Online Appendix to: Estimating the input of a Lévy-driven queue by Poisson sampling of the workload process

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Appendix A Proofs for Section 3

Proof of Lemma 4. We first prove (31). From (25),

$$\frac{\partial}{\partial \psi} J_n(\psi,\varphi)\bigg|_{\varphi=\varphi(\alpha),\psi=\psi(\xi)} = -\frac{\xi}{\xi-\varphi(\alpha)}\cdot\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\alpha}{\psi(\xi)^2}e^{-\psi(\xi)V_{i-1}}+\frac{\alpha V_{i-1}}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right) . \tag{79}$$

Applying the GPK formula (2), PASTA and the law of large numbers,

$$\begin{aligned}\frac{\partial}{\partial \psi} J_n(\psi(\xi),\varphi(\alpha)) &\to_{\text{as}} -\frac{\xi}{\xi-\varphi(\alpha)}\left(\frac{\alpha}{\psi(\xi)^2}\mathrm{E}e^{-\psi(\xi)V_{i-1}}+\frac{\alpha}{\psi(\xi)}\xi\,\mathrm{E}[Ve^{-\psi(\xi)V}]\right)\\ &= -\frac{\alpha\varphi'(0)}{(\xi-\varphi(\alpha))\psi'(\xi)} = \partial J_\psi \ ,\end{aligned} \tag{80}$$

as $n\to\infty$, where it is used that

$$\mathrm{E}[Ve^{-\alpha V}] = \frac{\varphi'(0)}{(\varphi(\alpha))^2}\big(\alpha\varphi'(\alpha)-\varphi(\alpha)\big) .$$

Now we prove (32). Following the same steps as above,

$$\frac{\partial}{\partial \varphi} J_n(\psi,\varphi)\bigg|_{\varphi=\varphi(\alpha),\psi=\psi(\xi)} \to_{\text{as}} -\frac{\alpha\varphi'(0)}{\varphi(\alpha)(\xi-\varphi(\alpha))} =: \partial J_\varphi \ , \tag{81}$$

as $n\to\infty$. □

Proof of Lemma 5. a. We first compute the asymptotic variance of $\sqrt{n}J_n(\psi(\xi),\varphi(\alpha))$ and then apply the martingale CLT of Lemma 2a to verify that the limiting distribution is normal. Let

$$Z_i := e^{-\alpha V_i} - \frac{\xi}{\xi-\varphi(\alpha)}\left(e^{-\alpha V_{i-1}}-\frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right) . \tag{82}$$

Then $J_n(\psi(\xi),\varphi(\alpha)) = \frac{1}{n}\sum_{i=1}^n Z_i$. By (4), $\mathrm{E}[Z_i|V_{i-1}]=0$. Therefore,

$$n\mathrm{Var}(J_n(\psi(\xi),\varphi(\alpha))) = \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^n Z_i\right)^2 = \frac{1}{n}\left(\sum_{i=1}^n \mathrm{E}Z_i^2 + 2\sum_{i=2}^n\sum_{j<i}\mathrm{E}Z_iZ_j\right) , \tag{83}$$

and as

$$\mathrm{E}Z_iZ_j = \mathrm{E}[\mathrm{E}(Z_iZ_j|(V_0,\ldots,V_{i-1}))] = \mathrm{E}[Z_j\mathrm{E}(Z_i|V_{i-1})] = 0 \quad \forall j<i \ , \tag{84}$$

we have that $n\mathrm{Var}(J_n(\psi(\xi),\varphi(\alpha))) = \frac{1}{n}\sum_{i=1}^n \mathrm{E}Z_i^2$. The second moment of Z_i is

$$\begin{aligned}
\mathrm{E}Z_i^2 &= \mathrm{E}\Bigg[e^{-2\alpha V_i} - \frac{2\xi}{\xi-\varphi(\alpha)}e^{-\alpha V_i}\left(e^{-\alpha V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right) \\
&\quad + \left(\frac{\xi}{\xi-\varphi(\alpha)}\right)^2\left(e^{-2\alpha V_{i-1}} - \frac{2\alpha}{\psi(\xi)}e^{-(\alpha+\psi(\xi))V_{i-1}} + \left(\frac{\alpha}{\psi(\xi)}\right)^2 e^{-2\psi(\xi)V_{i-1}}\right)\Bigg] \\
&= \mathrm{E}\Bigg[e^{-2\alpha V_i} + \left(\frac{\xi}{\xi-\varphi(\alpha)}\right)^2 e^{-2\alpha V_{i-1}} + \left(\frac{\xi\alpha}{\psi(\xi)(\xi-\varphi(\alpha))}\right)^2 e^{-2\psi(\xi)V_{i-1}} \\
&\quad - \frac{2\xi}{\xi-\varphi(\alpha)}e^{-\alpha V_i-\alpha V_{i-1}} + \frac{2\xi\alpha}{\psi(\xi)(\xi-\varphi(\alpha))}e^{-\alpha V_i-\psi(\xi)V_{i-1}} - \frac{2\xi^2\alpha}{\psi(\xi)(\xi-\varphi(\alpha))^2}e^{-(\alpha+\psi(\xi))V_{i-1}}\Bigg] .
\end{aligned} \tag{85}$$

The limit of $\mathrm{E}e^{-\alpha V_i}$ as $i\to\infty$ is given by PASTA and the GPK formula (2). The joint distribution of (V_i, V_{i-1}) is obtained by conditioning on V_{i-1} and applying (4), for any $\alpha > 0$ and $\beta > 0$,

$$\begin{aligned}
\mathrm{E}e^{-\alpha V_i-\beta V_{i-1}} &= \mathrm{E}\left[e^{-\beta V_{i-1}}\frac{\xi}{\xi-\varphi(\alpha)}\left(e^{-\alpha V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)\right] \\
&= \frac{\xi}{\xi-\varphi(\alpha)}\mathrm{E}\left[e^{-(\alpha+\beta)V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-(\psi(\xi)+\beta)V_{i-1}}\right] \\
&\xrightarrow{i\to\infty} \frac{\xi\varphi'(0)}{\xi-\varphi(\alpha)}\left(\frac{\alpha+\beta}{\varphi(\alpha+\beta)} - \frac{\alpha(\psi(\xi)+\beta)}{\psi(\xi)\varphi(\psi(\xi)+\beta)}\right) .
\end{aligned} \tag{86}$$

