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Complete Axiomatizations for XPath Fragments*

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Abstract. We provide *complete axiomatizations* for several fragments of XPath: sets of equivalences from which every other valid equivalence is derivable. Specifically, we axiomatize *downward single axis fragments* of Core XPath (that is, Core XPath($\downarrow$) and Core XPath($\downarrow^+$)) as well as the full Core XPath. We make use of techniques from modal logic.

XPath is a language for navigating through XML documents. In this paper, we consider the problem of finding complete axiomatizations for fragments of XPath. By an axiomatization we mean a finite set of valid equivalences between XPath expressions. These equivalences can be thought of as (undirected) rewrite rules. *Completeness* then means that any two equivalent expressions can be rewritten to each other using the given equivalences. Completeness tells us, in a mathematically precise way, that the given set of equivalences captures everything there is to say about semantic equivalence of XPath expressions.

We are aware of two complete axiomatizations for XPath fragments. The first is for the downward, positive and filter-free fragment of XPath [1], a rather limited fragment, and the second [5] concerns Core XPath 2.0, a very expressive language, with non-elementary complexity for query containment (see [4]).

In this paper, we study Core XPath 1.0, which was introduced in [7, 8] to capture the navigational core of XPath 1.0. Our main results are:

- A complete axiomatization for Core XPath($\downarrow^+$) and for Core XPath($\downarrow$), i.e., the fragments with only the descendant and only the child axis, respectively. The axiomatizations are complete both for node expressions and for path expressions.

[*] This technical report is the full version of a paper accepted for LID 2008 workshop. If you are kindly going to quote it in your work, please check http://www.dcs.bbk.ac.uk/~tadeusz/ for the most current version published.

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- An axiomatization for full Core XPath that is complete for node expressions. The completeness can be extended to path expressions as well, at the cost of introducing an additional rule of inference.

Proofs utilize techniques and known results from modal logic. We will describe the connection in more detail in Section 4.

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1 Preliminaries

1.1 XML Trees

We abstract away from atomic data attached to the individual elements, and view an XML document as a finite node-labeled sibling-ordered tree. More formally, given a countably infinite set $\mathsf{lab} = v_1, v_2, v_3 \ldots$ of node labels, we define an *XML tree* to be a structure $T = (D, R_\downarrow, R_\rightarrow, L)$, where

- $(N, R_\downarrow)$ is a finite tree (with N the set of nodes and $R_\downarrow$ the child relation),
- $R_\rightarrow$ is the successor relation of some linear ordering between siblings in the tree (in particular, for $R_{\rightarrow^+}$ the transitive closure of $R_\rightarrow$, we have that for any two distinct siblings x, y, either $xR_{\rightarrow^+}y$ or $yR_{\rightarrow^+}x$), and
- $L : N \rightarrow \wp(\mathsf{lab})$ labels the nodes with elements of lab.

We denote by $R_{\downarrow^+}$ and $R_{\rightarrow^+}$ the transitive closures of $R_\downarrow$ and $R_\rightarrow$, i.e., the *descendant* and *following-sibling* relations. $R_\leftarrow$, $R_{\leftarrow^+}$, $R_\uparrow$ and $R_{\uparrow^+}$ are the converses of, respectively, $R_\rightarrow$, $R_{\rightarrow^+}$, $R_\downarrow$ and $R_{\downarrow^+}$. The elements of lab correspond to XML tags. It is customary to require that each node satisfies precise one tag. We do not make this assumption, but its role will be discussed in some detail.

1.2 Core XPath, the Navigational Core of XPath 1.0

The syntax of Core XPath is defined as follows:

$$\begin{aligned}
\mathsf{Step} &:= \ \downarrow \mid \leftarrow \mid \uparrow \mid \rightarrow \\
\mathsf{Axis} &:= \ . \mid \mathsf{Step} \mid \mathsf{Step}^+ \\
\mathsf{PathEx} &:= \ \mathsf{Axis} \mid \mathsf{PathEx}\,[\mathsf{NodeEx}] \mid \mathsf{PathEx}/\mathsf{PathEx} \mid \mathsf{PathEx} \cup \mathsf{PathEx} \\
\mathsf{NodeEx} &:= \ v \mid \langle \mathsf{PathEx} \rangle \mid \neg \mathsf{NodeEx} \mid \mathsf{NodeEx} \vee \mathsf{NodeEx} \ \ (v \in \mathsf{lab})
\end{aligned}$$

In this paper, Greek letters $\phi, \psi \ldots$ range over elements of NodeEx and Roman capitals $A, B, C \ldots$ over elements of PathEx. s and a are metavariables ranging over, respectively, elements of Step and elements of Axis distinct than "."; thus,

Table 1. Semantics of Core XPath

$\llbracket . \rrbracket_{\mathsf{PExpr}}$	$:=$	$\{(x,x) \mid x \in D\}$
$\llbracket \mathsf{s} \rrbracket_{\mathsf{PExpr}}$	$:=$	R_{s}
$\llbracket \mathsf{s}^+ \rrbracket_{\mathsf{PExpr}}$	$:=$	the transitive closure of R_{s}
$\llbracket A/B \rrbracket_{\mathsf{PExpr}}$	$:=$	$\{(x,y) \mid \exists z$ such that $(x,z) \in \llbracket A \rrbracket_{\mathsf{PExpr}}$ and $(z,y) \in \llbracket B \rrbracket_{\mathsf{PExpr}}\}$
$\llbracket A \cup B \rrbracket_{\mathsf{PExpr}}$	$:=$	$\llbracket A \rrbracket_{\mathsf{PExpr}} \cup \llbracket B \rrbracket_{\mathsf{PExpr}}$
$\llbracket A\,[\phi] \rrbracket_{\mathsf{PExpr}}$	$:=$	$\{(x,y) \mid (x,y) \in \llbracket A \rrbracket_{\mathsf{PExpr}}$ and $y \in \llbracket \phi \rrbracket_{\mathsf{NExpr}}\}$
$\llbracket v \rrbracket_{\mathsf{NExpr}}$	$:=$	$\{x \mid v \in L(x)\}$
$\llbracket \langle \mathsf{PathEx} \rangle \rrbracket_{\mathsf{NExpr}}$	$:=$	$\{x \mid \exists y.(x,y) \in \llbracket \mathsf{PathEx} \rrbracket_{\mathsf{PExpr}}\}$
$\llbracket \neg\phi \rrbracket_{\mathsf{NExpr}}$	$:=$	$\{x \mid x \notin \llbracket \phi \rrbracket_{\mathsf{NExpr}}\}$
$\llbracket \phi_1 \vee \phi_2 \rrbracket_{\mathsf{NExpr}}$	$:=$	$\llbracket \phi_1 \rrbracket_{\mathsf{NExpr}} \cup \llbracket \phi_2 \rrbracket_{\mathsf{NExpr}}$

we can write $\mathsf{a} := \mathsf{s} \mid \mathsf{s}^+$. Note that we include the non-transitive sibling axes → and ← in the language. Also, we use angled brackets to distinguish path expressions from node expressions that test for the existence of a path. We use the following abbreviations:

true	for the node expression $\langle . \rangle$	$\perp$	for the path expression $.\,[\mathsf{false}]$
false	for the node expression $\neg\mathsf{true}$	$\phi \wedge \psi$	for the node expression $\neg(\neg\phi \vee \neg\psi)$

The semantics of Core XPath is defined in Table 1 by the functions $\llbracket \cdot \rrbracket_{\mathsf{PExpr}}$ and $\llbracket \cdot \rrbracket_{\mathsf{NExpr}}$ which take as input an XML tree and a path expression or node expression, and produce a binary relation over the set of nodes or a set of nodes, respectively. The XML tree is kept implicit in our notation.

For arbitrary a, we will denote by Core XPath(a) the fragment of Core XPath in which the only allowed axes are a and ".". We will be mostly interested in Core XPath($\downarrow^+$) and Core XPath($\downarrow$), as well as the full Core XPath language.

A Core XPath *node equivalence* is an expression of the form $\phi \equiv \psi$, where $\phi, \psi \in \mathsf{NodeEx}$, and *a* Core XPath *path equivalence* is an expression of the form $A \equiv B$, where $A, B \in \mathsf{PathEx}$. An equivalence is *a* Core XPath(a) *equivalence* if expressions on both sides belong to Core XPath(a). If in any XML tree it holds that $\llbracket \phi \rrbracket_{\mathsf{NExpr}} = \llbracket \psi \rrbracket_{\mathsf{NExpr}}$, respectively $\llbracket A \rrbracket_{\mathsf{PExpr}} = \llbracket B \rrbracket_{\mathsf{PExpr}}$, then we say that the equivalence is *valid*. We will use $A \subseteq B$ as shorthand for $A \cup B \equiv B$, and $\phi \leq \psi$ as shorthand for $\phi \vee \psi \equiv \psi$.

2 Single Axis Fragments of Core XPath

2.1 Axioms

Our basic axioms for Core XPath are presented in Table 2. We discuss them in some detail.

