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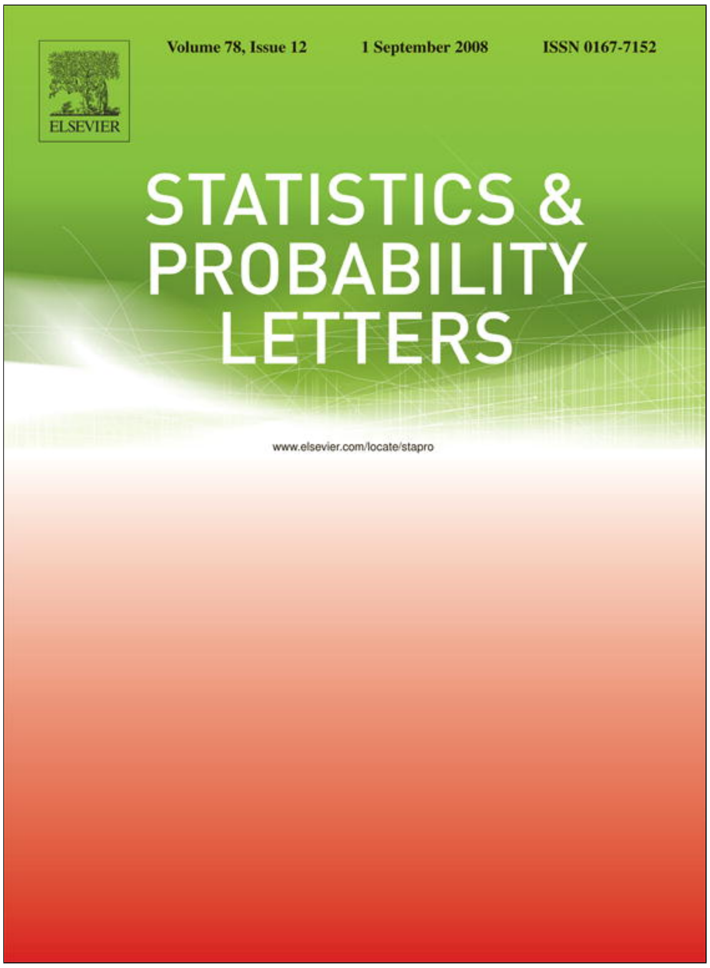
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Asymptotic behavior of the variance of the EWMA statistic for autoregressive processes

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Abstract

Serial correlation can seriously affect the performance of traditional control charts. Many authors have studied the effect of autocorrelation on EWMA control charts and have shown how to modify the control limits to account for autocorrelation. In this paper we compare three different estimation methods for the variance of the EWMA statistic that is adapted to autocorrelated data. This comparison is based on the asymptotic relative efficiency of the estimators.

1. Introduction

Today's manufacturing environment hardly resembles the period in which control chart methods were introduced. Due to high-tech measurement devices, the sampling rate may increase easily. Because of this increasing sampling rate, independence of successive observations – one of the fundamental assumptions for control charts – is often violated. This so-called serial correlation of the observations heavily affects the performance of the traditional Shewhart control charts. This justifies the development of control charts which take serial correlation into account.

As it turns out, two approaches for dealing with autocorrelated data can be distinguished. The first approach is to use standard control charts with adjusted control limits for autocorrelation (cf. Zhang (1998a,b), Jiang et al. (2000), Apley and Lee (2003), and Shiau and Hsu (2005)). The second approach consists of fitting a time series model, followed by a study of the residuals with standard control charts such as individual charts, $\bar{X}$ charts or Exponentially Weighted Moving Average (EWMA) charts (cf. Montgomery and Mastrangelo (1991), Wardell et al. (1994), and Koehler et al. (2001)). In this paper we restrict ourselves to EWMA charts with adjusted control limits.

Consider observations from a stationary process $X = (X_t, t = 0, 1, \ldots)$. The EWMA statistic at time t is defined by

$$W_t = \lambda X_t + (1 - \lambda) W_{t-1}, \quad t = 0, 1, \ldots \tag{1}$$

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where λ is a constant satisfying $0 < \lambda \leq 1$. The control limits of the EWMA chart converge to $\mu \pm L\sqrt{\alpha}$, where L is an appropriately chosen constant (often 3), $\mu = \mathrm{E}X$ and α is given by

$$\alpha = \lim_{t\to\infty} \mathrm{Var}(W_t).$$

Therefore, after an initial period, the control limits are determined by α.

We are interested in estimating α. Let $\rho(k)$ denote the autocorrelation of X at lag $k \in \mathbb{N}$. Zhang (1998a) approximates α by $\mathrm{Var}(W_M)$ for M sufficiently large. The expression obtained in this way depends on the autocorrelations at lags up till M, λ, and σ_X^2. Subsequently, Zhang (1998a) uses sample autocorrelations to estimate the first M autocorrelation coefficients $\rho(1), \ldots, \rho(M)$. Knowing that X is an AR(1) or AR(2) process, we can estimate α by a plug-in estimator, using the Yule–Walker estimators for the model parameters (cf. Brockwell and Davis (1991)). It is a natural question to ask which of these estimators should be preferred. It is the goal of the paper to answer this question. To analyze the performance of the estimators, we study their asymptotic properties as the number of observations tends to infinity. Their asymptotic distributions are used to study the relative efficiency of the estimators. It turns out that if X is an autoregressive process, the estimator based on plugging in the Yule–Walker estimators outperforms the estimator of Zhang (1998a) in terms of relative efficiency. Intuitively, this was to be expected since less parameters have to be estimated and more information is available. The analysis is checked by a simulation study.