Therefore,

$$\begin{aligned}
\lim_{i\to\infty}\mathrm{E}Z_i^2 &= \left(1 + \left(\frac{\xi}{\xi-\varphi(\alpha)}\right)^2\right)\frac{2\alpha\varphi'(0)}{\varphi(2\alpha)} + \left(\frac{\xi\alpha}{\psi(\xi)(\xi-\varphi(\alpha))}\right)^2 \cdot \frac{2\psi(\xi)\varphi'(0)}{\varphi(2\psi(\xi))} \\
&\quad - \frac{2\xi^2}{(\xi-\varphi(\alpha))^2}\left(\frac{2\alpha\varphi'(0)}{\varphi(2\alpha)} - \frac{\alpha(\alpha+\psi(\xi))\varphi'(0)}{\psi(\xi)\varphi(\alpha+\psi(\xi))}\right) \\
&\quad + \frac{2\xi^2\alpha}{\psi(\xi)(\xi-\varphi(\alpha))^2}\left(\frac{(\alpha+\psi(\xi))\varphi'(0)}{\varphi(\alpha+\psi(\xi))} - \frac{2\alpha\varphi'(0)}{\varphi(2\psi(\xi))}\right) \\
&\quad - \frac{2\xi^2\alpha}{\psi(\xi)(\xi-\varphi(\alpha))^2}\cdot\frac{(\alpha+\psi(\xi))\varphi'(0)}{\varphi(\alpha+\psi(\xi))} .
\end{aligned} \tag{87}$$

Rearranging terms we obtain $\lim_{i\to\infty}\mathrm{E}Z_i^2 = \sigma_\alpha^2$, as defined in (34). Therefore, we conclude that

$$n\mathrm{Var}\left(J_n(\psi(\xi),\varphi(\alpha))\right) = \frac{1}{n}\sum_{i=1}^n \mathrm{E}Z_i^2 \xrightarrow{n\to\infty} \sigma_\alpha^2 . \tag{88}$$

We now verify the conditions of Lemma 2a. Let

$$Z_{ni} := \frac{Z_i}{\sqrt{\mathrm{Var}(nJ_n(\psi(\xi),\varphi(\alpha)))}} = \frac{nJ_i(\psi(\xi),\varphi(\alpha)) - nJ_{i-1}(\psi(\xi),\varphi(\alpha))}{\sqrt{\mathrm{Var}(nJ_n(\psi(\xi),\varphi(\alpha)))}} , \tag{89}$$

and observe that $nJ_n(\psi(\xi),\varphi(\alpha))$ is a martingale as $\mathrm{E}[Z_i|V_{i-1}] = 0$ for every $i \geq 1$. Furthermore, for any n, $\mathrm{Var}(nJ_n(\psi(\xi),\varphi(\alpha))) > 0$ and $\mathrm{P}(Z_i < \infty) = 1$ as $\mathrm{E}Z_i = 0$ for all $i = 1,\ldots,n$. Hence, $\mathrm{E}\max_{1\leq i\leq n} Z_{ni}^2 < \infty$ and condition (21) holds. Above, we established that $n\mathrm{Var}(J_n(\psi(\xi),\varphi(\alpha)))$ goes to a constant, and thus

$$\mathrm{Var}(nJ_n(\psi(\xi),\varphi(\alpha))) = n^2\mathrm{Var}(J_n(\psi(\xi),\varphi(\alpha))) \to \infty , \tag{90}$$

and as $\mathrm{P}(Z_i < \infty) = 1$, we conclude that condition (22) is satisfied. Therefore, the Martingale CLT holds and $\sum_{i=1}^n Z_{ni} \to_{\mathrm{d}} \mathrm{N}(0,1)$. Finally, applying Slutsky's lemma yields

$$\sqrt{n}J_n(\psi(\xi),\varphi(\alpha)) = \sqrt{n}\sum_{i=1}^n Z_i = \sqrt{n\mathrm{Var}\left(J_n(\psi(\xi),\varphi(\alpha))\right)}\sum_{i=1}^n Z_{ni} \to_{\mathrm{d}} \sqrt{\sigma_\alpha^2}\mathrm{N}(0,1) . \tag{91}$$

b. In the previous part of the lemma we established that $J_n(\psi(\xi), \varphi(\alpha))$ satisfies the conditions of Lemma 2a. If we further assume that $\psi_n - \psi(\xi) \approx \frac{1}{n} M_n + R_n$ such that M_n satisfies the conditions of Lemma 2a, then

$$\sqrt{n}\,(\psi_n - \psi(\xi)) \approx \frac{M_n}{\sqrt{n}} \approx \sqrt{\frac{\mathrm{Var}(M_n)}{n}}\mathrm{N}(0,1)\;, \tag{92}$$

where $\sqrt{\mathrm{Var}(M_n)/n} \to \sigma_\xi$ by condition (ii). We can next apply Lemma 2b to conclude joint asymptotic normality:

$$\sqrt{n}\begin{pmatrix} J_n(\psi(\xi), \varphi(\alpha)) \\ \\ \psi_n - \psi(\xi) \end{pmatrix} \approx \sqrt{n}\begin{pmatrix} J_n(\psi(\xi), \varphi(\alpha)) \\ \\ \frac{M_n}{n} \end{pmatrix} \to_{\mathrm{d}} \mathrm{N}\left(0, S\right)\;, \tag{93}$$

if the covariance matrix S has finite terms. The variance terms are $S_{11} = \sigma_\alpha^2$ and

$$S_{22} = \lim_{n\to\infty} \mathrm{E}[M_n^2] = \lim_{n\to\infty} \mathrm{E}[(\psi_n - \psi(\xi))^2] = \sigma_\xi^2\;. \tag{94}$$

The asymptotic covariance is finite by condition (iii):

$$\begin{aligned} S_{12} = S_{21} &= \lim_{n\to\infty} \mathrm{E}\left[\sqrt{n} J_n(\psi(\xi), \varphi(\alpha)) \frac{M_n}{\sqrt{n}}\right] \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left[Z_i M_n\right] = \sigma_{\alpha,\psi}^2 < \infty\;. \end{aligned} \tag{95}$$

By the Cramér-Wold device this implies that any linear combination of $J_n(\psi(\xi), \varphi(\alpha))$ and $\psi_n - \psi(\xi)$ is also normal, and thus (36) follows.

