows that almost all solution trajectories of the auxiliary system are tangent to the horizontal axis at $(x, u) = (1, 0)$.

More precise information can be obtained by the following lemma, which can be verified by direct substitution.

**Lemma 2.** *The integral curves of the system $\dot{\xi} = \alpha\xi + \beta\eta$, $\dot{\eta} = \gamma\eta$ satisfy the relation*

$$\xi = C|\eta|^{\frac{\alpha}{\gamma}} - \frac{\beta}{\alpha - \gamma}\eta, \tag{27}$$

*where $C$ is an integration constant.* Substituting $\alpha = h'(1)$, $\beta = (n(1 - \sigma) - 1)/\sigma$ and $\gamma = (h'(1) - \rho)/\sigma$, this lemma yields that $\xi = C|\eta|^{1/\delta} + \ldots$, where

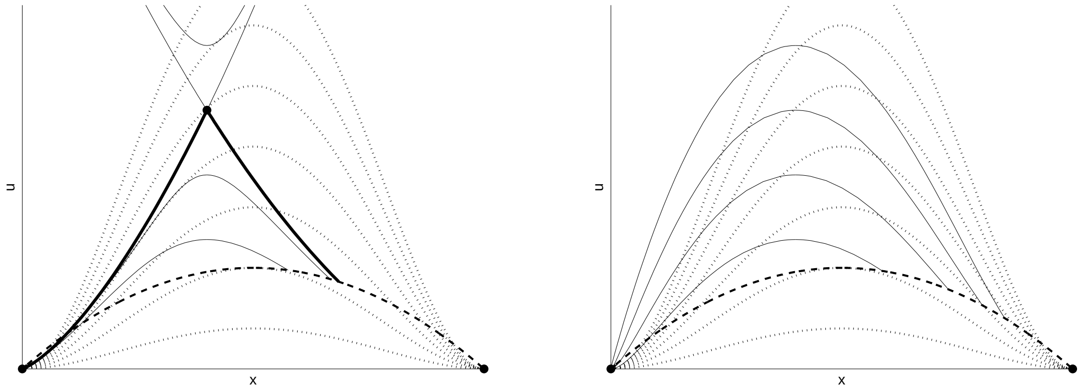
$$\delta = -\frac{\rho - h'(1)}{\sigma h'(1)} = \frac{1}{\sigma} - \frac{\rho}{\sigma h'(1)} > 1.$$

It follows that the integral curve through $(\xi_0, \eta_0)$ is to lowest order given as

$$\eta = \eta_0 \left( \frac{\xi}{\xi_0} \right)^\delta + \cdots .$$

The state dynamics at the equilibrium read as

$$\dot{\xi} = h'(1)\xi + \frac{n(1-\sigma) - 1}{\sigma} \eta_0 \left( \frac{\xi}{\xi_0} \right)^\delta + \cdots ;$$




(a) $\sigma = 0.8$

(b) $\sigma = 0.4$

Figure 7: *Solutions of the auxiliary system (solid) and level curves of the value function (dotted) in the symmetric two player case of the fishery model with production function* $h(x) = x(1-x)$ *and* $\rho = 0.2$.

as $\delta > 1$, it follows that $\xi(t) = \mathrm{e}^{h'(1)t}\xi_0 + \cdots$, and consequently $u(t) = \eta(t) = \eta_0\,\mathrm{e}^{\delta h'(1)t} + \cdots$. Recalling that $p = u^{-\sigma}$, we finally obtain

$$p(t) = p_0\,\mathrm{e}^{(\rho - h'(1))t}.$$

Hence, as $h'(1) < 0$, for all solutions of the auxiliary system that tend to the equilibrium

$$(x_0, u_0) = (1, 0),$$

the transversality condition $\lim_{t\to\infty} p(t)\,\mathrm{e}^{-\rho t} = 0$ is violated. ∎

Using the lemma, we have sketched the symmetric Markov perfect Nash equilibria in figure 7. A characteristic feature of these strategy equilibria is that if the initial fish stock is higher than the semi-stable threshold value $\bar{x}$ introduced above, it cannot be stabilised. Moreover, for these non-stabilisable initial stocks, we see that as the initial stock is larger, the eventually reached steady state stock grows smaller.