Idempotent Semirings Axioms The name comes from algebra. Idempotency is the property expressed by the axiom ISAx3. The natural numbers with addition and multiplication form a semiring, but not an idempotent one. Distributive lattices are natural examples of idempotent semirings if $\cap$ is denoted as / and

Table 2. Single Axis Axioms

Path Axiom Schemes for Idempotent Semirings

ISAx1	$(A \cup B) \cup C$	$\equiv A \cup (B \cup C)$
ISAx2	$A \cup B$	$\equiv B \cup A$
ISAx3	$A \cup A$	$\equiv A$
ISAx4	$A/(B/C)$	$\equiv (A/B)/C$
ISAx5	$./A$	$\equiv A$
	$A/.$	$\equiv A$
ISAx6	$A/(B \cup C)$	$\equiv A/B \cup A/C$
	$(A \cup B)/C$	$\equiv A/C \cup B/C$
ISAx7	$\bot$	$\subseteq A$

Path Axiom Schemes for Predicates

PrAx1	$A\,[\neg \langle B \rangle]\,/B$	$\equiv \bot$
PrAx2	$.\,[\langle . \rangle]$	$\equiv .$
PrAx3	$A\,[\phi \vee \psi]$	$\equiv A\,[\phi] \cup A\,[\psi]$
PrAx4	$(A/B)\,[\phi]$	$\equiv A/B\,[\phi]$

Node Axiom Schemes

NdAx1	ϕ	$\equiv \neg(\neg\phi \vee \psi) \vee \neg(\neg\phi \vee \neg\psi)$
NdAx2	$\langle .\,[\phi] \rangle$	$\equiv \phi$
NdAx3	$\langle A \cup B \rangle$	$\equiv \langle A \rangle \vee \langle B \rangle$
NdAx4	$\langle A/B \rangle$	$\equiv \langle A\,[\langle B \rangle] \rangle$

Axioms for Transitive Axes

TransAx1	$\langle \mathsf{s}^+\,[\phi] \rangle$	$\equiv \langle \mathsf{s}^+\,[\phi \wedge \neg \langle \mathsf{s}^+\,[\phi] \rangle]\, \rangle$
TransAx2	$\mathsf{s}^+/\mathsf{s}^+$	$\subseteq \mathsf{s}^+$

$\top$ as ".". *Tarski's relation algebras* [18, 19] and *Kleene algebras* [12, 13] interpret / and $\cup$ in the same way as we do (as composition and union of relations, respectively), hence both have idempotent semirings reducts.

Predicate Axioms In the one-sorted signature of Tarski's relation algebras [18, 19], which include intersection and complementation operators for binary relations, predicates can be treated as defined operations. This fact was used in [5] for the axiomatization of Core XPath 2.0. In XPath 1.0, there are less operations on relations available and predicates cannot be defined away.

Node Axioms NdAx1 is known in the algebraic community as the *Huntington equation* [11, 10, 14] and together with Der1 and Der2 from Table 3 force that the node expression connectives $\neg$ and $\vee$ form a Boolean algebra. The axioms NdAx2, NdAx3 and NdAx4 are counterparts of PrAx2, PrAx3 and PrAx4, respectively. This slight redundancy is the price one pays for working in a two-sorted signature.

Axioms for Transitive Axes TransAx1 is known by logicians as *The Löb Axiom*. It is valid on transitive structures which are *well-founded*, i.e., contain no infinite ascending s^+-chains and no s^+-cycles. Axiom TransAx2 forces transitivity for path expressions; the corresponding axiom for node expressions is already derivable from TransAx1. See [2, 20] for more information.

An Aside on Labelling Recall that in "real" XML trees, unlike the ones that we are using, each node satisfies exactly one label. In order to obtain completeness for this more restricted semantics, it would suffice to add a further axiom scheme:

$$v \wedge v' \equiv \bot \qquad \text{for } v, v' \in \mathsf{lab} \text{ distinct}$$

In what follows, however, we will not bother to add the above to our list of axioms. Thus, one may think of labels in the present setting as modelling both XML tag names and attribute-value pairs.

2.2 Rules and Derivations

Definition 1. *For P, Q both path expressions or both node expressions, we say that $P \equiv Q$ is* derivable *from a given set of axioms and axioms schemes if it can be obtained from them using the standard rules of equational logic:*

- $P \equiv P$
- *If $P \equiv Q$ then $Q \equiv P$*
- *If $P \equiv Q$ and $Q \equiv R$, then $P \equiv R$*
- *If $P \equiv Q$ and R' is obtained from R by replacing some occurrences of P by Q, then $R \equiv R'$.*

We will say that an expression Γ is consistent *relative to a given set of axioms if $\Gamma \equiv \bot$ is not derivable. An expression Γ is* provably equivalent *to Δ relative to a given set of axioms if $\Gamma \equiv \Delta$ is derivable from these axioms.*

Lemma 2. *All equivalences in Table 3 can be derived from those in Table 2.*

Proof. See Appendix. □

In the remainder of this section, derivability, consistency and provable equivalence will be considered relative to axioms in Table 2, unless stated otherwise. Also, when these notions are mentioned in the context of Core XPath(a) for any a, axioms involving axes other than a and "." are *not* allowed in the derivations.

2.3 Completeness for Core XPath($\downarrow^+$)

We prove the following completeness results:

Theorem 3 (Node Completeness for Core XPath($\downarrow^+$)). *A* Core XPath($\downarrow^+$) node *equivalence is valid iff it is derivable from the axioms in Table 2.*

Theorem 4 (Path Completeness for Core XPath($\downarrow^+$)). *A* Core XPath($\downarrow^+$) path *equivalence is valid iff it is derivable from the axioms in Table 2.*

Table 3. Some Examples of Derivable Equivalences

Der1	$\phi \vee \psi$	$\equiv \psi \vee \phi$
Der2	$\phi \vee (\psi \vee \chi)$	$\equiv (\phi \vee \psi) \vee \chi$
Der3	$A\,[\phi]$	$\equiv A/.\,[\phi]$
Der4	$A\,[\mathsf{true}]$	$\equiv A$
Der5	$A\,[\mathsf{false}]$	$\equiv \bot$
Der6	$(A \cup B)\,[\phi]$	$\equiv A\,[\phi] \cup B\,[\phi]$
Der7	ϕ	$\leq \mathsf{true}$
Der8	$\phi \vee \neg\phi$	$\equiv \mathsf{true}$
Der9	A	$\equiv A\,[\phi] \cup A\,[\neg\phi]$
Der10	$A\,[\phi]\,/.\,[\neg\phi]$	$\equiv \bot$
Der11{	$A/\bot$	$\equiv \bot$
	$\bot/A$	$\equiv \bot$
Der12	$A\,[\phi]\,[\neg\phi]$	$\equiv \bot$
Der13	$A\,[\phi \wedge \psi]\,[\neg\phi]$	$\equiv \bot$
Der14	$A\,[\phi]\,[\psi]\,[\neg\phi \vee \neg\psi]$	$\equiv \bot$
Der15	$A\,[\phi]\,[\psi]$	$\equiv A\,[\phi \wedge \psi]$
Der16	$A\,[\phi \wedge \psi]$	$\equiv A\,[\phi]\,/.\,[\psi]$
Der17	$.\,[\langle A \rangle]\,/A$	$\equiv A$

Throughout this section, we keep the number of labels fixed as m, that is, we assume all expressions use only the labels among $v_1, \ldots, v_m$. Also, we will represent XML trees as tuples $T = (D, R_{\downarrow^+}, L)$, where $R_{\downarrow^+}$ is the descendant axis. This convention allows to phrase some lemmas more conveniently. Note that, since $\mathsf{Core\ XPath}(\downarrow^+)$ does not provide means to refer to the sibling order, the latter may be chosen arbitrarily.

We begin with a definition of a subclass of the $\mathsf{Core\ XPath}(\downarrow^+)$ node expression, which we call *simple node expressions*:

$$\mathsf{siNode} := \mathsf{true} \mid v \mid \left\langle \downarrow^+ [\mathsf{siNode}] \right\rangle \mid \neg\mathsf{siNode} \mid \mathsf{siNode} \vee \mathsf{siNode}$$

where $v \in \mathsf{lab}$. Readers familiar with modal logic will realize that our $\left\langle \downarrow^+ [v] \right\rangle$ is just the modal formula $\Diamond v$, where $\Diamond$ is interpreted by $R_{\downarrow^+}$. For more on the connection with modal logic, see Section 4.

Lemma 5. *Every* $\mathsf{Core\ XPath}(\downarrow^+)$ *node expression is provably equivalent to a simple node expression.*

Proof. See Appendix. □

Recall that by "provably equivalent" we mean derivability from the axioms in Table 2.

We define *the degree* of a simple node expression as the maximal number of nested occurrences of $\left\langle \downarrow^+ [\cdot] \right\rangle$; i.e., the degree of a label is 0, the degree of a Boolean expression is the maximum of degrees of its Boolean components and the degree of $\left\langle \downarrow^+ [\phi] \right\rangle$ is the degree of ϕ plus one. The set of all simple node expressions of degree at most n is denoted by deg_n.