□

Proof of Lemma 6. As the proof is identical for all three second order terms, we restrict ourselves to the first of them:

$$\sqrt{n}(\varphi_n - \varphi(\alpha))^2 \partial_\varphi^2 J = (\varphi_n - \varphi(\alpha))\sqrt{n}(\varphi_n - \varphi(\alpha))\partial_\varphi^2 J, \quad \partial_\varphi^2 J := \frac{\partial^2}{\partial \varphi^2} J_n(\psi(\xi), \varphi(\alpha)). \tag{96}$$

By explicit evaluation it is straightforward to verify that $\partial_\varphi^2 J < \infty$, almost surely if $\mathrm{E}X(1) < 0$, i.e., the stationary workload V exists and is bounded almost surely. The consistency assumption $\psi_n \to_{\mathbb{P}} \psi(\xi)$ implies, by Theorem 1, that $\varphi_n - \varphi(\alpha) \to_{\mathbb{P}} 0$. The scaled error term $\sqrt{n}(\varphi_n - \varphi(\alpha))$ is almost surely bounded as the first two moments of the error have finite limits due to Lemma 5. Therefore the error term indeed diminishes to zero, as $n \to \infty$. □

Proof of Theorem 7. In Lemma 5 we verified the martingale CLT conditions of Lemma 2a for every element in $\boldsymbol{J}_n := (J_n(\psi(\xi), \varphi(\alpha_1)), \ldots J_n(\psi(\xi), \varphi(\alpha_p)))$. Along the same lines, it is shown that any linear combination of the entries of $\boldsymbol{J}_n$ converges to a normal random variable, so that by the Cramér-Wold device we conclude joint asymptotic normality. All that remains is to compute for every pair $\alpha \neq \beta$ the covariance

$$\lim_{n\to\infty} \mathrm{Cov}\Big[\sqrt{n}(\hat{\varphi}_n(\alpha;\psi_n) - \varphi(\alpha)), \sqrt{n}(\hat{\varphi}_n(\beta;\psi_n) - \varphi(\beta))\Big]. \tag{97}$$

By (28), this covariance has the same limit as

$$\frac{n}{\partial J_{\varphi,\alpha}\partial J_{\varphi,\beta}} \mathrm{Cov}\Big[J_n(\psi(\xi), \varphi(\alpha)) + \partial J_{\psi,\alpha}(\psi_n - \psi(\xi)), J_n(\psi(\xi), \varphi(\beta)) + \partial J_{\psi,\beta}(\psi_n - \psi(\xi))\Big]\;. \tag{98}$$

The covariance can be computed by considering the covariance of the different pairs,

$$\begin{aligned} &n\mathrm{Cov}\Big[J_n(\psi(\xi), \varphi(\alpha)) + \partial J_{\psi,\alpha}(\psi_n - \psi(\xi)), J_n(\psi(\xi), \varphi(\beta)) + \partial J_{\psi,\beta}(\psi_n - \psi(\xi))\Big] \\ &\quad = n\mathrm{Cov}\left[J_n(\psi(\xi), \varphi(\alpha)), J_n(\psi(\xi), \varphi(\beta))\right] \\ &\qquad + n\partial J_{\psi,\beta}\mathrm{Cov}\left[J_n(\psi(\xi), \varphi(\alpha)), \psi_n - \psi(\xi)\right] \\ &\qquad + n\partial J_{\psi,\alpha}\mathrm{Cov}\left[J_n(\psi(\xi), \varphi(\beta)), \psi_n - \psi(\xi)\right] \\ &\qquad + \partial J_{\psi,\alpha}\partial J_{\psi,\beta}\mathrm{Var}\left(\sqrt{n}(\psi_n - \psi(\xi))\right)\;. \end{aligned} \tag{99}$$

The limit of the latter three terms are $\sigma^2_{\alpha,\xi}$, $\sigma^2_{\beta,\xi}$ and σ^2_{ξ}, respectively. As $\mathrm{E}J_n(\psi(\xi),\varphi(\alpha)) = \mathrm{E}J_n(\psi(\xi),\varphi(\beta)) = 0$,

$$\mathrm{Cov}\left(J_n(\psi(\xi),\varphi(\alpha)), J_n(\psi(\xi),\varphi(\beta))\right) = \mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))J_n(\psi(\xi),\varphi(\beta))\right] \ . \tag{100}$$

Similar to the arguments made in the proof of Lemma 5, by conditioning on V_{i-1} and applying (4) we obtain that, for every $i > j$,

$$\mathrm{E}\Bigg[\left(e^{-\alpha V_i} - \frac{\xi}{\xi-\varphi(\alpha)}\left(e^{-\alpha V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)\right) \tag{101}$$

$$\cdot\left(e^{-\beta V_j} - \frac{\xi}{\xi-\varphi(\beta)}\left(e^{-\beta V_{j-1}} - \frac{\beta}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}\right)\right)\Bigg] = 0 \ , \tag{102}$$

and similarly by conditioning on V_{j-1} for every $i < j$ we get a zero expectation for the respective terms. Hence, by (25),

$$\begin{aligned} &n\mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))J_n(\psi(\xi),\varphi(\beta))\right] \\ &\quad = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\Bigg[\left(e^{-\alpha V_i} - \frac{\xi}{\xi-\varphi(\alpha)}\left(e^{-\alpha V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)\right) \\ &\qquad\qquad \cdot\left(e^{-\beta V_i} - \frac{\xi}{\xi-\varphi(\beta)}\left(e^{-\beta V_{i-1}} - \frac{\beta}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)\right)\Bigg] \ . \end{aligned} \tag{103}$$

