Solving large structured Markov Decision Problems for perishable inventory management and traffic control

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Appendix E

Extrapolating tabulated functions

A tabulated function is a function whose values are known and tabulated for a finite number of points on a grid, rather than set by an analytical expression.

E.1 1-dimensional tabulated functions

Consider a tabulated function $V(x) : \mathbb{N} \to \mathbb{R}$, for which the function values are known for $x \in \{0, 1, \ldots, B\}$. Through quadratic extrapolation based on three of the known values one can estimate the function value $V(y)$ for $y \in \{B+1, B+2, \ldots\}$.

E.1.1 Indirect extrapolation through a quadratic function

One approach for quadratic extrapolation is to compute the coefficients of the quadratic function $f(x)$ through the three points $V(B)$, $V(B-1)$ and $V(B-2)$, and estimate $V(y)$ by $f(y)$. We call this the indirect approach, since it requires the computation of the coefficients a, b, and c of the function $f(x) = a \cdot y^2 + b \cdot y + c$. The computation of these coefficients follow from the next theorem:

Theorem E.1.1. *Given a function $f(x) = ay^2 + by + c$, with unknown coefficients a, b, and c, but known function values $f(B)$, $f(B-1)$, and $f(B-2)$, for some $B \in \mathbb{R}$, then a, b, and c are set by:*

$$a = [f(B) - 2f(B-1) + f(B-2)]/2,$$
$$b = [f(B) - f(B-2)]/2 - 2a(B-1), \text{ and}$$
$$c = f(B) - aB^2 - bB.$$

Proof. The coefficients follow from the following observations:

- Coefficient c follows directly from the definition of $f(B)$.
- Coefficient b follows from the first order derivative of the curve $f(x) = ax^2 + bx + c$ through the points $(B, f(B))$, $(B-1, f(B-1))$ and $(B-2, f(B-2))$: $f'(x) = 2ax + b$. It is easily shown that $f'(x) = [f(x+1) - f(x-1)]/2$ for any x. Hence $2ax + b = [f(x+1) - f(x-1)]/2$ and thus $b = [f(x+1) - f(x-1)]/2 - 2ax$. Once a is known b is computed by evaluating the expression for $x = B - 1$.
- Coefficient a follows from the second order derivative of $f(x)$: $f''(x) = 2a$. We claim $f''(x) = [f(x+1) - f(x)] - [f(x) - f(x-1)]$ for all x. (The proof follows from straight forward calculus.) Hence $a = [f(x+1) - 2f(x) + f(x-1)]/2$ for any x. Evaluating this expression for $x = B - 1$ gives the value of a.

□

Indirect extrapolation formula

The suggested extrapolation formula for $V(y)$ is thus:

$$V(y) \approx a \cdot y^2 + b \cdot y + c \tag{E.1}$$

with

$a = [V(B) - 2V(B-1) + V(B-2)]/2$,
$b = [V(B) - V(B-2)]/2 - 2a(B-1)$, and
$c = V(B) - aB^2 - bB$.

If V indeed appears to be a quadratic function than this approximation is exact.

E.1.2 Alternative: direct quadratic extrapolation

An alternative approximation of $V(y)$ extrapolates V using (directly) the known function values $V(B)$, $V(B-\Delta)$ and $V(B-2\Delta)$ (with $\Delta = y - B$), rather then first computing the coefficients of a (quadratic) function:

$$V(y) \approx 3 \cdot V(B) - 3 \cdot V(B-\Delta) + V(B-2\Delta) \tag{E.2}$$

This extrapolation works fine whenever $V(B)$, $V(B-\Delta)$ and $V(B-2\Delta)$ are known: thus when $B \leq y \leq \lfloor \frac{3B}{2} \rfloor$. (For $y \leq B$ the extrapolation is meaning less, since the function values are known for $x \in \{0, 1, \ldots, B\}$. For $y > \lfloor \frac{3B}{2} \rfloor$ one may apply a trick by recursively extrapolating function values.)

The proof that the approximation indeed corresponds to quadratic extrapolation comes from the proof of the following theorem.

Theorem E.1.2. *Given a quadratic function $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, then for any $y, B \in \mathbb{R}$ and $\Delta = y - B$ the following relation holds:*

$$f(y) = 3 \cdot f(B) - 3 \cdot f(B-\Delta) + f(B-2\Delta) \tag{E.3}$$

Proof. For $y = B$, and thus $\Delta = 0$, the theorem clearly holds. A formal proof requires straight forward calculus to show the following equality (in which the constant c has already been canceled):

$$a(B+\Delta)^2 + b(B+\Delta) = 3(aB^2 + bB) - 3(a(B-\Delta)^2 + b(B-\Delta)) + a(B-2\Delta)^2 + b(B-2\Delta)$$

By simplifying the expressions and further cancelation of terms that are on both sides of the = sign, one proofs the equality. □

Example

Suppose we know the function values of a tabulated function $V(x)$ for $x = 0, 1, \ldots, 5 (= B)$, as given in the table below, and we are interested in an estimate of $V(7)$.

x	0	1	2	3	4	5	6	7
$V(x)$	5	10	19	32	49	70	?	?