For any finite set $s \subseteq \{1, \ldots, k\}$, define $k \setminus s$ to be $\{1, \ldots, k\} - s$. Let

$$\mathsf{NFN}_0 = \Big\{ \bigwedge_{i \in s} v_i \ \wedge \bigwedge_{i \in m \setminus s} \neg v_i \ \Big|\ s \subseteq \{1, \ldots, m\} \Big\}.$$

Assume NFN_i are defined for all i smaller than $n \geq 1$. Let $f(n)$ be the cardinality of $\bigcup_{i<n} \mathsf{NFN}_i$. Fix an enumeration of this set $\alpha_1, \dots \alpha_{f(n)}$ and define the auxiliary set NFN'_n of all elements of the form

$$\beta \ \wedge \ \bigwedge_{i \in s} \langle \downarrow^+ [\alpha_i] \rangle \ \wedge \bigwedge_{i \in f(n) \setminus s} \neg \langle \downarrow^+ [\alpha_i] \rangle$$

for arbitrary $\beta \in \mathsf{NFN}_0$ and $s \subseteq \{1, \dots, f(n)\}$. For any ϕ of the above form and for any $\alpha_i \in \bigcup_{i<n} \mathsf{NFN}_i$, we say that α_i *is positive* in ϕ if $i \in s$, and otherwise α_i *is negative* in ϕ.

We define NFN_n as the set of *consistent* node expressions in NFN'_n (relative to axioms in Table 2). Lemma 7 below shows that elements of NFN_n can be thought of as normal forms for node expressions of degree n. For $\phi, \phi' \in \mathsf{NFN}_n$, we write $\phi \preceq_{\downarrow^+} \phi'$ if $\phi \wedge \langle \downarrow^+ [\phi'] \rangle$ is consistent.

Lemma 6. *If $\phi, \phi' \in \mathsf{NFN}_n$ and $\phi \preceq_{\downarrow^+} \phi'$ then every $\chi \in \bigcup_{i<n} \mathsf{NFN}_i$ positive in ϕ' is also positive in ϕ.*

Proof. Assume the contrary, i.e., some χ is positive in ϕ' but negative in ϕ. No formula of the form

$$\gamma \wedge \neg \langle \downarrow^+ [\chi] \rangle \wedge \langle \downarrow^+ [\langle \downarrow^+ [\chi] \rangle \wedge \gamma'] \rangle$$

is consistent with axioms in Table 2 (recall they include TransAx2). Details of the derivation are left to the reader. It follows that $\phi \wedge \langle \downarrow^+ [\phi'] \rangle$ must be inconsistent, which contradicts $\phi \preceq_{\downarrow^+} \phi'$. □

Incidentally, this argument relies on the transitivity of the $\downarrow^+$ axis.

If for $\phi, \phi' \in \mathsf{NFN}_n$, we have $\phi \preceq_{\downarrow^+} \phi'$ and some $\chi \in \bigcup_{i<n} \mathsf{NFN}_i$ is positive in ϕ but negative in ϕ', then we write $\phi \prec_{\downarrow^+} \psi$. It follows from the definition that $\prec_{\downarrow^+}$ is well-founded: there is no infinite descending chain of NFN_n expressions.

Lemma 7.

- *Every simple node expression in deg_n is provably equivalent to a disjunction of (zero or more) elements of NFN_n.*
- *For every pair of distinct elements $\phi, \psi \in \mathsf{NFN}_n$, $\phi \wedge \psi$ is inconsistent.*
- *Every element of NFN_n is satisfiable.*

Proof. (sketch) We only sketch a proof of the third item. Take any $\phi \in \mathsf{NFN}_n$. We construct an XML tree satisfying ϕ at the root as follows. The domain D of our XML tree consists of all sequences of elements of NFN_n of the form $(\beta_1, \dots, \beta_k)$, with $\beta_1 = \phi$ and for any $j < k$, $\beta_j \prec_{\downarrow^+} \beta_{j+1}$. Note that there are only finitely many such sequences. For $x, y \in D$, we say $xR_{\downarrow^+}y$ if the sequence x is an initial segment of the sequence y (recall the convention about trees in the present proof). Finally, the labelling function L labels the node $(\beta_1, \dots, \beta_k)$ with v if v is positive in β_k. It can be shown by induction that the root of the XML tree obtained this way, i.e., (ϕ), indeed satisfies ϕ. We call this tree *the canonical tree* of ϕ. □

Theorem 3 now follows:

Proof (of Theorem 3). We restrict ourselves to the difficult direction. Suppose that $\phi \equiv \psi$ is valid, where ϕ, ψ are arbitrary $\mathsf{Core\ XPath}(\downarrow^+)$ node expressions. By Lemma 5 and the first item of Lemma 7, for large enough n, ϕ is provably equivalent to some disjunction ϕ' of NFN_n expressions, and ψ is equivalent to some disjunction ψ' of NFN_n expressions. It follows by the remaining items of Lemma 7 that ϕ' and ψ' must be identical (up to the ordering of the disjuncts): if one contains a disjunct which does not appear in the other, this disjunct is satisfiable and wherever it is satisfied, no other disjunct may hold, a contradiction. Hence, ϕ' and ψ' are provably equivalent, and therefore so are ϕ and ψ. □

Next, we will proceed to prove Theorem 4

Definition 8. NFP_n *is the set of path expressions of the form*

$$S = .\,[\beta_1]\,/{\downarrow^+}\,[\beta_2]\,/\ldots/{\downarrow^+}\,[\beta_\ell]\,,$$

where $\ell \geq 1$, each $\beta_i \in \mathsf{NFN}_n$, and $\beta_i \preceq_{\downarrow^+} \beta_j$ for $i < j$.

Note that we use the weak order $\preceq_{\downarrow^+}$ here, not the strict order $\prec_{\downarrow^+}$ used in the construction of the models in the node completeness proof.

Lemma 9. *For every path expression A, there exists suitably large n s.t. for every $n' \geq n$, A is equivalent to a disjunction of elements of $\mathsf{NFP}_{n'}$.*

Proof. Repeated use of Lemma 7, Der3, Der16, Der17 and TransAx2. □

Definition 10. *Given two XML trees $T_1 = (D, R_{\downarrow^+}, L)$ and $T_2 = (D', R'_{\downarrow^+}, L')$ with roots r and r', respectively, and D, D' disjoint, we define their* transitive root union $T_1 \rhd T_2$ *as the XML tree $(D \cup D', R_{\downarrow^+} \cup R'_{\downarrow^+} \cup (\{r\} \times D'), L \cup L')$. That is, the root of the second becomes a child of the root of the first.*

For any $S \in \mathsf{NFP}_n$ of the form

$$S = .\,[\beta_1]\,/{\downarrow^+}\,[\beta_2]\,/\ldots/{\downarrow^+}\,[\beta_\ell]\,,$$

we define the canonical tree *of S as the structure*

$$T = T_1 \rhd (T_2 \rhd (\ldots \rhd T_\ell)\ldots),$$

where each T_i is the canonical tree of β_i as defined in the proof of Theorem 3.

Lemma 11. *Let $S \in \mathsf{NFP}_n$ and its canonical tree T be as in Definition 10 above and for each $i \leq \ell$, let r_i be the root of the tree T_i. Then $(r_1, r_\ell) \in [\![S]\!]_{\mathsf{PExpr}}$. Moreover, for any $S' = .[\beta'_1]/{\downarrow^+}[\beta'_2]/\ldots/{\downarrow^+}[\beta'_{\ell'}] \in \mathsf{NFP}_n$, $(r_1, r_\ell) \in [\![S']\!]_{\mathsf{PExpr}}$ iff $(\beta'_1, \ldots, \beta'_{\ell'})$ is a subsequence of $(\beta_1, \ldots, \beta'_{\ell'})$ s.t. $\beta_1 = \beta'_1$ and $\beta'_{\ell'} = \beta_\ell$.*

Proof. It is enough to prove the "moreover" part. The "if" direction is by direct verification (recall that if $\beta_i \preceq_{\downarrow^+} \beta_j$, then every formula negative in β_i is also negative in β_j). For the converse, recall that by Lemma 7, no two distinct elements of NFN_n can be true at the same point. If β'_1 is distinct from β_1, there is nothing to prove. Let $i \geq 2$ be the smallest number s.t. $(\beta'_1 = \beta_1, \beta'_2, \dots, \beta'_i)$ is not a subsequence of $(\beta_1, \beta_2, \dots, \beta_l)$. That is, $(\beta'_1 = \beta_1, \beta'_2, \dots, \beta'_{i-1}) = (\beta_1, \beta_{g(2)}, \dots, \beta_{g(i-1)})$ for some strictly increasing g, but $\beta'_i = \beta_r$ for no r s.t. $g(i-1) < r \leq k$. Note that we can always assume that $g(i)$ is chosen to be minimal, that is, e.g., that for no i properly contained between 1 and $g(2)$, β_i was equal to β'_2, for no i properly contained between $g(2)$ and $g(3)$, β_i was equal to β'_2 etc.