We compute the expectation for every $i \geq 1$,

$$\begin{aligned} &\mathrm{E}\Bigg[\left(e^{-\alpha V_i} - \frac{\xi}{\xi-\varphi(\alpha)}\left(e^{-\alpha V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)\right)\left(e^{-\beta V_i} - \frac{\xi}{\xi-\varphi(\beta)}\left(e^{-\beta V_{i-1}} - \frac{\beta}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)\right)\Bigg] \\ &= \mathrm{E}\Bigg[e^{-(\alpha+\beta)V_i} + \frac{\xi^2}{(\xi-\varphi(\alpha))(\xi-\varphi(\beta))}e^{-(\alpha+\beta)V_{i-1}} + \frac{\alpha\beta\xi^2}{\psi(\xi)^2(\xi-\varphi(\alpha))(\xi-\varphi(\beta))}e^{-2\psi(\xi)V_{i-1}} \\ &\quad - \frac{\beta\xi^2}{\psi(\xi)(\xi-\varphi(\alpha))(\xi-\varphi(\beta))}e^{-(\alpha+\psi(\xi))V_{i-1}} - \frac{\xi}{\xi-\varphi(\beta)}e^{-\alpha V_i - \beta V_{i-1}} + \frac{\beta\xi}{\psi(\xi)(\xi-\varphi(\beta))}e^{-\alpha V_i - \psi(\xi)V_{i-1}} \\ &\quad - \frac{\alpha\xi^2}{\psi(\xi)(\xi-\varphi(\alpha))(\xi-\varphi(\beta))}e^{-(\beta+\psi(\xi))V_{i-1}} - \frac{\xi}{\xi-\varphi(\alpha)}e^{-\beta V_i - \alpha V_{i-1}} + \frac{\alpha\xi}{\psi(\xi)(\xi-\varphi(\alpha))}e^{-\beta V_i - \psi(\xi)V_{i-1}}\Bigg] \ , \end{aligned} \tag{104}$$

and by applying the GPK formula (2), with the additional conditioning step used before for the joint limiting distribution of (V_{i-1}, V_i), we obtain the limit of the expectation,

$$\begin{aligned} &\frac{(\alpha+\beta)\varphi'(0)}{\varphi(\alpha+\beta)} + \frac{\xi^2\varphi'(0)}{(\xi-\varphi(\alpha))(\xi-\varphi(\beta))}\Bigg(\frac{\alpha+\beta}{\varphi(\alpha+\beta)} + \frac{2\alpha\beta}{\psi(\xi)\varphi(2\psi(\xi))} - \frac{\beta(\alpha+\psi(\xi))}{\psi(\xi)\varphi(\alpha+\psi(\xi))} - \frac{\alpha(\beta+\psi(\xi))}{\psi(\xi)\varphi(\beta+\psi(\xi))} \\ &\qquad - 2\left(\frac{\alpha+\beta}{\varphi(\alpha+\beta)} - \frac{\alpha(\beta+\psi(\xi))}{\psi(\xi)\varphi(\beta+\psi(\xi))} - \frac{\beta(\alpha+\psi(\xi))}{\psi(\xi)\varphi(\alpha+\psi(\xi))} + \frac{2\alpha\beta}{\psi(\xi)\varphi(2\psi(\xi))}\right)\Bigg). \end{aligned} \tag{105}$$

Rearranging the terms yields the desired expression for $\sigma^2_{J,\alpha,\beta}$. □

Appendix B Proofs for Section 4

Proof of Lemma 10. The smoothness is due to the fact that $\ell_n(\psi)$, $\ell_n'(\psi)$ and $\ell_n''(\psi)$ are all continuous functions of ψ. Furthermore, for every $\xi < \psi < \infty$ the functions are well defined and finite and therefore bounded on any compact interval. □

Proof of Lemma 11. The first derivative of the empirical log-likelihood function is

$$\ell_n'(\psi) := \frac{1}{n} L_n'(\psi) = \frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{\psi} + V_{i-1}}{1 - \frac{\xi}{\psi} e^{-\psi V_{i-1}}} \left(\frac{\xi}{\psi} e^{-\psi V_{i-1}} - Y_i \right) , \tag{106}$$

for any $\psi > \xi$ (where we note that it is otherwise not always well defined). The limiting (time-average) distribution of the pair (Y_n, V_{n-1}) is given by the distribution of a stationary workload observation and the probability to find the system empty in an exponentially distributed time after the observation. Thus, the joint distribution is characterized by $V = V(\infty)$, combined with conditional probability $\mathrm{P}(Y = 1 \,|\, V) = \frac{\xi}{\psi} e^{-\psi V}$. Applying PASTA for the sampling process $N(t)$,

$$\ell_n'(\psi) = \frac{1}{N(t_n)} \ell_{N(t_n)}'(\psi) \to_{\mathrm{as}} \mathrm{E}\ell_1'(\psi) = \mathrm{E}\left[\frac{\frac{1}{\psi} + V}{1 - \frac{\xi}{\psi} e^{-\psi V}} \left(\frac{\xi}{\psi} e^{-\psi V} - Y \right) \right] , \tag{107}$$

and by iterating the expectation we obtain (48),

$$\ell_n'(\psi) \to_{\mathrm{as}} \mathrm{E}(\mathrm{E}(\ell_1'(\psi)|V)) = \mathrm{E}\left[\frac{\frac{1}{\psi} + V}{1 - \frac{\xi}{\psi} e^{-\psi V}} \left(\frac{\xi}{\psi} e^{-\psi V} - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V} \right) \right] . \tag{108}$$