This paper is organized as follows. In the next section we present a few preliminary results on autoregressive processes. In Section 3 we present the variances of the EWMA statistic as in Zhang (1998a), Schmid (1997), and Vermaat et al. (2008). In Section 4 we introduce the estimators for these variances. In the subsequent section we will study their asymptotic distributions and their relative efficiencies. Section 6 contains a simulation study on the effect of model misspecification. We finish this paper with some concluding remarks.

2. Preliminaries on second order autoregressive processes

In this section we summarize some results on autoregressive processes that we will need later on. Let $A = (A_t, t = 0, 1, \ldots)$ be a white-noise series, i.e. a series of zero mean, uncorrelated random variables. A second order autoregressive process (an AR(2) process) $X = (X_t, t = 0, 1, \ldots)$ is defined via the recursive relation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + A_t, \quad t = 1, 2, \ldots. \tag{2}$$

Define the quadratic function ϕ by $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. If

$$\phi(z) \neq 0 \quad \text{for all } z \in \mathbb{C} \text{ with } |z| \leq 1, \tag{3}$$

then the unique stationary AR(2) process exists and is given by $X_t = \sum_{j=0}^{\infty} \psi_j A_{t-j}$ where $\{\psi_j\}_{j=0}^{\infty}$ is a sequence of numbers that is absolutely summable and which can be determined from ϕ by the relation $\psi(z) = 1/\phi(z) = \sum_{j=0}^{\infty} \psi_j z^j$ (see Brockwell and Davis (1991), Theorem 3.1.1). The latter implies that the relationship between A and X is *causal* (cf. Brockwell and Davis (1991), Definition 3.1.3). The set of values of (ϕ_1, ϕ_2) for which $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| \leq 1$ is given by

$$\{(\phi_1, \phi_2) \in \mathbb{R}^2 : \phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1 \text{ and } -1 < \phi_2 < 1\}.$$

Of course, the case that $\phi_2 = 0$ corresponds to a first order autoregressive process.

We recall that the autocovariance function of a stationary time series is given by $\gamma(h) = \mathrm{cov}(X_{t+h}, X_t)$, $h = 0, 1, \ldots$. The autocorrelation function is given by $\rho(h) = \gamma(h)/\gamma(0)$.

3. The variance of the EWMA statistic

In this section we give expressions for the variance of the EWMA statistic. Zhang (1998a) derived the variance of the EWMA statistic W_t at time t as

$$\alpha_t = \mathrm{Var}(W_t) = \frac{\lambda}{2-\lambda}\sigma_X^2 \left(1 - (1-\lambda)^{2t} + 2\sum_{k=1}^{t-1} \rho(k)(1-\lambda)^k (1 - (1-\lambda)^{2(t-k)})\right). \tag{4}$$

If $t \to \infty$, this becomes

$$\alpha := \lim_{t\to\infty} \alpha_t = \frac{\lambda}{2-\lambda}\left(2\sum_{k=0}^{\infty}\rho(k)(1-\lambda)^k - 1\right)\sigma_X^2. \tag{5}$$

The following lemma shows that if we assume that X is a causal stationary AR(2) process, α can be expressed in the AR(2) model parameters.

Lemma 1. *Suppose X is a causal stationary AR(2) process as defined in (2). Then*

$$\alpha = \frac{\lambda}{2-\lambda}\left(\frac{\phi_1(1+\phi_2)(\lambda-1)+(\phi_2-1)(1+\phi_2(\lambda-1)^2)}{(1-\phi_2)(-1+\phi_1(1-\lambda)+\phi_2(\lambda-1)^2)}\right)\sigma_X^2. \tag{6}$$

In particular; if $\phi_2 = 0$ (i.e. an AR(1) process), then

$$\alpha = \frac{\lambda}{2-\lambda}\left(\frac{1+\phi_1(1-\lambda)}{1-\phi_1(1-\lambda)}\right)\sigma_X^2. \tag{7}$$

Proof. The autocorrelation function $\rho(k)$ for an AR(2) process satisfies the second order difference equation

$$\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2), \quad k > 0 \tag{8}$$

with starting values $\rho_0 = 1$ and $\rho_1 = \phi_1/(1-\phi_2)$. The general solution of (8) can be found in Brockwell and Davis (1991) p. 108. As it turns out, this solution depends on the zeros of the characteristic equation corresponding to (8): $\pi^2 - \phi_1\pi - \phi_2 = 0$. These zeros are given by

$$\nu_1 = \left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)/2, \qquad \nu_2 = \left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)/2. \tag{9}$$