So it means that $\beta'_1 = \beta_1$, $\beta'_2 = \beta_{g(2)}$, $\dots$, $\beta'_{i-1} = \beta_{g(i-1)}$ are true at, respectively, r_1, $r_{g(2)}$, $\dots$, $r_{g(i-1)}$ but for no $j > g(i-1)$, β'_i does hold at r_i. But this means that $(r_1, r_k) \notin [\![S']\!]_{\mathsf{PExpr}}$. □

Lemma 12. *For any*

$$\begin{aligned} S' &= .\,[\beta'_1]\,/{\downarrow^+}\,[\beta'_2]\,/\dots/{\downarrow^+}\,[\beta'_{\ell'}] \\ S_1 &= .\,\left[\beta^1_1\right]/{\downarrow^+}\left[\beta^1_2\right]/\dots/{\downarrow^+}\left[\beta^1_{\ell(1)}\right] \qquad\qquad \in \mathsf{NFP}_n, \\ &\dots \\ S_k &= .\,\left[\beta^k_1\right]/{\downarrow^+}\left[\beta^k_2\right]/\dots/{\downarrow^+}\left[\beta^k_{\ell(k)}\right], \end{aligned}$$

S' is contained in $S_1 \cup \dots \cup S_k$ iff for some $i \leq k$, $(\beta^i_1, \dots, \beta^i_{\ell(i)})$ is a subsequence of $(\beta'_1, \dots, \beta'_{\ell'})$ s.t. $\beta'_1 = \beta^i_1$ and $\beta'_{\ell'} = \beta^i_{\ell(i)}$.

Proof. The "if" direction follows by a direct calculation. The "only if" is a consequence of Lemma 11, the validity of TransAx2 and soundness of the axioms: if $S' \neq S_i$ for every i, then $(r_1, r_{\ell'})$ in the canonical tree belongs to $[\![S']\!]_{\mathsf{PExpr}}$, but not to $[\![S_1]\!]_{\mathsf{PExpr}} \cup \dots \cup [\![S_k]\!]_{\mathsf{PExpr}}$. □

Finally, we prove Theorem 4:

Proof (of Theorem 4). Follows from Lemma 9 and Lemma 12. See the proof of Theorem 3 above for an analogous reasoning. □

2.4 Completeness for Core XPath(↓)

Similar results can be proved for Core XPath(↓):

Theorem 13 (Node Completeness for Core XPath(↓)). *A* Core XPath(↓) node *equivalence is valid iff it is derivable from the axioms in Table 2.*

Theorem 14 (Path Completeness for Core XPath(↓)). *A* Core XPath(↓) path *equivalence is valid iff it is derivable from the axioms in Table 2.*

Recall that derivations for Core XPath($\downarrow$) are not allowed to contain expressions involving other axes than $\downarrow$ and ".". Hence, this time the axioms for transitive axes are not included. As these were the only axioms in Table 2 involving specific axes, we obtain

Corollary 15. *A valid* Core XPath($\downarrow$) *equivalence remains valid after* $\downarrow$ *is uniformly replaced by an arbitrary path expression* A.

In other words, there are no Core XPath($\downarrow$)-specific validities.

Proof (of Theorems 13 and 14, sketch). We only highlight the most important differences with the proofs of Theorems 3 and 4. We define an ordering $\prec_\downarrow$ not just on NFN_n, but on the whole set $\bigcup_{i\leq n}\mathsf{NFN}_i$, relating normal form node expressions of degree i to ones of degree $i-1$. More precisely, we say that $\phi \prec_\downarrow \phi'$ if for some $i \leq n$, $\phi \in \mathsf{NFN}_i$, $\phi' \in \mathsf{NFN}_{i-1}$ and ϕ' is positive in ϕ. Canonical trees based on this order are defined similarly as in the proof of Lemma 7, this time using the child relation rather than the descendant relation. That is, we make $(\beta_1, \ldots, \beta_k)$ a parent of $(\beta_1, \ldots, \beta_k, \beta_{k+1})$ in the canonical tree. In the normal form for path expressions, the degree of node expressions in the sequence decreases. The definition of canonical trees for normal form path expressions does not require essential changes. The proof of the analogue of Lemma 11, finally, even becomes a bit simpler. □

2.5 Comments and Further Work

We conjecture that complete axiomatizations for the six remaining single axis fragments can be obtained by adding instances of the following axiom schemes, where s ranges over $\{\rightarrow, \leftarrow, \uparrow\}$:

- For intransitive axes: $\mathsf{s}\,[\neg\phi] \equiv .\,[\neg\,\langle \mathsf{s}\,[\phi]\rangle]\,/\mathsf{s}$.
- For transitive ones: $.\,[\langle \mathsf{s}^+\,[\phi]\rangle]\,/\mathsf{s}^+ \equiv \mathsf{s}^+\,[\phi] \cup \mathsf{s}^+\,[\phi]\,/\mathsf{s}^+ \cup \mathsf{s}^+\,[\langle \mathsf{s}^+\,[\phi]\rangle]$.

We are presently working on proofs of these results, and on extensions to some restricted combinations of axes.

We note that some of the equivalences proved above would cease to be XPath validities if path or node expressions are allowed to contain subexpressions which do not belong to Core XPath. Take the following two queries:

- $A =$ `following-sibling::*[child::`v`][position()=1]` and
- $B =$ `following-sibling::*[child::`v` and position()=1]`.

$A \equiv B$ is not a valid equivalence, even though it can be thought of as an instance of Der15. It is because `position()=1` is not a Core XPath expression, although some expressions containing it can be simulated.

Table 4. Additional Axioms for Core XPath

TreeAx1{	$\mathsf{s}^+/\mathsf{s} \cup \mathsf{s}$	$\equiv \mathsf{s}^+$	
	$\mathsf{s}/\mathsf{s}^+ \cup \mathsf{s}$	$\equiv \mathsf{s}^+$	
TreeAx2	$\mathsf{s}\,[\phi]\,/\mathsf{s}^{-1}$	$\equiv .\,[\langle \mathsf{s}\,[\phi]\rangle]$	for $\mathsf{s} \in \{\leftarrow, \rightarrow, \downarrow\}$
TreeAx3	$\uparrow[\phi]\,/\!\downarrow$	$\equiv (\leftarrow^+ \cup \rightarrow^+ \cup\, .)\,[\langle \uparrow[\phi]\rangle]$	
TreeAx4{	$\leftarrow^+$	$\equiv \leftarrow^+[\langle \uparrow \rangle]$	
	$\rightarrow^+$	$\equiv \rightarrow^+[\langle \uparrow \rangle]$	

Table 5. Equivalences of Blackburn, Meyer-Viol, de Rijke [3]

LOFT0	(boolean axioms)		
LOFT1{	$\langle \mathsf{s}\,[\mathsf{false}]\rangle$	$\equiv \mathsf{false}$	
	$\langle \mathsf{s}\,[\phi \vee \psi]\rangle$	$\equiv \langle \mathsf{s}\,[\phi]\rangle \vee \langle \mathsf{s}\,[\psi]\rangle$	
LOFT2	ϕ	$\leq \neg \left\langle \mathsf{s}\left[\neg \left\langle \mathsf{s}^{-1}\,[\phi]\right\rangle\right]\right\rangle$	
LOFT3	$\langle \mathsf{s}\,[\neg\phi]\rangle \wedge \langle \mathsf{s}\,[\phi]\rangle$	$\equiv \mathsf{false}$	for $\mathsf{s} \in \{\uparrow, \leftarrow, \rightarrow\}$
LOFT4	$\langle \mathsf{s}\,[\phi]\rangle \vee \left\langle \mathsf{s}\left[\left\langle \mathsf{s}^+\,[\phi]\right\rangle\right]\right\rangle$	$\equiv \left\langle \mathsf{s}^+\,[\phi]\right\rangle$	
LOFT5	$\neg \langle \mathsf{s}\,[\phi]\rangle \wedge \left\langle \mathsf{s}^+\,[\phi]\right\rangle$	$\leq \left\langle \mathsf{s}^+\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle$	
LOFT6	$\langle \mathsf{s}\,[\mathsf{true}]\rangle$	$\leq \left\langle \mathsf{s}^+\,[\neg \langle \mathsf{s}\,[\mathsf{true}]\rangle]\right\rangle$	
LOFT7	(TransAx1 for $\downarrow^+$ and $\rightarrow^+$)		
LOFT8	$\neg \langle \downarrow[\phi]\rangle$	$\geq \left\langle \downarrow\left[\neg \langle \leftarrow \rangle \wedge \neg\phi \wedge \neg \left\langle \rightarrow^+\,[\phi]\right\rangle\right]\right\rangle$	
LOFT9	$\langle \downarrow[\phi]\rangle$	$\leq \langle \downarrow[\neg \langle \leftarrow \rangle]\rangle \wedge \langle \downarrow[\neg \langle \rightarrow \rangle]\rangle$	
LOFT10	$\neg \langle \uparrow \rangle$	$\leq \neg \langle \leftarrow \rangle \wedge \neg \langle \rightarrow \rangle$	