The same argument yields (49). □

Proof of Lemma 13. From

$$\begin{aligned} \mathrm{E}\ell_n'(\psi(\xi)) &= \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left[\frac{\frac{1}{\psi(\xi)} + V_{i-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}}} \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} - Y_i \right) \right] \\ &= \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left[\frac{\frac{1}{\psi(\xi)} + V_{i-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}}} \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} \right) \right] = 0 , \end{aligned} \tag{109}$$

we find

$$\mathrm{Var}(\ell_n'(\psi(\xi))) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{E}\left[\frac{\frac{1}{\psi(\xi)} + V_{i-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}}} \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} - Y_i \right) \right]^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij} , \tag{110}$$

where $s_{ij} :=$

$$\mathrm{E}\left[\frac{\frac{1}{\psi(\xi)} + V_{i-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}}} \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} - Y_i \right) \right] \left[\frac{\frac{1}{\psi(\xi)} + V_{j-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{j-1}}} \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{j-1}} - Y_j \right) \right] . \tag{111}$$

By conditioning on V_{i-1} and using the fact that $\mathrm{P}(Y_i = 1 | V_{i-1}) = \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{j-1}}$, where $\{Y_i | V_{i-1}\}$ is independent of (V_{j-1}, Y_j) for any $j < i$, we conclude that $s_{ij} = 0$ for all $i < j$. Similarly, conditioning on V_{j-1} yields that $s_{ij} = 0$ for all $j > i$. We find that

$$\mathrm{Var}(\ell_n'(\psi(\xi))) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{E}\left[\frac{\frac{1}{\psi(\xi)} + V_{i-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}}} \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} - Y_i \right) \right]^2 . \tag{112}$$

Observe that, for $i = 1, \ldots, n$,

$$\begin{aligned} m_i(v) &:= \mathrm{E}\left[\left(\frac{\frac{1}{\psi(\xi)} + V_{i-1}}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}}} \right)^2 \left(\frac{\xi}{\psi(\xi)} e^{-\psi(\xi) V_{i-1}} - Y_i \right)^2 \Big| V_{i-1} = v \right] \\ &= \left(\frac{\frac{1}{\psi(\xi)} + v}{1 - \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) v}} \right)^2 \left(\frac{\xi^2}{\psi(\xi)^2} e^{-2\psi(\xi) v} - 2 \frac{\xi}{\psi(\xi)} e^{-\psi(\xi) v} \mathrm{E}[Y_i | V_{i-1} = v] + \mathrm{E}[Y_i^2 | V_{i-1} = v] \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\frac{1}{\psi(\xi)}+v}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}}\right)^2 \left(\frac{\xi^2}{\psi(\xi)^2}e^{-2\psi(\xi)v} - 2\frac{\xi^2}{\psi(\xi)^2}e^{-2\psi(\xi)v} + \frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}\right) \\
&= \frac{\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}\left(\frac{1}{\psi(\xi)}+v\right)^2}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}} \ . \qquad (113)
\end{aligned}
$$

Applying PASTA once more yields

$$
n\mathrm{Var}(\ell_n'(\psi(\xi))) = \frac{1}{N(t_n)}\sum_{i=1}^{N(t_n)} \mathrm{E}\, m_i(V(t_{i-1})) \to_{\mathrm{as}} \mathrm{E}\, m_1(V) = I_\xi \ . \qquad (114)
$$

Next we apply the martingale CLT of Lemma 2, using the same notation as before for the corresponding martingale difference terms. For $i \in \{1,\ldots,n\}$, we denote

$$
Z_i := \frac{\left(\frac{1}{\psi(\xi)}+V_{i-1}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}} - Y_i\right)}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}} \ , \quad Z_{ni} := \frac{Z_i}{\sqrt{\mathrm{Var}(L_n'(\psi(\xi)))}}. \qquad (115)
$$

As $\mathrm{E}Z_i = 0$, $L_n'(\psi(\xi)) = \sum_{i=1}^n Z_i$ is a martingale with respect to $\{V_1,\ldots,V_n\}$. For any n, $\mathrm{Var}(L_n'(\psi(\xi))) > 0$ and $\mathrm{P}(Z_i < \infty) = 1$ as $\mathrm{E}Z_i = 0$ for all $i \in \{1,\ldots,n\}$. Hence $\mathrm{E}\max_{1\le i\le n} Z_{ni}^2 < \infty$ so that the condition (21) holds. We have established that

$$
\frac{1}{n}\mathrm{Var}(L_n'(\psi(\xi))) = n\mathrm{Var}(\ell_n'(\psi(\xi))) \to_{\mathrm{as}} I_\xi < \infty \ , \qquad (116)
$$

and therefore $\sqrt{\mathrm{Var}(L_n'(\psi(\xi)))} \to_{\mathrm{as}} \infty$. Furthermore, as $\mathrm{P}(Z_i < \infty) = 1$,

$$
\max_{1\le i\le n} |Z_{ni}| = \max_{1\le i\le n} \frac{|Z_i|}{\sqrt{\mathrm{Var}(L_n'(\psi(\xi)))}} \to_{\mathrm{as}} 0 \qquad (117)
$$

and thus condition (22) is satisfied as well. Therefore, by Lemma 2,

$$
\frac{L_n'(\psi(\xi))}{\sqrt{\mathrm{Var}(L_n'(\psi(\xi)))}} \to_{\mathrm{d}} \mathrm{N}(0,1) \ . \qquad (118)
$$

Recall that $\ell_n'(\psi(\xi)) = n^{-1} L_n'(\psi(\xi))$, so that

$$
\frac{L_n'(\psi(\xi))}{\sqrt{\mathrm{Var}(L_n'(\psi(\xi)))}} = \frac{n\ell_n'(\psi(\xi))}{\sqrt{\mathrm{Var}(n\ell_n'(\psi(\xi)))}} = \frac{\sqrt{n}\ell_n'(\psi(\xi))}{\sqrt{n\mathrm{Var}(\ell_n'(\psi(\xi)))}} \ . \qquad (119)
$$