Hence, we can distinguish three cases: (i) both zeros are real, (ii) there is one real zero with multiplicity two, (iii) both zeros are conjugate complex numbers. Some tedious computations yield that the solution to (8) is given by

$$\rho(k) = \begin{cases} \dfrac{\nu_2^k(\phi_1+\nu_1(\phi_2-1)) - \nu_1^k(\phi_1+\nu_2(\phi_2-1))}{(\nu_1-\nu_2)(\phi_2-1)} & \text{if } \nu_1 \neq \nu_2 \text{ are real} \\ \dfrac{\nu^{k-1}(-k\phi_1+\nu(k-1+\phi_2(1-k)))}{\phi_2-1} & \text{if } \nu_1 = \nu_2 = \nu \text{ is real,} \\ c\nu_1^k + \bar{c}\,\overline{\nu_1}^k \quad \text{if } \nu_1 = \overline{\nu_2} \text{ are complex,} \end{cases} \tag{10}$$

where $c = (1 - i\phi_1(1+\phi_2)(1-\phi_2)^{-1}|\phi_1^2 + 4\phi_2|^{-1/2})/2$. Note that condition (3) ensures that $\max\{|\nu_1|, |\nu_2|\} < 1$. Hence, substituting (10) into (5) yields a convergent series. After some thorough calculations we find that

$$\sum_{k=0}^{\infty}\rho(k)(1-\lambda)^k = \frac{-1+\phi_2(1+(\lambda-1)\phi_1)}{(\phi_2-1)(1+(\lambda-1)\phi_1-(1-\lambda)^2\phi_2)}.$$

Substituting this expression into (5) gives the result. □

Remark. (i) Denote the variance of A by σ_A^2. The variance of the marginal distribution of X is related to σ_A^2 by (cf. Box et al. (1994), p. 62)

$$\sigma_X^2 = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_A^2}{(1-\phi_2)^2-\phi_1^2}.$$

Substituting this expression into (7), we recover the result of Schmid (1997): for a causal stationary AR(1) process α is given by

$$\alpha = \left(\frac{\lambda}{2-\lambda}\right)\frac{\sigma_A^2}{(1-\phi_1^2)}\frac{1+\phi_1(1-\lambda)}{1-\phi_1(1-\lambda)}. \tag{11}$$

(ii) The resulting variance in (6) is also derived in Vermaat et al. (2008). Moreover, in their paper also an expression for α_t in the AR(2) model parameters is derived.

4. Estimators for the variance of the EWMA statistic

In this section we define three estimators for α.

Zhang's estimator: For a stationary time series Zhang (1998a) approximates α by α_M (cf. (4)), where M is an integer greater than 25, setting $(1-\lambda)^{2M} \approx 0$. This gives the following approximation for α (cf. (5))

$$\alpha^{(Z)} = \frac{\lambda}{2-\lambda}\sigma_X^2 \left(1 + 2\sum_{k=1}^{M} \rho(k)(1-\lambda)^k (1-(1-\lambda)^{2(M-k)})\right). \tag{12}$$

We suppress the dependence on M in the notation. We estimate $\alpha^{(Z)}$ by a plug-in estimator $\hat{\alpha}_n^{(Z)}$, which is defined by

$$\hat{\alpha}_n^{(Z)} = \frac{\lambda}{2-\lambda}\hat{\gamma}_n(0) \left(1 + 2\sum_{k=1}^{M} \hat{\rho}_n(k)(1-\lambda)^k (1-(1-\lambda)^{2(M-k)})\right).$$

Here $\hat{\rho}_n(k)$ is the sample autocorrelation at lag k, which is defined by $\hat{\rho}_n(k) = \hat{\gamma}_n(k)/\hat{\gamma}_n(0)$, where

$$\hat{\gamma}_n(k) = \frac{1}{n}\sum_{t=1}^{n-k}(X_t - \bar{X}_n)(X_{t+k} - \bar{X}_n) \qquad k = 0, 1, 2, \ldots.$$

Schmid's estimator: For a causal stationary AR(1) process, Schmid (1997) estimates α by (cf. (11))

$$\hat{\alpha}_n^{(S)} = \frac{\lambda}{2-\lambda}\left(\frac{1+\hat{\phi}_1(1-\lambda)}{1-\hat{\phi}_1(1-\lambda)}\right)\hat{\gamma}_n(0), \tag{13}$$

where $\hat{\phi}_1 = \hat{\gamma}_n(1)/\hat{\gamma}_n(0)$ is the Yule–Walker estimator for ϕ_1.

Vermaat's estimator: For a causal stationary AR(2) process, Vermaat et al. (2008) estimate α by (cf. (6))

$$\hat{\alpha}_n^{(V)} = \frac{\lambda}{2-\lambda}\left(\frac{\hat{\phi}_1(1+\hat{\phi}_2)(\lambda-1) + (\hat{\phi}_2 - 1)(1+\hat{\phi}_2(\lambda-1)^2)}{(1-\hat{\phi}_2)(-1+\hat{\phi}_1(1-\lambda)+\hat{\phi}_2(\lambda-1)^2)}\right)\hat{\gamma}_n(0). \tag{14}$$