3 Full Core XPath

We will now extend the axiomatization to full Core XPath. Table 4 presents a new group of axioms governing interactions between axes; the symbol s^{-1} denotes the converse of s. As in the previous cases, we focus first on completeness for node expressions. Using the same notion of derivability as before (cf. Section 2), we have:

Theorem 16 (Node Completeness for Full Core XPath). *A* Core XPath node *equivalence is valid iff it is derivable from the axioms given in Tables 2 and 4.*

Proof (sketch). As before, we first reduce all node expressions to simple ones. Next, we observe that these simple Core XPath node expressions can be seen as notational variants of formulas of the *logic of finite trees (LOFT)* [3]. Finally, we show that equivalences axiomatizing LOFT are derivable when written as simple node expressions. They are given in Table 5; the reader is asked to recall that both $\phi \leq \psi$ and $\psi \geq \phi$ abbreviate $\phi \vee \psi \equiv \psi$. See Appendix for the derivations. Completeness follows now from the completeness result in [3]. □

In fact, the proof given in [3] could be rewritten in the present setting, replacing everywhere modal formulas with simple node expressions. There is, however, little point of doing so as the proof is rather long and except for this syntactic transformation, no changes would be required. See also the comment at the end of Section 4.

The node completeness result can be lifted to path expressions by introducing an extra inference rule with syntactic side conditions, which we call the *Sep* rule, and which is closely related to the *separability rule* in [16].

(Sep) If $\langle A\,[v]\rangle \equiv \langle B\,[v]\rangle$ and v does not occur in A and B, then $A \equiv B$.

Corollary 17 (Non-orthodox Path Completeness for Full Core XPath). *A* Core XPath path equivalence *is valid iff it is derivable from the axioms in Tables 2 and 4 using the standard rules of equational logic plus the* Sep *rule.*

Proof. Suppose $A \equiv B$. Pick any v not occurring in A and B. Then the node expressions $\langle A\,[v]\rangle$ and $\langle B\,[v]\rangle$ are also equivalent. Hence, by Theorem 16, their equivalence is derivable (in the standard sense) from the axioms in Tables 2 and 4. A single application of the *Sep* rule now yields a derivation of $A \equiv B$. □

Such an axiomatization, however, is not very satisfactory as an axiomatization for path expressions. A purely equational one would be preferable both from the mathematical point of view (see, e.g., [17] and [21] for a justification) and from the point of view of potential applications for query optimization.

We should also point out that the rule *Sep* as formulated here is not sound when used in combination with the axiom scheme $v \wedge v' \equiv \bot$ discussed in Section 2.1. The problem can be solved, but every available solution requires some additional complications, e.g., in the formulation of the *Sep* rule. This is one more reason why a purely equational axiomatization would be of interest.

4 Discussion

We have given complete axiomatizations for Core XPath($\downarrow^+$), Core XPath($\downarrow$) and full Core XPath. We hope that these axiomatization will be of help in obtaining sets of effective rewrite rules for query optimization in these fragments. The query equivalence problem is known to be PSpace-complete for Core XPath($\downarrow^+$) and Core XPath($\downarrow$), and ExpTime-complete for full Core XPath. Even the latter is considerably better than the non-elementary complexity of query equivalence for Core XPath 2.0 [4], which was previously axiomatized in [5].

As we already noted, our results and techniques are closely connected to ones in modal logic. In particular, the simple node expressions of Core XPath($\downarrow$) and Core XPath($\downarrow^+$) correspond to formulas of the modal logics **K** and **GL** [2], respectively, and the simple node expressions of Core XPath correspond to formulas of the modal logic LOFT [3]. Consequently, Theorem 3 and 13 can also be derived from known completeness results for these logics. The main reason why we have given an explicit proof of Theorem 3 is that it provides the basis for the proof of the more interesting Theorem 4. The normal forms we use are inspired by [6, 15]. Theorem 14 is closely related to a completeness result for *dynamic relation algebras* in [9]. As for the full Core XPath language, we proved completeness by a direct reduction to a known completeness result for LOFT [3]. The proof in [3] was based on a constructive variant of the *filtration* technique, in general more popular in modal logic than normal forms. We believe the first step to obtain an equational axiomatization for path equivalences of full Core XPath would be to find a normal form completeness proof for node equivalences, analogous to those for single axis fragments given in this paper.

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Appendix

4.1 Proof of Lemma 2

- Der1 Use NdAx2, NdAx3, ISAx2.
- Der2 Use NdAx2, NdAx3, ISAx1. As was observed in the main paper, from this moment on we can use all the boolean equivalences for $\neg$ and $\vee$.
- Der3

$$\begin{aligned} A\,[\phi] &\equiv (A/.)\,[\phi] && \text{by ISAx5}\\ &\equiv A/.\,[\phi] && \text{by PrAx4} \end{aligned}$$

- Der4

$$\begin{aligned} A\,[\mathsf{true}] &\equiv A/.\,[\mathsf{true}] && \text{by Der3}\\ &\equiv A/. && \text{by PrAx2}\\ &\equiv A && \text{by ISAx5} \end{aligned}$$

- Der5

$$\begin{aligned} A\,[\mathsf{false}] &\equiv A\,[\mathsf{false}]\,/. && \text{by ISAx5}\\ &\equiv \bot && \text{by PrAx1} \end{aligned}$$

- Der6

$$\begin{aligned} (A \cup B)\,[\phi] &\equiv ((A \cup B)/.)\,[\phi] && \text{by ISAx5}\\ &\equiv (A \cup B)/.\,[\phi] && \text{by PrAx4}\\ &\equiv A/.\,[\phi] \cup B/.\,[\phi] && \text{by ISAx6}\\ &\equiv (A/.)\,[\phi] \cup (B/.)\,[\phi] && \text{by PrAx4}\\ &\equiv A\,[\phi] \cup B\,[\phi] && \text{by ISAx5} \end{aligned}$$

One of consequences we will use particularly often is *monotonicity*:

$$A \subseteq B \text{ implies } \begin{cases} A\,[\phi] \subseteq B\,[\psi] & \text{for any } \phi, \psi \in \mathsf{NodeEx} \text{ by Der6}\\ \langle A \rangle \leq \langle B \rangle & \text{by NdAx3} \end{cases}$$

$$\phi \leq \psi \text{ implies } A\,[\phi] \subseteq A\,[\psi] \;\text{ for any } A \in \mathsf{PathEx} \text{ by PrAx3}$$

- Der7. $.\,[\neg\langle . \rangle] \subseteq .\,[\phi]$ is an instance of ISAx7. Monotonicity yields $\langle .\,[\neg\langle . \rangle] \rangle \leq \langle .\,[\phi] \rangle$. An application of NdAx2 on both sides yields the dual form of Der7.
- Der8. Follows from Der7 by boolean laws.

- Der9

$$\begin{aligned} A &\equiv A\left[\mathsf{true}\right] && \text{by Der4}\\ &\equiv A\left[\phi \vee \neg\phi\right] && \text{by Der8}\\ &\equiv A\left[\phi\right] \cup A\left[\neg\phi\right] && \text{by PrAx3} \end{aligned}$$

- Der10

$$\begin{aligned} A\left[\phi\right]/.\left[\neg\phi\right] &\equiv A\left[\neg\neg\phi\right]/.\left[\neg\phi\right] && \text{boolean}\\ &\equiv A\left[\neg\left\langle .\left[\neg\phi\right]\right\rangle\right]/.\left[\neg\phi\right] && \text{by NdAx2}\\ &\equiv \bot && \text{by PrAx1} \end{aligned}$$

- Der11

$$\begin{aligned} A/\bot &\equiv A\left[\langle . \rangle\right]/.\left[\neg\langle . \rangle\right] && \text{by Der4}\\ &\equiv \bot && \text{by Der10} \end{aligned}$$

$$\begin{aligned} \bot/A &\equiv A\left[\mathsf{false}\right]/A && \text{by Der5}\\ &\subseteq A\left[\neg\langle A \rangle\right]/A && \text{by Der7 and monotonicity}\\ &\equiv \bot && \text{by PrAx2} \end{aligned}$$

- Der12

$$\begin{aligned} A\left[\phi\right]\left[\neg\phi\right] &\equiv A\left[\phi\right]/.\left[\neg\phi\right] && \text{by Der3}\\ &\equiv \bot && \text{by Der10} \end{aligned}$$

- Der13

$$\begin{aligned} A\left[\phi \wedge \psi\right]\left[\neg\phi\right] &\subseteq A\left[\phi\right]\left[\neg\phi\right] && \text{monotonicity}\\ &\equiv \bot && \text{by Der12} \end{aligned}$$