Because $\sqrt{n\mathrm{Var}(\ell_n'(\psi(\xi)))} \to_{\mathrm{as}} \sqrt{I_\xi}$, and applying Slutsky's lemma yields $\sqrt{n}\ell_n'(\psi(\xi)) \to_{\mathrm{d}} \sqrt{I_\xi}\mathrm{N}(0,1)$. □

Proof of Lemma 14. Taking derivative of $\ell_n'(\psi)$, as given in (46), yields

$$
\begin{aligned}
\ell_1''(\psi) = &-\frac{1}{\left(1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_0}\right)^2}\Bigg[\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_0}\left(\frac{1}{\psi(\xi)}+V_0\right)^2(1-Y_1) \\
&+ \frac{1}{\psi(\xi)^2}\left(1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_0}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_0} - Y_1\right)\Bigg] \qquad (120) \\
= &-\frac{\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_0}\left(\frac{1}{\psi(\xi)}+v\right)^2}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_0}} \ .
\end{aligned}
$$

The conditional expectation for any initial workload $V_0 = v$ is

$$
\begin{aligned}
\mathrm{E}[\ell_1''(\psi(\xi))|V_0=v] &= -\frac{1}{\left(1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}\right)^2}\left[\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}\left(\frac{1}{\psi(\xi)}+v\right)^2(1-\mathrm{P}_{\psi(\xi)}(Y_1|V_0=v))\right.\\
&\quad\left.+\frac{1}{\psi(\xi)^2}\left(1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}-\mathrm{P}_{\psi(\xi)}(Y_1|V_0=v)\right)\right]\\
&= -\frac{\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}\left(\frac{1}{\psi(\xi)}+v\right)^2}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}}\ .
\end{aligned}
\tag{121}
$$

Denote the limit of the second derivative, that is given by (49) in Lemma 11, at value ψ by $k_2(\psi)$. Recall that by (8), $\mathrm{P}_{\psi(\xi)}(Y_1|V_0=v) = \frac{\xi}{\psi(\xi)}e^{-\psi(\xi)v}$, and thus by taking expectation with respect to the stationary workload we obtain

$$
\ell_n''(\psi) \to_{\mathrm{as}} \mathrm{E}\left[\mathrm{E}[\ell_1''(\psi(\xi))|V]\right] = k_2(\psi(\xi)) = -I_\xi\ . \tag{122}
$$

Now consider the difference between D_n and $-I_\xi$:

$$
\begin{aligned}
|\,D_n-(-I_\xi)\,| &= \left|\int_0^1\left[\ell_n''(t\hat\psi_n+(1-t)\psi(\xi))+I_\xi\right]\,\mathrm{d}t\right|\\
&\le \int_0^1\left|\ell_n{}''(t\hat\psi_n+(1-t)\psi(\xi))+I_\xi\right|\,\mathrm{d}t\\
&\le \sup_{t\in[0,1]}\left|\ell_n{}''(t\hat\psi_n+(1-t)\psi(\xi))+I_\xi\right|\ .
\end{aligned}
\tag{123}
$$

By the consistency of the MLE established in Theorem 8, $t\hat\psi_n+(1-t)\psi(\xi)\to_{\mathrm{as}}\psi(\xi)$ for any $t\in[0,1]$. Hence, for any $\epsilon>0$ there exists some N such that $|t\hat\psi_n+(1-t)\psi(\xi)-\psi(\xi)|<\epsilon$, for all $n\ge N$ almost surely, and therefore

$$
\begin{aligned}
|D_n+I_\xi| &\le \sup_{\psi:\ |\psi-\psi(\xi)|<\epsilon}\left|\ell_n{}''(\psi)+I_\xi\right|\\
&\le \sup_{\psi:\ |\psi-\psi(\xi)|<\epsilon}\left|\ell_n{}''(\psi)-k_2(\psi)\right| + \sup_{\psi:\ |\psi-\psi(\xi)|<\epsilon}\left|k_2(\psi)+I_\xi\right|\ .
\end{aligned}
\tag{124}
$$

By Lemma 11, $\ell_n''(\psi)\to_{\mathrm{as}} k_2(\psi)$. Consequently, the first term goes to zero almost surely.
Let $\delta(\epsilon) := \sup_{\psi:\ |\psi-\psi(\xi)|<\epsilon}|k_2(\psi)+I_\xi|$, then (as k_2 is continuous and bounded on finite intervals) we have that $\delta(\epsilon)\to 0$ as $\epsilon\to 0$. We conclude that

$$
|D_n+I_\xi| \to_{\mathrm{as}} 0\ , \tag{125}
$$

and the result follows. □

Proof of Theorem 15. Let $\Omega_\xi(\alpha) := \xi/(\xi-\varphi(\alpha))$ and

$$
\Xi_{\xi,i}(\alpha) := e^{-\alpha V_{i-1}} - \frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}. \tag{126}
$$

As $\mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))\right]=0$,

$$
\mathrm{Cov}(\sqrt{n}J_n(\psi(\xi),\varphi(\alpha),\sqrt{n}(\hat\psi_n-\psi(\xi))) = n\mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))(\hat\psi_n-\psi(\xi))\right]\ . \tag{127}
$$

Recall that as $n\to\infty$, we almost surely have $\sqrt{n}(\hat\psi_n-\psi(\xi))\approx\sqrt{n}\ell_n'(\psi(\xi))/I_\xi$, and therefore

$$
\lim_{n\to\infty}\mathrm{Cov}(\sqrt{n}J_n(\psi(\xi),\varphi(\alpha)),\sqrt{n}(\hat\psi_n-\psi(\xi))) = \frac{1}{I_\xi}\lim_{n\to\infty}n\mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))\ell_n'(\psi(\xi))\right]\ . \tag{128}
$$

Direct computations yield that for any finite n the expectation $\mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))\,\ell_n'(\psi(\xi))\right]$ equals

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[\frac{\left(e^{-\alpha V_i}-\Omega_\xi(\alpha)\Xi_{\xi,i}(\alpha)\right)\left(\frac{1}{\psi(\xi)}+V_{j-1}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}-Y_j\right)}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}}\right]. \tag{129}$$