Here $\hat{\phi}_1$ and $\hat{\phi}_2$ are the Yule–Walker estimators for ϕ_1 and ϕ_2 respectively. The latter are given by

$$\hat{\phi}_1 = \frac{\hat{\gamma}_n(1)(\hat{\gamma}_n(0) - \hat{\gamma}_n(2))}{\hat{\gamma}_n(0)^2 - \hat{\gamma}_n(1)^2} \quad \text{and} \quad \hat{\phi}_2 = \frac{\hat{\gamma}_n(0)\hat{\gamma}_n(2) - \hat{\gamma}_n(1)^2}{\hat{\gamma}_n(0)^2 - \hat{\gamma}_n(1)^2}. \tag{15}$$

Hence, for a first order autoregressive process we have two estimators that can be used to derive the control limits of the EWMA-chart: $\hat{\alpha}_n^{(Z)}$ and $\hat{\alpha}_n^{(S)}$. In the next section we compare these estimators by studying their asymptotic distribution. Similarly, we will compare $\hat{\alpha}_n^{(Z)}$ and $\hat{\alpha}_n^{(V)}$ for second order autoregressive processes.

As Zhang (1998a) reported, the advantage of using the approximate variance as in (12) hinges on the fact that no modelling efforts are required. Only a number of autocorrelations has to be estimated to obtain an estimator for the variance of the EWMA statistic. However:

1. In practice it is valuable to know whether the process under study exhibits autocorrelation. Modelling the autocorrelation structure of a process gives insight into the working of the process. This may generate improvement actions besides the monitoring purpose.
2. With a modern software it is relatively easy to fit a time series model for a given series.
3. Zhang (1998a) reported that for certain values of the parameters of the time series model other control charts perform better than his proposal, in the sense that they have better average run length (ARL) properties for signalling out-of-control situations, like a shift in the mean. So, to choose the most appropriate control chart we need to estimate the time series parameters.
4. As we can see the estimator $\hat{\alpha}_n^{(V)}$ uses only estimators for ϕ_1, ϕ_2, and σ_X^2, while the estimator $\hat{\alpha}_n^{(Z)}$ uses up to M estimates for the autocorrelations and an estimate for the process variance σ_X^2. Hence, Zhang's approach might be very prodigal with parameters, whereas the approach via the estimated model parameters is parsimonious.

5. The estimators $\hat{\alpha}_n^{(S)}$ and $\hat{\alpha}_n^{(V)}$ are asymptotically more efficient for AR(1) respectively AR(2) processes than the estimator $\hat{\alpha}_n^{(Z)}$, which is shown in Section 5 of this paper.

We illustrate the difference in the two approaches by analyzing the asymptotic behavior of the estimators for the control limits.

5. Asymptotics of the estimators

In this section we analyze the asymptotic behavior of the estimators $\hat{\alpha}_n^{(Z)}$,$\hat{\alpha}_n^{(S)}$, and $\hat{\alpha}_n^{(V)}$ as defined in (12)–(14) respectively.

5.1. Asymptotic distributions

The following result will be used in the sequel. We will use the abbreviation IID to denote Independent and Identically Distributed.

Proposition 1. *Suppose X is a causal AR(p) process generated by the IID sequence A. Assume* $\mathrm{E}(A^4) < \infty$ *and let* $\eta = \mathrm{E}(A^4)/\sigma_A^4$. *For any non-negative integer k,*

$$\sqrt{n}\left(\begin{bmatrix}\hat{\gamma}_n(0)\\ \hat{\gamma}_n(1)\\ \vdots\\ \hat{\gamma}_n(k)\end{bmatrix} - \begin{bmatrix}\gamma(0)\\ \gamma(1)\\ \vdots\\ \gamma(k)\end{bmatrix}\right) \xrightarrow{d} N_{k+1}(0, \Omega_{k+1})$$

where $\xrightarrow{d}$ denotes convergence in distribution and

$$\Omega_{k+1} = \left[(\eta-3)\gamma(q)\gamma(r) + \sum_i (\gamma(i)\gamma(i-q+r) + \gamma(i+r)\gamma(i-q))\right]_{q,r=0,\dots,k}. \tag{16}$$

Proof. See Brockwell and Davis (1991), p. 230–231. □

Theorem 1. *Suppose X is a causal AR(p) process generated by the IID sequence A. Assume* $\mathrm{E}(A^4) < \infty$ *and let* $\eta = \mathrm{E}(A^4)/\sigma_A^4$, *then*

$$\sqrt{n}(\hat{\alpha}_n^{(Z)} - \alpha^{(Z)}) \xrightarrow{d} N(0, V^{(Z)}),$$

where $V^{(Z)}$ is given by $V^{(Z)} = \boldsymbol{c}'\Omega_{M+1}\boldsymbol{c}$ for $\boldsymbol{c}$ as in (17) and Ω_{M+1} as in (16).