- Der14

$$\begin{aligned} A\left[\phi\right]\left[\psi\right]\left[\neg\phi \vee \neg\psi\right] &\equiv A\left[\phi\right]\left[\psi\right]\left[\neg\phi\right] \cup A\left[\phi\right]\left[\psi\right]\left[\neg\psi\right] && \text{by PrAx3}\\ &\subseteq A\left[\phi\right]\left[\mathsf{true}\right]\left[\neg\phi\right] \cup A\left[\mathsf{true}\right]\left[\psi\right]\left[\neg\psi\right] && \text{by Der7 and monotonicity}\\ &\equiv A\left[\phi\right]\left[\neg\phi\right] \cup A\left[\psi\right]\left[\neg\psi\right] && \text{by Der4}\\ &\equiv \bot && \text{by Der12} \end{aligned}$$

- Der15
  First, let us derive

$$\begin{aligned} A\,[\phi\wedge\psi] &\equiv A\,[\phi\wedge\psi]\,[\phi]\cup A\,[\phi\wedge\psi]\,[\neg\phi] && \text{by Der9}\\ &\equiv A\,[\phi\wedge\psi]\,[\phi]\cup\bot && \text{by Der13}\\ &\equiv A\,[\phi\wedge\psi]\,[\phi] && \text{by ISAx7} \end{aligned}$$

$A\,[\phi\wedge\psi]\equiv A\,[\phi\wedge\psi]\,[\psi]$ is derived analogously. Thus, using monotonicity we get $A\,[\phi\wedge\psi]\subseteq A\,[\phi]\,[\psi]$. Conversely,

$$\begin{aligned} A\,[\phi]\,[\psi] &\equiv A\,[\phi]\,[\psi]\,[\phi\wedge\psi]\cup A\,[\phi]\,[\psi]\,[\neg\phi\vee\neg\psi] && \text{by Der9}\\ &\equiv A\,[\phi]\,[\psi]\,[\phi\wedge\psi]\cup\bot && \text{by Der14}\\ &\equiv A\,[\phi]\,[\psi]\,[\phi\wedge\psi] && \text{by ISAx7} \end{aligned}$$

Using Der7, monotonicity and Der4 we get that $A\,[\phi]\,[\psi]\subseteq A\,[\phi\wedge\psi]$ and Der15 is proved.

- Der16

$$\begin{aligned} A\,[\phi\wedge\psi] &\equiv A\,[\phi]\,[\psi] && \text{by Der15}\\ &\equiv A\,[\phi]\,/.\,[\psi] && \text{by Der3} \end{aligned}$$

- Der17

$$\begin{aligned} A &\equiv ./A && \text{by ISAx5}\\ &\equiv (.\,[\langle A\rangle]\cup .\,[\neg\langle A\rangle])/A && \text{by Der9}\\ &\equiv .\,[\langle A\rangle]\,/A\cup .\,[\neg\langle A\rangle]\,/A && \text{by ISAx6}\\ &\equiv .\,[\langle A\rangle]\,/A\cup\bot && \text{by PrAx1}\\ &\equiv .\,[\langle A\rangle]\,/A && \text{by ISAx7} \end{aligned}$$

4.2 Proof of Lemma 5

We provide a translation $(\cdot)^{\mathsf{s}}:\mathsf{NodeEx}\mapsto\mathsf{siNode}$ which is constant for elements of siNode. This mapping uses an auxiliary mapping $(\cdot)^{\mathsf{s}}:\mathsf{PathEx}\mapsto(\mathsf{siNode}\mapsto\mathsf{siNode})$ assigning to every path expression a unary function defined on simple node expressions. As domains of both mappings are disjoint, we use the same symbol with no risk of confusion. Their definitions are given in Table 6. It is easy to show from the axioms that this translation always yields a provably equivalent node expression. Moreover, the axioms proving equivalence are only those occurring in Table 2 and thus not dependent on a chosen a.

Hence, the Lemma can be reformulated as follows:

> For every $A\in\mathsf{PathEx}$ and for every $\phi\in\mathsf{NodeEx}$, $\langle A\rangle\equiv A^{\mathsf{s}}(\mathsf{true})$ and $\phi\equiv\phi^{\mathsf{s}}$ are provable.

Table 6. Translation of $\mathsf{Core\ XPath(a)}$ node expressions into simple node expressions

v^{s}	$:=$	v	$.^{\mathsf{s}}(\phi)$	$:=$	ϕ
$(\neg\phi)^{\mathsf{s}}$	$:=$	$\neg\phi^{\mathsf{s}}$	$\mathsf{a}^{\mathsf{s}}(\phi)$	$:=$	$\langle \mathsf{a}\,[\phi]\rangle$
$(\phi \vee \psi)^{\mathsf{s}}$	$:=$	$\phi^{\mathsf{s}} \vee \psi^{\mathsf{s}}$	$(A \cup B)^{\mathsf{s}}(\phi)$	$:=$	$A^{\mathsf{s}}(\phi) \vee B^{\mathsf{s}}(\phi)$
$\langle A\rangle^{\mathsf{s}}$	$:=$	$A^{\mathsf{s}}(\langle . \rangle)$	$(A\,[\psi])^{\mathsf{s}}(\phi)$	$:=$	$A^{\mathsf{s}}(\psi^{\mathsf{s}} \wedge \phi)$
			$(A/B)^{\mathsf{s}}(\phi)$	$:=$	$A^{\mathsf{s}}(B^{\mathsf{s}}(\phi))$

By Der4, this in turn is implied by the following

> For every $A \in \mathsf{PathEx}$ and for every $\phi \in \mathsf{NodeEx}$, $\langle A\,[\phi]\rangle \equiv A^{\mathsf{s}}(\phi)$ and $\phi \equiv \phi^{\mathsf{s}}$ are provable.

Inductive steps for node expressions are obvious, hence we focus only on inductive steps for path expressions.

- $A = .$: by NdAx2.
- $A = \mathsf{a} \in \mathsf{siAxis}$: by definition of $(\cdot)^{\mathsf{s}}$.
- $A = B \cup C$:

$$\begin{aligned}
\langle (B \cup C)\,[\phi]\rangle &\equiv \langle B\,[\phi] \cup C\,[\phi]\rangle && \text{by Der6}\\
&\equiv \langle B\,[\phi]\rangle \vee \langle C\,[\phi]\rangle && \text{by NdAx3}\\
&\equiv B^{\mathsf{s}}(\phi) \vee C^{\mathsf{s}}(\phi) && \text{by IH}\\
&\equiv (B \cup C)^{\mathsf{s}}(\phi) && \text{by definition of } (\cdot)^{\mathsf{s}}
\end{aligned}$$

- $A = B/C$:

$$\begin{aligned}
\langle (B/C)\,[\phi]\rangle &\equiv \langle B/C\,[\phi]\rangle && \text{by PrAx4}\\
&\equiv \langle B\,[\langle C\,[\phi]\rangle]\rangle && \text{by NdAx4}\\
&\equiv B^{\mathsf{s}}(C^{\mathsf{s}}(\phi)) && \text{by IH}\\
&\equiv (B/C)^{\mathsf{s}}(\phi) && \text{by definition of } (\cdot)^{\mathsf{s}}
\end{aligned}$$

- $A = B\,[\psi]$:

$$\begin{aligned}
\langle B\,[\psi]\,[\phi]\rangle &\equiv \langle B\,[\psi^{\mathsf{s}}]\,[\phi]\rangle && \text{by IH on } \mathsf{NodeEx}\\
&\equiv \langle B\,[\psi^{\mathsf{s}} \wedge \phi]\rangle && \text{by Der15}\\
&\equiv B^{\mathsf{s}}(\psi^{\mathsf{s}} \wedge \phi) && \text{by IH on } \mathsf{PathEx}\\
&\equiv (B\,[\psi])^{\mathsf{s}}(\phi) && \text{by definition of } (\cdot)^{\mathsf{s}}
\end{aligned}$$

Proof of Theorem 16

We begin by deriving a number of auxiliary references shown in Table 7. We use an additional abbreviation: $\mathsf{s}^* = . \cup \mathsf{s}^+$.