**Asymmetric strategies.** Here the assumption is dropped that the players play symmetric strategies; for simplicity, we restrict to the two–player case $n = 2$ and assume that $\sigma = 1/2$ holds. Then

$$G_i(x, p) = \frac{2}{p_i} + p_i\left(h(x) - \frac{1}{p_1^2} - \frac{1}{p_2^2}\right)$$

The system (13) takes the form

$$\begin{pmatrix} h(x) - \dfrac{1}{p_1^2} - \dfrac{1}{p_2^2} & 2\dfrac{p_1}{p_2^3} \\ 2\dfrac{p_2}{p_1^3} & h(x) - \dfrac{1}{p_1^2} - \dfrac{1}{p_2^2} \end{pmatrix} \begin{pmatrix} \dfrac{\mathrm{d}p_1}{\mathrm{d}x} \\ \dfrac{\mathrm{d}p_2}{\mathrm{d}x} \end{pmatrix} = (\rho - h(x)) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Using the relation $p_i = u_i^{-1/2}$, this system takes the form

$$\begin{pmatrix} h - u_1 - u_2 & 2u_1 \\ 2u_2 & h - u_1 - u_2 \end{pmatrix} \begin{pmatrix} \dfrac{\mathrm{d}u_1}{\mathrm{d}x} \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}x} \end{pmatrix} = 2(h' - \rho) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

It is convenient to consider instead of $u_1$ and $u_2$ the quantities $v = u_1 + u_2$ and $w = u_1 - u_2$; the systems then takes the simpler form

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= 2\frac{h' - \rho}{\Delta}\left(hv - 2v^2 + w^2\right), \\ \frac{\mathrm{d}w}{\mathrm{d}x} &= 2\frac{h' - \rho}{\Delta}(h - v)w, \end{aligned}$$

with $\Delta(x) = h^2 - 2hv + w^2 = (h - v)^2 + w^2 - v^2$. The auxiliary system to this system of equations reads as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= \Delta = h^2 - 2hv + w^2, \\ \frac{\mathrm{d}v}{\mathrm{d}s} &= 2(h' - \rho)\left(hv - 2v^2 + w^2\right), \\ \frac{\mathrm{d}w}{\mathrm{d}s} &= 2(h' - \rho)(h - v)w. \end{aligned}$$

Note that the plane $w = 0$, corresponding to the symmetric case $u_1 = u_2$, is invariant under the flow of the auxiliary system; in other words, that case is nested in the present one.

We will not give a full analysis of this system, leaving that to future work. However, we would like to point out one consequence of the equation $\dot{w} = 2(h' - \rho)(h - v)w$. Recall that $\dot{x} = h - v$; hence, if the system is on a time path for which the stock decreases, the factor $h - v < 0$, and the sign of $\dot{w}/w$ is the opposite of $h' - \rho$.

In the example above, the factor $h' - \rho$ is positive for small $\rho$ and small $x$, and it follows that then the differences between strategies decay exponentially if the stock decreases towards an equilibrium close to $x = 0$. Conversely, if $\rho$ sufficiently large, differences between strategies increase exponentially, which can be seen as a mad scramble to exploit the last remnants of the stock.

# 5 Conclusions

In this article, a framework has been elaborated to find necessary conditions for Markov Nash equilibrium strategies in differential games with a single state variable. The Nash equilibria have been characterized as solutions of a system of explicit first order ordinary differential equations, usually nonlinear.

By analyzing a series of classical examples, we have shown that this characterization can be used to find both direct analytic information, by integration of the equations, and indirect qualitative information, by a geometric analysis of the solution curves of an auxiliary system in the phase space.

Additionally, we have addressed the issues of continuity and differentiability of Markov strategies in this class of differential games. In particular, in the shallow lake model, we have shown the existence of a non-continuous Markov-perfect Nash equilibrium. Our simple approach is capable enough to deliver interesting insights into a large class of capital accumulation games.

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