Proof. Recall the definition of $\hat{\alpha}_n^{(Z)}$ in (12). Write $\hat{\alpha}_n^{(Z)} = \boldsymbol{c}'\hat{\boldsymbol{\gamma}}_n := \boldsymbol{c}'(\hat{\gamma}_n(0), \dots, \hat{\gamma}_n(M))'$ for $\boldsymbol{c} \in \mathbb{R}^{M+1}$ defined by

$$\boldsymbol{c} = \frac{\lambda}{2-\lambda}\begin{bmatrix}1\\ 2(1-\lambda)(1-(1-\lambda)^{2(M-1)})\\ 2(1-\lambda)^2(1-(1-\lambda)^{2(M-2)})\\ \vdots\\ 2(1-\lambda)^{M-1}(1-(1-\lambda)^2)\\ 0\end{bmatrix}. \tag{17}$$

Let $\boldsymbol{\gamma} = (\gamma(0), \dots, \gamma(M))'$. Using Proposition 1 and applying the Cramér–Wold device (cf. Pollard (2002) p. 202) gives

$$\sqrt{n}(\hat{\alpha}_n^{(Z)} - \alpha^{(Z)}) = \sqrt{n}(\boldsymbol{c}'\hat{\boldsymbol{\gamma}}_n - \boldsymbol{c}'\boldsymbol{\gamma}) \xrightarrow{d} N(0, \boldsymbol{c}'\Omega_{M+1}\boldsymbol{c}). \quad \square$$

Theorem 2. *Suppose X is a causal AR(1) process generated by the IID sequence A. Assume $\mathrm{E}(A^4) < \infty$ and let $\eta = \mathrm{E}(A^4)/\sigma_A^4$, then*

$$\sqrt{n}(\hat{\alpha}_n^{(S)} - \alpha) \xrightarrow{d} N(0, V^{(S)}).$$

where $V^{(S)}$ is given by $V^{(S)} = \boldsymbol{b}'\Omega_2\boldsymbol{b}$ with $\boldsymbol{b}$ defined by

$$\boldsymbol{b} = \lambda(2-\lambda)^{-1}\left(\frac{2\gamma_0 + \gamma_1(1-\lambda)}{\gamma_0 - \gamma_1(1-\lambda)} - \frac{\gamma_0^2 + \gamma_0\gamma_1(1-\lambda)}{(\gamma_0 - \gamma_1(1-\lambda))^2}, \frac{2\gamma_0^2(1-\lambda)}{(\gamma_0 - \gamma_1(1-\lambda))^2}\right)'$$

and Ω_2 as in (16).

Proof. If we define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \frac{\lambda}{2-\lambda}\left(\frac{x^2 + xy(1-\lambda)}{x - y(1-\lambda)}\right),$$

then

$$\sqrt{n}(\hat{\alpha}_n^{(S)} - \alpha) = \sqrt{n}(f(\hat{\gamma}_n(0), \hat{\gamma}_n(1)) - f(\gamma(0), \gamma(1))).$$

By the Delta-method (cf. Pollard (2002) p. 184), together with Proposition 1, this converges weakly to a normal distribution with variance $\nabla f(\gamma_0, \gamma_1)\Omega_1\nabla f(\gamma(0), \gamma(1))'$. One can verify that the gradient $\nabla f(\gamma(0), \gamma(1))$ equals the vector b. $\square$

Theorem 3. *Suppose X is a causal AR(2) process generated by the IID sequence A. Assume $\mathrm{E}(A^4) < \infty$ and let $\eta = \mathrm{E}(A^4)/\sigma_A^4$, then*

$$\sqrt{n}(\hat{\alpha}_n^{(V)} - \alpha) \xrightarrow{d} N(0, V^{(V)}),$$

where $V^{(V)}$ is given by $V^{(V)} = \boldsymbol{d}'\Omega_3\boldsymbol{d}$ for $\boldsymbol{d} = \nabla h(\gamma(0), \gamma(1), \gamma(2))'$ as in (18) and Ω_3 as in (16).

Proof. Substituting (15) into (14) we obtain $\hat{\alpha}_n^{(V)} = h(\hat{\gamma}_n(0), \hat{\gamma}_n(1), \hat{\gamma}_n(2))$, with $h : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$h(x, y, z) = \left(\frac{\lambda}{2-\lambda}\right)\frac{x^3 + 2y^3(\lambda-1) - x^2(\lambda-1)(y + z(1-\lambda)) - xy(z(\lambda-1) + y(2 + (\lambda-2)\lambda))}{x^2 + x(\lambda-1)(y + z(1-\lambda)) + y(z(1-\lambda) + y(\lambda-2)\lambda)}.$$

An application of Proposition 1, combined with the Delta method, gives

$$\sqrt{n}(\hat{\alpha}_n^{(V)} - \alpha) \xrightarrow{d} N(0, \nabla h(\gamma(0), \gamma(1), \gamma(2))\Omega_3\nabla h(\gamma(0), \gamma(1), \gamma(2))').$$