Table 7. Additional auxiliary equivalences

Der18	$.\,[\langle A\rangle]\,/.\,[\phi]\,/A$	$\equiv .\,[\phi]\,/A$	
Der19	$\langle A/B\rangle$	$\leq \langle A\rangle$	
Der20	$\phi \wedge \psi$	$\equiv \langle .\,[\phi]\,/.\,[\psi]\rangle$	
Der21	$\langle A\,[\mathsf{false}]\rangle$	$\equiv \mathsf{false}$	
Der22	$\langle A\,[\phi \vee \psi]\rangle$	$\equiv \langle A\,[\phi]\rangle \vee \langle A\,[\psi]\rangle$	
Der23	$\langle A\,[\phi \wedge \psi]\rangle \wedge \neg \langle A\,[\psi]\rangle$	$\equiv \mathsf{false}$	
Der24	$\langle A\,[\phi]\rangle \wedge \neg \langle A\,[\psi]\rangle$	$\leq \langle A\,[\phi \wedge \neg\psi]\rangle$	
Der25	$.\,[\langle \mathsf{s}\,[\phi]\rangle]$	$\subseteq \mathsf{s}\,[\phi]\,/\mathsf{s}^{-1}$	
Der26	$\mathsf{s}/\mathsf{s}^{-1}$	$\equiv .\,[\langle \mathsf{s}\rangle]$	for $\mathsf{s} \in \{\leftarrow, \rightarrow, \downarrow\}$
Der27	$\uparrow/\downarrow$	$\equiv (\leftarrow^{+} \cup \rightarrow^{+} \cup\, .)\,[\langle\uparrow\rangle]$	
Der28	$\langle \mathsf{s}\rangle$	$\equiv \left\langle \mathsf{s}^{+}\right\rangle$	
Der29	$\langle \mathsf{s}\,[\phi]\rangle$	$\equiv \left\langle \mathsf{s}^{+}\,[\phi \wedge \neg \langle \mathsf{s}\,[\phi]\rangle]\right\rangle$	
Der30	$\langle \mathsf{s}^{*}\,[\neg \langle \mathsf{s}\,[\mathsf{true}]\rangle]\rangle$	$\equiv \mathsf{true}$	

- Der18—from Der16, commutativity of boolean $\wedge$ and Der17.
- Der19

$$\begin{aligned}
\langle A/B\rangle &\equiv \langle A\,[\langle B\rangle]\rangle && \text{by NdAx4}\\
&\leq \langle A\,[\mathsf{true}]\rangle && \text{monotonicity}\\
&\equiv \langle A\rangle && \text{by Der4}
\end{aligned}$$

- Der20—from NdAx2 and Der16.
- Der21—from monotonicity and NdAx2.
- Der22—from NdAx3 and PrAx3.
- Der23

$$\begin{aligned}
\langle A\,[\phi \wedge \psi]\rangle \wedge \neg \langle A\,[\psi]\rangle &\leq \langle A\,[\phi \wedge \psi]\rangle \wedge \neg \langle A\,[\psi \wedge \psi]\rangle && \text{monotonicity}\\
&\equiv \mathsf{false} && \text{by dual of Der8}
\end{aligned}$$

- Der24

$$\begin{aligned}
&\langle A\,[\phi]\rangle \wedge \neg \langle A\,[\psi]\rangle \equiv\\
&\equiv \langle A\,[(\phi \wedge \neg\psi) \vee (\phi \wedge \psi)]\rangle \wedge \neg \langle A\,[\psi]\rangle && \text{boolean}\\
&\equiv (\langle A\,[\phi \wedge \neg\psi]\rangle \vee \langle A\,[\phi \wedge \psi]\rangle) \wedge \neg \langle A\,[\psi]\rangle && \text{by Der22}\\
&\equiv (\langle A\,[\phi \wedge \neg\psi]\rangle \wedge \neg \langle A\,[\psi]\rangle) \vee (\langle A\,[\phi \wedge \psi]\rangle \wedge \neg \langle A\,[\psi]\rangle) && \text{boolean}\\
&\equiv \langle A\,[\phi \wedge \neg\psi]\rangle \wedge \neg \langle A\,[\psi]\rangle && \text{by Der23}\\
&\leq \langle A\,[\phi \wedge \neg\psi]\rangle && \text{boolean}
\end{aligned}$$

- Der25 For $\mathsf{s} \in \{\leftarrow, \rightarrow, \downarrow\}$ by TreeAx2 ($\subseteq$ can be even replaced by $\equiv$ then). For $\mathsf{s} = \uparrow$ by TreeAx3.
- Der26 and Der27—from TreeAx2 and TreeAx3, respectively, using Der4.

- Der28

$$\begin{aligned}\left\langle \mathsf{s}^+ \right\rangle &\equiv \left\langle \mathsf{s} \cup \mathsf{s}/\mathsf{s}^+ \right\rangle && \text{by TreeAx1}\\ &\equiv \langle \mathsf{s} \rangle \vee \left\langle \mathsf{s}/\mathsf{s}^+ \right\rangle && \text{by NdAx3}\\ &\equiv \langle \mathsf{s} \rangle && \text{by Der19}\end{aligned}$$

- Der29

$$\begin{aligned}\langle \mathsf{s}\,[\phi] \rangle &\leq \left\langle \mathsf{s}^+\,[\phi] \right\rangle && \text{by LOFT4}\\ &\leq \left\langle \mathsf{s}^+ \left[\phi \wedge \neg \left\langle \mathsf{s}^+\,[\phi] \right\rangle\right] \right\rangle && \text{by TransAx1}\\ &\leq \left\langle \mathsf{s}^+\,[\phi \wedge \neg \langle \mathsf{s}\,[\phi] \rangle] \right\rangle && \text{by LOFT4.}\end{aligned}$$

  The use of LOFT4 does not lead to a vicious circle, see its derivation below.
- Der30—follows from Der29 and boolean axioms. Recall that $\mathsf{s}^* = . \cup \mathsf{s}^+$.

Now we can derive the LOFT axioms themselves.

- LOFT0. See the remark on Der2 above.
- LOFT1—already proved, as an instance of Der21 and Der22.
- LOFT2. By boolean reasoning, it is equivalent to

$$\left\langle \mathsf{s} \left[\neg \left\langle \mathsf{s}^{-1}\,[\phi] \right\rangle\right] \right\rangle \wedge \phi \equiv \mathsf{false}.$$

  This in turn follows from

$$\begin{aligned}\left\langle \mathsf{s} \left[\neg \left\langle \mathsf{s}^{-1}\,[\phi] \right\rangle\right] \right\rangle \wedge \phi &\equiv \left\langle . \left[\left\langle . \left[\left\langle \mathsf{s} \left[\neg \left\langle \mathsf{s}^{-1}\,[\phi] \right\rangle\right] \right\rangle\right] / . \,[\phi] \right\rangle\right] \right\rangle && \text{by Der20}\\ &\leq \left\langle . \left[\left\langle \mathsf{s} \left[\neg \left\langle \mathsf{s}^{-1}\,[\phi] \right\rangle\right] / \mathsf{s}^{-1} / . \,[\phi] \right\rangle\right] \right\rangle && \text{by Der25}\\ &\equiv \left\langle . \left[\left\langle \mathsf{s} \left[\neg \left\langle \mathsf{s}^{-1}\,[\phi] \right\rangle\right] / \mathsf{s}^{-1}\,[\phi] \right\rangle\right] \right\rangle && \text{by Der3}\\ &\equiv \mathsf{false} && \text{by PrAx1}\end{aligned}$$

- LOFT3

$$\begin{aligned}\langle \mathsf{s}\,[\neg\phi] \rangle \wedge \langle \mathsf{s}\,[\phi] \rangle &\equiv \langle . \,[\langle \mathsf{s}\,[\neg\phi] \rangle] / . \,[\langle \mathsf{s}\,[\phi] \rangle] \rangle && \text{by Der20}\\ &\leq \left\langle . \left[\left\langle \mathsf{s}\,[\neg\phi] / \mathsf{s}^{-1} / \mathsf{s}\,[\phi] / \mathsf{s}^{-1} \right\rangle\right] \right\rangle && \text{by Der25}\\ &\equiv \left\langle . \left[\left\langle \mathsf{s}\,[\neg\phi] / . \left[\left\langle \mathsf{s}^{-1} \right\rangle\right] / . \,[\phi] / \mathsf{s}^{-1} \right\rangle\right] \right\rangle && \text{by Der26}\\ &\equiv \left\langle . \left[\left\langle \mathsf{s}\,[\neg\phi] / . \,[\phi] / \mathsf{s}^{-1} \right\rangle\right] \right\rangle && \text{by Der17}\\ &\equiv \left\langle . \left[\left\langle \mathsf{s}\,[\mathsf{false}] / \mathsf{s}^{-1} \right\rangle\right] \right\rangle && \text{by Der16}\\ &\equiv \mathsf{false} && \text{by Der5}\end{aligned}$$

– LOFT4

$$\begin{aligned}
\left\langle \mathsf{s}^{+}\,[\phi]\right\rangle &\equiv \left\langle (\mathsf{s} \cup \mathsf{s}/\mathsf{s}^{+})\,[\phi]\right\rangle && \text{by TreeAx1}\\
&\equiv \left\langle \mathsf{s}\,[\phi] \cup (\mathsf{s}/\mathsf{s}^{+})\,[\phi]\right\rangle && \text{by Der6}\\
&\equiv \left\langle \mathsf{s}\,[\phi] \cup \mathsf{s}/\mathsf{s}^{+}\,[\phi]\right\rangle && \text{by PrAx4}\\
&\equiv \langle \mathsf{s}\,[\phi]\rangle \vee \left\langle \mathsf{s}/\mathsf{s}^{+}\,[\phi]\right\rangle && \text{by NdAx3}\\
&\equiv \langle \mathsf{s}\,[\phi]\rangle \vee \left\langle \mathsf{s}\left[\left\langle \mathsf{s}^{+}\,[\phi]\right\rangle\right]\right\rangle && \text{by NdAx4}
\end{aligned}$$

