Iteratively algebraic orders
Rodenburg, P.H.

Citation for published version (APA):
Iteratively algebraic orders

P.H. Rodenburg

Institute for Informatics, University of Amsterdam


Introduction

In a preliminary version of [1], Grätzer asked if there exists a lattice that is isomorphic to its lattice of ideals and in which not every ideal is principal. This question was answered in the negative by D. Higgs [2].

A related question was posed by H.-E. Hoffmann in [3]: whether an algebraic order (poset) whose compact elements again form an algebraic order and so on, can have noncompact elements. It was answered in the negative by M.H. Albert [4].

Having forgotten a crucial element of Higgs’ proof, and unable to understand the proof in [4], I invented a simpler proof, which I present below, after a brief rehearsal of definitions. At the end I will indicate what baffled me.

Substance

This paper is about ordered sets, that is, sets with a (partial) ordering relation \( \leq \) on them. We write \( x < y \) if \( x \leq y \) and \( x \neq y \), and \( x \prec y \) if \( y \) is an upper cover of \( x \), that is, \( x < y \) and if \( x \leq z \leq y \) then \( z \) is either \( x \) or \( y \). A chain is a linearly ordered set.

Definition 1. A subset \( X \) of an ordered set \( L \) is directed if every finite subset of \( X \) has an upper bound in \( L \).

In particular, the void subset of a directed set has an upper bound, so directed sets are nonvoid.

Definition 2. An ordered set \( L \) is upwards complete if every directed \( X \subseteq L \) has a supremum in \( L \).

The supremum of \( X \) is denoted by \( \bigvee X \). We write \( (X)_L \), omitting the subscript if it can be derived from the context, for
\[ \{y \in L\} \text{ for some } x \in X, y \leq x. \]
Instead of \((\{x\})\), we write \((x)\). Dually we have \([X]\) and \([x]\).

Definition 3. An element \( k \) of an ordered set \( L \) is compact if for every directed \( X \subseteq L \), \( k \leq \bigvee X \) implies \( k \in (X) \).

We denote the set of compact elements of an ordered set \( L \) by \( K(L) \). We put \( K^0(L) = L \), \( K^{n+1}(L) = K(K^n(L)) \).
**Definition 4.** An ordered set $L$ is algebraic if it is upwards complete and for every $x \in L$, $(x] \cap K(L)$ is directed and $x$ is its supremum. It is iteratively algebraic if for all $n$, $K^n(L)$ is algebraic.

**Theorem.** If an ordered set $A$ is iteratively algebraic, $A = K(A)$.

**Proof.** Assume $A$ is iteratively algebraic, and $A \neq K(A)$. Since the supremum of a chain of noncompact elements is noncompact, by Zorn’s Lemma, $A$ contains a maximal noncompact element $m$. Clearly, $[m]$ satisfies the ACC — the supremum of an infinitely ascending chain cannot be compact. Let

$$m \prec c_1 \prec c_2 \prec \ldots \prec c_n$$

$(n \geq 0)$ be a maximal chain in $[m]$. The element $m$ is the supremum of a set $C$ of compact elements. We know that $m$ is not in $K(A)$; but $C \subseteq K(A)$, and since $K(A)$ is algebraic, $C$ has a supremum in $K(A)$, which must be $c_1$. So $n$ was not zero after all. But now $c_1$ is noncompact in $K(A)$. Let $m' = c_i$ be the greatest element of $\{c_1, \ldots, c_n\}$ that is noncompact in $K(A)$. Then we have a maximal chain

$$m' \prec c_{i+1} \prec c_{i+2} \prec \ldots \prec c_n$$

$(n \geq i)$ in $[m']_{K(A)}$. Then as before, $n$ cannot be $i$; but proceeding to $K^2(A)$, we shall get an even shorter chain, and eventually a contradiction. So $A = K(A)$.

**Corollary.** If a lattice $L$ is isomorphic to its ideal lattice, all its ideals are principal.

**Proof.** As an ordered set, the ideal lattice $\text{Idl}(L)$ is algebraic; so likewise $L$ is algebraic. $(L$ will even be an algebraic lattice if it has a 0.) The compact elements of $\text{Idl}(L)$ are the principal ideals. The sublattice of principal ideals is obviously isomorphic to $L$, so $L \cong K(L)$, which implies that $L$ is iteratively algebraic. Then by the Theorem, all the elements of $L$, and hence all the elements of $\text{Idl}(L)$, are compact.

**Discussion**

The proof of the theorem certainly owes to Higgs, but omits his main idea: a construction of double sequences of compact elements, based on the observation that a lower cover of an ideal generated by a compact element must be principal. Albert [4] claims to prove the theorem, but his conclusion that $A = K(A)$, after a transfinite lopping off of maximal elements, appears right out of the blue. Hansoul [5] suggests a proof of the dual of Albert’s theorem along the lines of [2].

**References**