Here, the gradient $\nabla h : \mathbb{R}^3 \to \mathbb{R}^3$ of h has elements

$$\begin{aligned}
\frac{\partial h(x,y,z)}{\partial x} &= u_\lambda(x,y,z)x^4 + 2x^3(\lambda-1)(y+z-z\lambda) + 2xy\{-z^2(\lambda-1)^3 + yz(\lambda-1)^4 \\
&\quad - y^2(\lambda-1)(2+(\lambda-2)\lambda)\} + x^2(y+z(\lambda-1))\{-z(\lambda-1)^3 + y(1+3(\lambda-2)\lambda)\} \\
&\quad + y^2\{z^2(\lambda-1)^2 + 2yz(\lambda-1)(2+(\lambda-2)\lambda) - y^2(2+(\lambda-2)\lambda(4+(\lambda-2)\lambda))\} \\
\frac{\partial h(x,y,z)}{\partial y} &= u_\lambda(x,y,z)\Big(2(\lambda-1)\{-x^4x^2(2y^2 + 2yz(\lambda-1) + z^2(\lambda-1)^2) - 2x^3y(\lambda-1) + \\
&\quad 2xy^2(\lambda-1)(y+z-z\lambda) + y^3(-2z(\lambda-1) + y(\lambda-2)\lambda)\}\Big) \\
\frac{\partial h(x,y,z)}{\partial z} &= u_\lambda(x,y,z)2(\lambda-1)^2(x^2-y^2)^2,
\end{aligned} \tag{18}$$

where,

$$u_\lambda(x,y,z) = \lambda(2-\lambda)^{-1}\left(x^2 + x(\lambda-1)(y+z-z\lambda) + y(z - z\lambda + y(\lambda-2)\lambda)\right)^{-2}. \quad \square$$

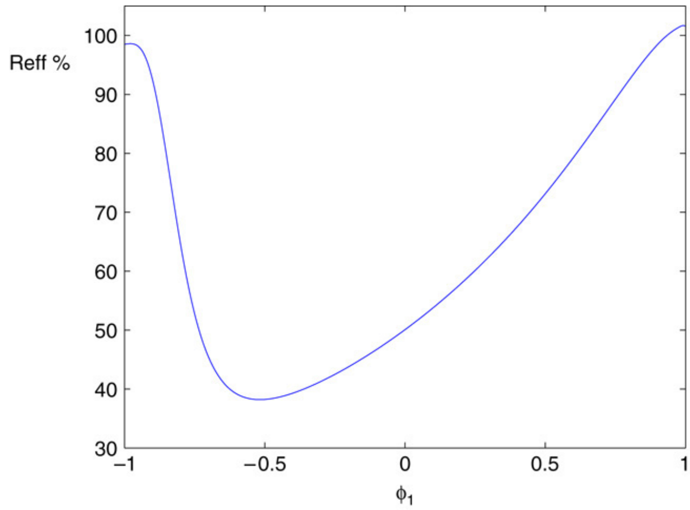

Fig. 1. Plot of the asymptotic relative efficiency of Schmid's estimator (cf. (13)) over Zhang's estimator (cf. (12)) for a stationary AR(1) process.

Remark. For autoregressive processes the Yule–Walker estimators are asymptotically equivalent to the maximum likelihood estimators based on a Gaussian likelihood (cf. Brockwell and Davis (1991) p. 240). That is, the rescaled differences of the estimators subtracted by their true value converge weakly to a multivariate normal distribution with the same variance. This implies efficiency of the Yule–Walker estimators. Our estimators are obtained by plugging in these efficient estimators.

5.2. *Asymptotic relative efficiencies*

In this section we study the efficiency of $\hat{\alpha}_n^{(S)}$ over $\hat{\alpha}_n^{(Z)}$ for an AR(1) process and $\hat{\alpha}_n^{(V)}$ and $\hat{\alpha}_n^{(Z)}$ for an AR(2) process. Note that $\alpha_n^{(S)}$ and $\hat{\alpha}_n^{(V)}$ are consistent estimators for α while $\hat{\alpha}_n^{(Z)}$ is not.

Suppose that the process under study is a causal stationary first order autoregressive process (this is equivalent to $|\phi_1| < 1$). We define the asymptotic relative efficiency for the estimators $\hat{\alpha}_n^{(Z)}$ and $\hat{\alpha}_n^{(S)}$ by

$$\mathrm{Reff}(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(S)}) = \frac{V^{(S)}}{V^{(Z)}} = \frac{\boldsymbol{b}'\Omega_2\boldsymbol{b}}{\boldsymbol{c}'\Omega_{M+1}\boldsymbol{c}}.$$

In this definition we neglect the fact that $\hat{\alpha}_n^{(Z)}$ estimates $\alpha^{(Z)}$, whereas $\hat{\alpha}_n^{(S)}$ estimates α. We use the current definition of relative efficiency purely to compare the asymptotic variances of the estimators. In Fig. 1 we have plotted the asymptotic relative efficiency for different ϕ_1s in the stationary region of an AR(1) process with $M = 25$. As we can see $\mathrm{Reff}(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(S)})$ varies from 0.4 to 1. Only at the boundaries of the stationary region $\mathrm{Reff}(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(S)}) > 1$ the estimator of Zhang (1998a) (cf. (12)) is more efficient.

For a causal stationary AR(2) process we define the asymptotic relative efficiency for the estimators $\hat{\alpha}_n^{(Z)}$ and $\hat{\alpha}_n^{(V)}$ by

$$\mathrm{Reff}(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(V)}) = \frac{V^{(V)}}{V^{(Z)}} = \frac{\boldsymbol{d}'\Omega_3\boldsymbol{d}}{\boldsymbol{c}'\Omega_{M+1}\boldsymbol{c}}.$$

In Fig. 2 we have drawn contour lines of $\mathrm{Reff}(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(V)})$ for different ϕ_1 and ϕ_2 in the stationary region of an AR(2) process, i.e. $\phi_1 + \phi_2 < 1$, $\phi_1 - \phi_2 < 1$, and $-1 < \phi_2 < 1$ and with $M = 25$. The asymptotic relative efficiency varies from .56 to 1. Again $\mathrm{Reff}(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(V)}) > 1$ at the boundaries of the region of stationarity.