Conditioning on $(V_0,\ldots,V_{i-1})$, all terms in the above display with $i>j$ can be written as

$$\frac{\left(\frac{1}{\psi(\xi)}+V_{j-1}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}-Y_j\right)}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}}\mathrm{E}\left[e^{-\alpha V_i}-\Omega_\xi(\alpha)\Xi_{\xi,i}(\alpha)\middle|V_{i-1}\right]=0\ , \tag{130}$$

and for the terms such that $i<j$, conditioning on $(V_0,\ldots,V_{j-1})$ yields

$$\left(e^{-\alpha V_i}-\Omega_\xi(\alpha)\Xi_{\xi,i}(\alpha)\right)\mathrm{E}\left[\frac{\left(\frac{1}{\psi(\xi)}+V_{j-1}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}-Y_j\right)}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{j-1}}}\middle|V_{j-1}\right]=0\ . \tag{131}$$

For $j=i$, we compute the expectation conditioned on V_{i-1},

$$\begin{aligned}
&\mathrm{E}\left[\frac{\left(e^{-\alpha V_i}-\Omega_\xi(\alpha)\Xi_{\xi,i}(\alpha)\right)\left(\frac{1}{\psi(\xi)}+V_{i-1}\right)\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}-Y_i\right)}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}}\middle|V_{i-1}\right]\\
&=\mathrm{E}\Bigg[\frac{\frac{1}{\psi(\xi)}+V_{i-1}}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}}\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\mathrm{E}[e^{-\alpha V_i}|V_{i-1}]-\mathrm{E}[Y_ie^{-\alpha V_i}]\right)\\
&\quad-\Omega_\xi(\alpha)\left(\frac{\Xi_{\xi,i}(\alpha)\left(\frac{1}{\psi(\xi)}+V_{i-1}\right)\mathrm{E}\left[\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}-Y_i\middle|V_{i-1}\right]}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}}\right)\Bigg]\\
&=\mathrm{E}\left[\frac{\frac{1}{\psi(\xi)}+V_{i-1}}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}}\left(\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\mathrm{E}[e^{-\alpha V_i}|V_{i-1}]-\mathrm{P}(Y_i=1,V_i=0)\right)\right]\\
&=\mathrm{E}\left[\frac{\xi^2\left(\frac{1}{\psi(\xi)}+V_{i-1}\right)e^{-\psi(\xi)V_{i-1}}}{\psi(\xi)(\xi-\varphi(\alpha))\left(1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}\right)}\left(\Xi_{\xi,i}(\alpha)-\frac{\xi-\varphi(\alpha)}{\xi}\right)\right]\ ,
\end{aligned} \tag{132}$$

where the second equality follows from the fact that given V_{i-1}, Y_i is a Bernoulli random variable with probability $\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}$ and equals one only when $V_{i-1}=0$. Finally, by applying PASTA we conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\mathrm{Cov}(\sqrt{n}J_n(\psi(\xi),\varphi(\alpha),\sqrt{n}(\hat{\psi}_n-\psi(\xi)))\ =\frac{1}{I_\xi}\lim_{n\to\infty}n\mathrm{E}\left[J_n(\psi(\xi),\varphi(\alpha))\ell_n'(\psi(\xi))\right]\\
&=\frac{\xi^2}{\psi(\xi)(\xi-\varphi(\alpha))I_\xi}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left[\frac{\left(\frac{1}{\psi(\xi)}+V_{i-1}\right)e^{-\psi(\xi)V_{i-1}}}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V_{i-1}}}\left(\Xi_{\xi,i}(\alpha)-\frac{\xi-\varphi(\alpha)}{\xi}\right)\right]\\
&=\frac{\xi^2}{\psi(\xi)(\xi-\varphi(\alpha))I_\xi}\mathrm{E}\left[\frac{\left(\frac{1}{\psi(\xi)}+V\right)e^{-\psi(\xi)V}}{1-\frac{\xi}{\psi(\xi)}e^{-\psi(\xi)V}}\left(e^{-\alpha V}-\frac{\alpha}{\psi(\xi)}e^{-\psi(\xi)V}-\frac{\xi-\varphi(\alpha)}{\xi}\right)\right]\ .
\end{aligned} \tag{133}$$

This completes the proof. □

Appendix C Proofs for Section 5

Proof of Proposition 16. Consider the expected workload after an exponential time $T \sim \exp(\xi)$ given that the initial workload was v,

$$\begin{aligned}
\mathrm{E}(V(T)|V(0)=v) &= \mathrm{E}\left[V(T)\mathbf{1}\left(\inf_{0\le s\le T} X(s) \ge -v\right)|V(0)=v\right] \\
&\quad + \mathrm{E}\left[V(T)\mathbf{1}\left(\inf_{0\le s\le T} X(s) < -v\right)|V(0)=v\right] \\
&= \mathrm{E}\left[(v+X(T))\mathbf{1}\left(\inf_{0\le s\le T} X(s) \ge -v\right)|V(0)=v\right] \\
&\quad + \mathrm{P}\left(\inf_{0\le s\le T} X(s) < -v\right)\mathrm{E}[V(T)|V(0)=0] \\
&\le v + \mathrm{E}X(T) + \mathrm{P}\left(\inf_{0\le s\le T} X(s) < -v\right)\mathrm{E}[V(T)|V(0)=0] \ ,
\end{aligned}$$

yielding, with the last step being due to [11, Lemma 6.2],

$$\begin{aligned}
\mathrm{E}(V(T)-v\,|\,V(0)=v) - \mathrm{E}X(T) &\le \mathrm{P}\left(\inf_{0\le s\le T} X(s) < -v\right)\mathrm{E}[V(T)\,|\,V(0)=0] \\
&= e^{-\psi(\xi)v}\mathrm{E}[V(T)|V(0)=0] \ . \qquad (134)
\end{aligned}$$