Suppose that we are not aware of nor interested in the exact relationship between x and $V(x)$, then

1. we can apply the direct approach:
   $$V(7) \approx 3 \cdot V(5) - 3 \cdot V(5-2) + V(5-4) = 3 \cdot 70 - 3 \cdot 32 + 10 = 124.$$

2. we may apply the indirect approach and we thus first compute the value of the coefficients a, b, and c:

   $$a = [V(5) - 2V(5-1) + V(5-2)]/2 = [70 - 2 \cdot 49 + 32]/2 = 2,$$
   $$b = [V(5) - V(5-2)]/2 - 2a(5-1) = [70-32]/2 - 8 \cdot 2 = 3, \text{ and}$$
   $$c = V(5) - 5^2 a - 5b = 70 - 25 \cdot 2 - 5 \cdot 3 = 5.$$

   Hence $V(y) \approx 2y^2 + 3y + 5$ and $V(7) \approx 2 \cdot 7^2 + 3 \cdot 7 + 5 = 124.$ the same results in more computations.

3. we can compute $V(7)$ recursively by using the direct approach for Δ fixed to 1 (independent of B):
   $$V(7) \approx 3 \cdot V(6) - 3 \cdot V(6-1) + V(6-2) = 3 \cdot V(6) - 3 \cdot 70 + 49,$$

   which requires the computation of $V(6)$:
   $$V(6) \approx 3 \cdot V(5) - 3 \cdot V(5-1) + V(5-2) = 3 \cdot 70 - 3 \cdot 49 + 32 = 95.$$

Evaluation

Since all given points are on the curve of the quadratic function $f(x) = 2x^2 + 3x + 5$, the results of the three quadratic extrapolations do coincide. In general this will only be the case for the last two extrapolations that are based on the last three known function values, while the points used for the first extrapolation depends on the overflow Δ.

Although the direct approach needs the value of Δ it seems to be a very efficient approach, since it does not require the computation of the coefficients a, b, and c. Unless Δ needs to be computed anyway, it becomes worth to compute the coefficients once and store them in memory when many extrapolations have to be executed.

A great advantage of the direct approach is that it can easily be extended to extrapolate a multi-dimensional tabulated function.

E.2 n-dimensional tabulated functions

Consider a tabulated function $V(\mathbf{x}) = V(x_1, \ldots, x_n)$ for which the (real) values are known on the n-dimensional $(0, B)$-grid. Hence $V(\mathbf{x})$ is known when all elements of vectors $\mathbf{x}$ are in $\{0, 1, \ldots, B\}$.

Through quadratic extrapolation one can estimate the function value $V(\mathbf{y})$ when $\mathbf{y}$ shows 'overflow' in one or more dimensions: when one or more elements exceed B. The extrapolation formula is displayed in the following equation:

$$V(\mathbf{y}) \approx 3 \cdot V(\mathbf{z}) - 3 \cdot V(\mathbf{z} - \mathbf{\Delta}) + V(\mathbf{z} - 2\mathbf{\Delta}) \tag{E.4}$$

where $\mathbf{z} = (\min\{B, y_1\}, \ldots, \min\{B, y_F\})$ and $\mathbf{\Delta} = (y_1 - z_1, \ldots, y_F - z_F)$.

When the function increases quadratically in each direction at which overflow might happen, than the approximation is exact.

Example

The following points all lay on the hyper curve of the quadratic form $(2x_1 + 3x_2)^2 + 7$.

x_1	0	0	0	1	1	1	2	2	2	3	1	3
x_2	0	1	2	0	1	2	0	1	2	1	3	3
$V(x_1, x_2)$	7	16	43	11	32	71	23	56	107	?	?	?

Suppose we are not aware of the exact form the curve through the points and we wish to estimate the last 3 missing value in the table below:

$V(3, 1)$, $V(1, 3)$ and $V(3, 3)$ can be estimated using Equation (E.4):

- $V(3, 1) \approx 3 \cdot V(2, 1) - 3 \cdot V(1, 1) + V(0, 1) = 88$
- $V(1, 3) \approx 3 \cdot V(1, 2) - 3 \cdot V(1, 1) + V(1, 0) = 128$
- $V(3, 3) \approx 3 \cdot V(2, 2) - 3 \cdot V(1, 1) + V(0, 0) = 232$

The estimates indeed correspond to the values set by the quadratic form.