Effect of overfitting. Suppose X is a (stationary, causal) first order autoregressive time series, but we estimate α by $\hat{\alpha}_n^{(V)}$, which is "designed" for a second order autoregressive process. In Fig. 3 we have plotted

$$\mathrm{Reff}(\hat{\alpha}_n^{(V)}, \hat{\alpha}_n^{(S)}) = \frac{V^{(S)}}{V^{(V)}} = \frac{\boldsymbol{b}'\Omega_2\boldsymbol{b}}{\boldsymbol{d}'\Omega_3\boldsymbol{d}}$$

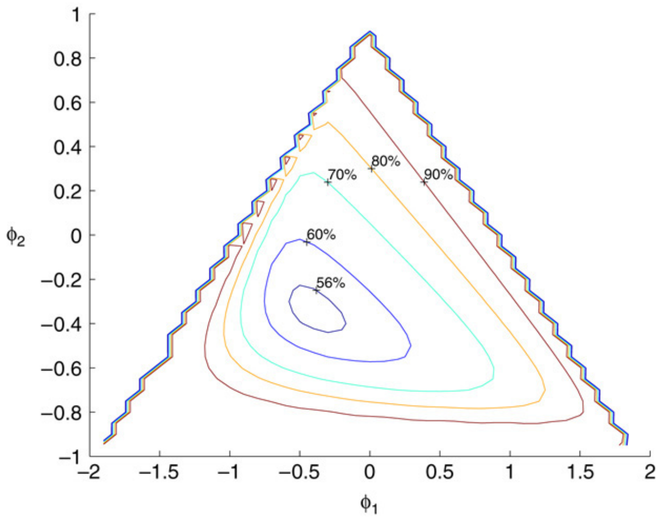

Fig. 2. Contour plot of the asymptotic relative efficiency of Vermaat's estimator (cf. (14)) over Zhang's estimator (cf. (12)) for a stationary AR(2) process.

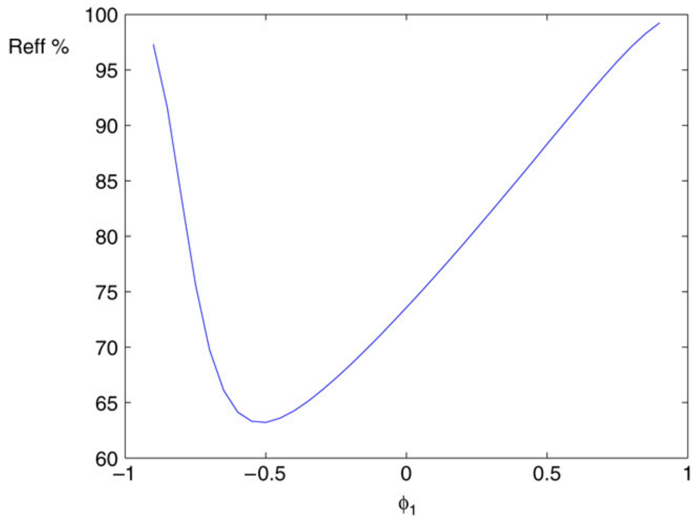

Fig. 3. Plot of the asymptotic relative efficiency Schmid's estimator (cf. (13)) over Vermaat's estimator (cf. (14)) for a stationary AR(1) process.

for $\phi_1 \in (-1, 1)$. As we can see the $\text{Reff}(\hat{\alpha}_n^{(V)}, \hat{\alpha}_n^{(S)}) < 1$ for all $\phi_1 \in (-1, 1)$. Thus, overfitting by one degree results in a loss of efficiency, which can be up to 35%.

6. Simulations and the effect of misspecification

We have studied the asymptotic properties in the previous section by simulation. The obtained analytical results are confirmed by the simulation.

The simulation study was done as follows. We simulate $\sqrt{n}(\hat{\alpha}_n - \alpha)$ 1000 times for all three estimators. The length of the time series n is 1000. For the estimation of α we have chosen to set $\lambda = 0.2$ and the noise to be gaussian with zero mean and standard deviation one. The sample means of the 1000 simulations of $\sqrt{n}(\alpha_n^{(Z)} - \alpha^{(Z)})$, $\sqrt{n}(\alpha_n^{(S)} - \alpha)$, and $\sqrt{n}(\alpha_n^{(V)} - \alpha)$ are denoted by $\overline{\sqrt{n}(\alpha_n^{(Z)} - \alpha^{(Z)})}$, $\overline{\sqrt{n}(\alpha_n^{(S)} - \alpha)}$, and $\overline{\sqrt{n}(\alpha_n^{(V)} - \alpha)}$ respectively and their sample variances are denoted by $V_n^{(Z)}$, $V_n^{(S)}$, and $V_n^{(V)}$ respectively.