Consequently, for any threshold $\tau > 0$ and all $v \ge \tau$,

$$\mathrm{E}(V(T)-v\,|\,V(0)=v) - \mathrm{E}X(T) \le \mathrm{E}[V(T)\,|\,V(0)=0]e^{-\psi(\xi)\tau} \ . \qquad (135)$$

The expected bias of $\hat{\vartheta}_m(\tau)$ is

$$\begin{aligned}
b_m(\vartheta;\tau) = \mathrm{E}\left[\hat{\vartheta}_m(\tau) - \vartheta\right] &= \mathrm{E}\left[\frac{\xi}{m}\sum_{j=1}^{m}(V_{i(j)} - V_{i(j)-1}) - (-\varphi'(0))\right] \\
&= \frac{\xi}{m}\sum_{j=1}^{m}\mathrm{E}\left[(V_{i(j)} - V_{i(j)-1}) - \mathrm{E}X(T)\right] \ . \qquad (136)
\end{aligned}$$

For every $j = 1\ldots,m$, conditioning on $V_{i(j)-1} = v$ (with $v \ge \tau$), applying (135) yields

$$\mathrm{E}\left[(V_{i(j)} - V_{i(j)-1}) - \mathrm{E}X(T)\middle|V_{i(j)-1} = v\right] \le \mathrm{E}[V(T)|V(0)=0]e^{-\psi(\xi)\tau} \ , \qquad (137)$$

and the bound (68) follows by total expectation. □

Proof of Proposition 17. For any sample $\boldsymbol{V} = (V_0,\ldots,V_n)$ such that $n \ge m$, let

$$f(x,n) := \frac{\xi(V_n - V_0)}{n} - \frac{\xi}{n}\sum_{i=1}^{n}\frac{e^{-xV_{i-1}}}{x} \ , \qquad (138)$$

then the estimator $\tilde{\psi}_m$ is given by solving (66), i.e., $\hat{\vartheta}_m(\tau) = f(\tilde{\psi}_m, M(m,\tau))$. By (6), for any $\{M(m,\tau) = n\}$,

$$\mathrm{E}[f(\psi(\xi),n) - f(\tilde{\psi}_m,n)] = \mathrm{E}\left[\hat{\vartheta}_m(\tau) + \varphi'(0)\right] \ , \qquad (139)$$

thus, the upper bound (68) for the bias of $\hat{\varphi}_m(\tau)$ we obtain yields

$$\mathrm{E}[f(\psi(\xi),n) - f(\tilde{\psi}_m,n)] \le \xi e^{-\psi(\xi)\tau}\mathrm{E}[V(T)|V(0)=0] \ . \qquad (140)$$

As $f(x,n)$ is continuous and monotone decreasing with x, this also means that the bias of $f(\tilde{\psi}_n,n)$ from $f(\psi(\xi),n)$ is bounded. The bound further holds when taking expectation over the possible values of $M(m,\tau)$. Moreover, if the threshold τ is increased, then the bias vanishes, and by continuity so does the bias of the estimator for $\tilde{\psi}$. □

Proof of Proposition 18. We define

$$K(t) := \sum_{i=1}^{N(t)} \mathbf{1}(V(t_i) \geq \tau), \tag{141}$$

where $N(t)$ is the Poisson sampling process and t_i are the event times. By PASTA, as $t \to \infty$,

$$\frac{K(t)}{N(t)} = \frac{1}{N(t)} \sum_{i=1}^{N(t)} \mathbf{1}(V(t_i) \geq \tau) \to_{\mathbb{P}} \mathrm{P}(V > \tau) \ . \tag{142}$$

Observe that $m = K(t_{M(m,\tau)})$, so that for $M(m, \tau) = n$,

$$\begin{aligned} \hat{\vartheta}_m(\tau) &= \frac{\xi}{K(t_n)} \sum_{i=1}^{N(t_n)} (V(t_i) - V(t_{i-1}))\mathbf{1}(V(t_{i-1}) \geq \tau) \\ &= \xi \frac{N(t_n)}{K(t_n)} \cdot \frac{1}{N(t_n)} \sum_{i=1}^{N(t_n)} (V(t_i) - V(t_{i-1}))\mathbf{1}(V(t_{i-1}) \geq \tau) \ . \end{aligned} \tag{143}$$

The term $N(t_n)/K(t_n)$ converges to $(\mathrm{P}(V > \tau))^{-1}$ as $n \to \infty$, and this also holds for $m \to \infty$ as $M(m, \tau) \to \infty$ almost surely in this case. By PASTA, the limiting joint distribution of the indicator and the increment is given by $(V(T) - V, \mathbf{1}(V \geq \tau))$, where $T \sim \exp(\xi)$. Therefore, we can apply Slutsky's lemma to conclude that the limiting distribution of the product is given by the product of the limiting distributions,

$$\hat{\vartheta}_m(\tau) \to_{\mathbb{P}} \xi \frac{\mathrm{E}\left[\mathrm{E}[(V(T) - V)\mathbf{1}(V \geq \tau) \,|\, V]\right]}{\mathrm{P}(V > \tau)} \ . \tag{144}$$

By first conditioning on $V = v$ and then plugging in the explicit expectation of $\mathrm{E}[V(T) - v]$ given in (6) we obtain that for $v \geq \tau$

$$\mathrm{E}\left[(V(T) - V)\mathbf{1}(V \geq \tau)\Big| V = v\right] = \frac{e^{-\psi(\xi)v}}{\psi(\xi)} - \frac{\varphi'(0)}{\xi} \ , \tag{145}$$

and 0 else. Then the asymptotic estimator is

$$\vartheta(\tau) = \frac{\xi}{\mathrm{P}(V > \tau)} \int_{\tau}^{\infty} \left[\frac{e^{-\psi(\xi)v}}{\psi(\xi)} - \frac{\varphi'(0)}{\xi}\right] \mathrm{d}F_V(v) = \frac{\xi \mathrm{E}\left[e^{-\psi(\xi)V}\mathbf{1}(V \geq \tau)\right]}{\psi(\xi)\mathrm{P}(V > \tau)} - \varphi'(0) \ . \tag{146}$$

□