– LOFT5 (see [20]). By boolean reasoning, it boils down to proving that

$$t{:=}\neg\,\langle \mathsf{s}\,[\phi]\rangle \wedge \left\langle \mathsf{s}^{+}\,[\phi]\right\rangle \wedge \neg\left\langle \mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle \equiv \mathsf{false}.$$

This is proven by first observing that

$$\begin{aligned}
t &\equiv \neg(\langle \mathsf{s}\,[\phi]\rangle \vee \langle \mathsf{s}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\rangle)\wedge\\
&\qquad \wedge \left\langle \mathsf{s}/\mathsf{s}^{+}\,[\phi]\right\rangle \wedge \neg\left\langle \mathsf{s}/\mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle && \text{by LOFT4}\\
&\equiv \neg\,\langle \mathsf{s}\,[\phi \vee (\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle)]\rangle \wedge \left\langle \mathsf{s}/\mathsf{s}^{+}\,[\phi]\right\rangle \wedge\\
&\qquad \wedge \neg\left\langle \mathsf{s}/\mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle && \text{by Der22}\\
&\equiv \neg\,\langle \mathsf{s}\,[\phi \vee \langle \mathsf{s}\,[\phi]\rangle]\rangle \wedge \left\langle \mathsf{s}/\mathsf{s}^{+}\,[\phi]\right\rangle \wedge \neg\left\langle \mathsf{s}/\mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle && \text{boolean}\\
&\equiv \neg\,\langle \mathsf{s}\,[\phi \vee \langle \mathsf{s}\,[\phi]\rangle]\rangle \wedge \left\langle \mathsf{s}\left[\left\langle \mathsf{s}^{+}\,[\phi]\right\rangle\right]\right\rangle \wedge\\
&\qquad \wedge \neg\left\langle \mathsf{s}\left[\left\langle \mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle\right]\right\rangle && \text{by NdAx4}\\
&\leq \left\langle \mathsf{s}\left[\neg(\phi \vee \langle \mathsf{s}\,[\phi]\rangle) \wedge \left\langle \mathsf{s}^{+}\,[\phi]\right\rangle \wedge \neg\left\langle \mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle\right]\right\rangle && \text{by Der24}\\
&\leq \left\langle \mathsf{s}\left[\neg\,\langle \mathsf{s}\,[\phi]\rangle \wedge \left\langle \mathsf{s}^{+}\,[\phi]\right\rangle \wedge \neg\left\langle \mathsf{s}^{+}\,[\neg\phi \wedge \langle \mathsf{s}\,[\phi]\rangle]\right\rangle\right]\right\rangle && \text{monotonicity}\\
&= \langle \mathsf{s}\,[t]\rangle\,.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
t &\leq \langle \mathsf{s}\,[t]\rangle && \text{by the above}\\
&\leq \left\langle \mathsf{s}^{+}\,[t \wedge \neg\,\langle \mathsf{s}\,[t]\rangle]\right\rangle && \text{Der29}\\
&\equiv \langle \mathsf{s}\,[\mathsf{false}]\rangle && \text{by the above}\\
&\equiv \mathsf{false} && \text{by Der5}
\end{aligned}$$

– LOFT6—already proved, as an instance of Der29.

– LOFT7 does not need to be proved, being an instance of an axiom.

– LOFT8 By boolean reasoning, it is equivalent to

$$\langle\downarrow[\neg\,\langle\leftarrow\rangle \wedge \neg\,\langle\rightarrow^{*}[\phi]\rangle]\rangle \wedge \langle\downarrow[\phi]\rangle \equiv \mathsf{false}.$$

$$\begin{aligned}
&\langle\downarrow[\neg\langle\leftarrow\rangle\wedge\neg\langle\rightarrow^{*}[\phi]\rangle]\rangle\wedge\langle\downarrow[\phi]\rangle\equiv\\
&\qquad\equiv\langle.\,[\langle\downarrow[\neg\langle\leftarrow\rangle\wedge\neg\langle\rightarrow^{*}[\phi]\rangle]\rangle]\,/.\,[\langle\downarrow[\phi]\rangle]\rangle && \text{by Der20}\\
&\qquad\equiv\langle\downarrow[\neg\langle\leftarrow\rangle\wedge\neg\langle\rightarrow^{*}[\phi]\rangle]\,/\uparrow/\downarrow[\phi]\,/\uparrow\rangle && \text{by Der26}\\
&\qquad\equiv\left\langle\downarrow[\neg\langle\leftarrow\rangle\wedge\neg\langle\rightarrow^{*}[\phi]\rangle]\,/(\leftarrow^{+}\cup\rightarrow^{+}\cup\,.)\,[\langle\uparrow\rangle]\,[\phi]\,/\uparrow\right\rangle && \text{by Der27}\\
&\qquad\equiv\left\langle\downarrow[\neg\langle\leftarrow\rangle\wedge\neg\langle\rightarrow^{*}[\phi]\rangle]\,/(\leftarrow^{+}\cup\rightarrow^{+}\cup\,.)\,[\phi]\,/\uparrow\right\rangle && \text{by Der17}\\
&\qquad\leq\left\langle(\downarrow[\neg\langle\leftarrow\rangle]\,/\leftarrow^{+}/\uparrow)\cup(\downarrow[\neg\langle\rightarrow^{*}[\phi]\rangle]\,/\rightarrow^{*}[\phi]\,/\uparrow\right\rangle && \text{by ISAx6}\\
&\qquad\equiv\mathsf{false} && \text{by PrAx1}
\end{aligned}$$

- LOFT9

  $$\begin{aligned}
  \langle\downarrow[\phi]\rangle&\leq\langle\downarrow[\mathsf{true}]\rangle && \text{monotonicity}\\
  &\equiv\langle\downarrow[\langle\leftarrow^{*}[\neg\langle\leftarrow[\mathsf{true}]\rangle]\rangle]\rangle && \text{by Der30}\\
  &\equiv\langle\downarrow[\neg\langle\leftarrow\rangle]\rangle\vee\left\langle\downarrow\left[\left\langle\leftarrow^{+}[\neg\langle\leftarrow\rangle]\right\rangle\right]\right\rangle && \text{by Der22}
  \end{aligned}$$

  But now

  $$\begin{aligned}
  \left\langle\downarrow\left[\left\langle\leftarrow^{+}[\neg\langle\leftarrow\rangle]\right\rangle\right]\right\rangle&\equiv\left\langle\downarrow\left[\left\langle\leftarrow^{+}[\uparrow]\,[\neg\langle\leftarrow\rangle]\right\rangle\right]\right\rangle && \text{by TreeAx4}\\
  &\leq\langle\downarrow[\langle\uparrow/\downarrow[\neg\langle\leftarrow\rangle]\rangle]\rangle && \text{by Der27}\\
  &\equiv\langle\downarrow/\uparrow/\downarrow[\neg\langle\leftarrow\rangle]\rangle && \text{by NdAx4}\\
  &\equiv\langle.\,[\downarrow]\,/\downarrow[\neg\langle\leftarrow\rangle]\rangle && \text{by Der26}\\
  &\equiv\langle\downarrow[\neg\langle\leftarrow\rangle]\rangle && \text{by Der17.}
  \end{aligned}$$

  Thus, we got $\langle\downarrow[\phi]\rangle\leq\langle\downarrow[\neg\langle\leftarrow\rangle]\rangle$.
  The proof of $\langle\downarrow[\phi]\rangle\leq\langle\downarrow[\neg\langle\rightarrow\rangle]\rangle$ is analogous.
- LOFT10. First observe that by boolean reasoning, it is equivalent to

  $$\langle\leftarrow\rangle\vee\langle\rightarrow\rangle\leq\langle\uparrow\rangle\,.$$

  $$\begin{aligned}
  \langle\leftarrow\rangle\vee\langle\rightarrow\rangle&\equiv\left\langle\leftarrow^{+}\right\rangle\vee\left\langle\rightarrow^{+}\right\rangle && \text{by Der28}\\
  &\equiv\left\langle\leftarrow^{+}\cup\rightarrow^{+}\right\rangle && \text{by NdAx3}\\
  &\equiv\left\langle\leftarrow^{+}[\langle\uparrow\rangle]\cup\rightarrow^{+}[\langle\uparrow\rangle]\right\rangle && \text{by TreeAx4}\\
  &\equiv\left\langle(\leftarrow^{+}\cup\rightarrow^{+})\,[\langle\uparrow\rangle]\right\rangle && \text{by Der6}\\
  &\leq\langle\uparrow/\downarrow\rangle && \text{by Der27}\\
  &\leq\langle\uparrow\rangle && \text{by Der19}
  \end{aligned}$$