The empirical relative efficiency between Zhang's and Schmid's estimator is defined by

$$\text{Reff}_n(\hat{\alpha}_n^{(Z)}, \hat{\alpha}_n^{(S)}) = \frac{V_n^{(S)}}{V_n^{(Z)}}. \tag{19}$$

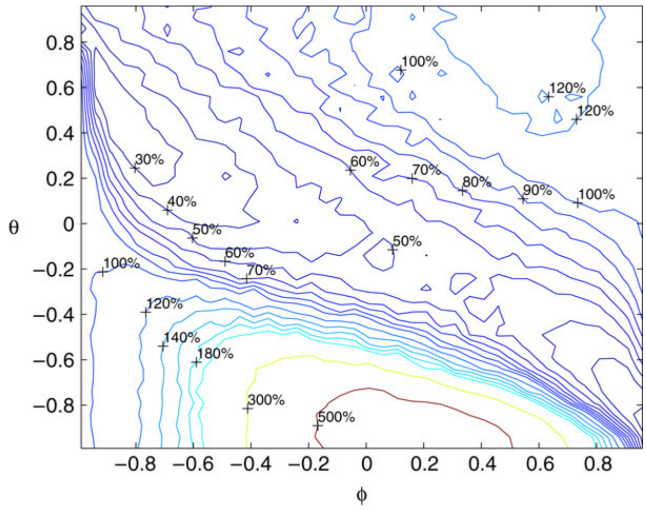

Fig. 4. Contour plot of the simulated asymptotic relative efficiency of Schmid's estimator (cf. (13)) over Zhang's estimator (cf. (12)) for a stationary ARMA(1,1) process.

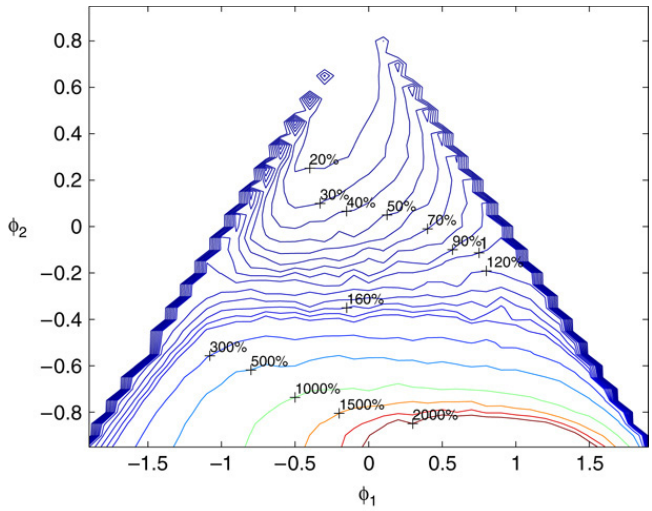

Fig. 5. Contour plot of the simulated asymptotic relative efficiency of Schmid's estimator (cf. (13)) over Zhang's estimator (cf. (12)) for a stationary AR(2) process.

The empirical relative efficiency between Zhang's and Vermaat's estimator is defined mutatis mutandis as in (19).

To study the effect of misspecification, we consider three cases to judge the performance of the Zhang's, Schmid's and Vermaat's estimator.

(i) Suppose that the data are generated by a stationary causal ARMA(1, 1) process, i.e. the process dynamics are given by the relation $X_t = \phi X_{t-1} + \theta A_{t-1} + A_t$, where (A_t) is a sequence of independent standard Normally distributed random variables and $(\phi, \theta) \in (-1, 1)^2$. Zhang's estimator is designed to handle this case adequately, whereas both Schmid's as Vermaat's estimator are not. Fig. 4 shows a contour plot of the simulated asymptotic relative efficiency of Schmid's estimator over Zhang's estimator (cf. (19)). From this figure we can conclude that roughly only in the region $(\phi, \theta) \in (-0.5, 0.5) \times (-1, -0.5)$ Zhang's estimator outperforms Schmid's estimator strongly.

(ii) Consider the same setting as under (i), but now compare Vermaat's and Zhang's estimator. The resulting contour plot is not included, since the pattern of lines in the plot is very similar to that of Fig. 4. The only notable difference is that the values corresponding to the lines are smaller in the region where Zhang's estimator performs better (around twice as small).

(iii) Suppose the data are from a stationary AR(2) process. Fig. 5 shows contour lines of the simulated asymptotic relative efficiency of Schmid's estimator over Zhang's estimator within the stationary region of the AR(2) process. If ϕ_2 is negative, then Zhang's estimator turns out to be much more efficient.

7. Conclusions

We have derived the asymptotic distributions for three estimators for the variance of the EWMA statistic introduced by Zhang (1998a), Schmid (1997) and Vermaat et al. (2008). The asymptotic relative efficiency of the estimator by Zhang (1998a) is compared to those by Schmid (1997) and Vermaat et al. (2008). The estimators of Schmid (1997) and Vermaat et al. (2008) are more efficient for AR(1) respectively AR(2) processes. Hence, for processes that can be modelled by an AR(1) or an AR(2) process, we would advocate the EWMA control chart based on Schmid's and Vermaat's estimators respectively.

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