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Domains of Existence for Finely Holomorphic Functions

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Abstract We show that fine domains in \mathbb{C} with the property that they are Euclidean F_σ and G_δ , are in fact fine domains of existence for finely holomorphic functions. Moreover *regular* fine domains are also fine domains of existence. Next we show that fine domains such as $\mathbb{C} \setminus \mathbb{Q}$ or $\mathbb{C} \setminus (\mathbb{Q} \times i\mathbb{Q})$, more specifically fine domains V with the properties that their complement contains a non-empty polar set E that is of the first Baire category in its Euclidean closure K and that $(K \setminus E) \subset V$, are *not* fine domains of existence.

Keywords Finely holomorphic function · Domain of existence

Mathematics Subject Classification (2010) 30G12 · 30A14 · 31C40

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1 Introduction

It was already known to Weierstrass that every domain Ω in \mathbb{C} is a *domain of existence*, roughly speaking, it admits a holomorphic function f that cannot be extended analytically at any boundary point of Ω . In his thesis [2] Borel showed, however, that it may be that f can be (uniquely) extended to a strictly larger set X in an "analytic" way, albeit that X is no longer Euclidean open. This eventually led Borel to the introduction of his Cauchy domains and monogenic functions, cf. [3]. Finely holomorphic functions on fine domains in \mathbb{C} as introduced by the first named author, are the natural extension and setting for Borel's ideas, see [11], also for some historic remarks on Borel's work. In this paper we will study *fine domains of existence*, roughly speaking, fine domains in \mathbb{C} that admit a finely holomorphic function that cannot be extended as a finely holomorphic function at any fine boundary point. Definition 3.1 contains a precise definition. Here the results are different from both the classical Weierstrass case of one variable and the classical several variable case, where Hartogs showed that there exist domains $D \subsetneq D^*$ in \mathbb{C}^n with the property that every holomorphic function on D extends to the larger domain D^* .

Our results are as follows. In Section 3 we show that every fine domain that is a Euclidean F_σ as well as a Euclidean G_δ is a fine domain of existence. Euclidean domains are of this form and therefore are fine domains of existence. We also show that *regular* fine domains are fine domains of existence. Noting that on Euclidean domains holomorphic and finely holomorphic functions are the same, our Theorem 3.9 includes Weierstrass' theorem. In Section 4, however, we show that fine domains V with the property that their complement contains a non-empty polar set E that is of the first Baire category in its Euclidean closure K (in particular, E has no Euclidean isolated points) and $(K \setminus E) \subset V$, are *not* fine domains of existence.

Pyrh  showed in [16] that the unit disc is a fine domain of existence, and that as a corollary of the proof, the same holds for simply connected Euclidean domains by the Riemann mapping theorem.

The starting point of our research was the following observation. Edlund [5] showed that every closed set $F \subset \mathbb{C}$ admits a continuous function $f : F \rightarrow \mathbb{C}$ such that the graph of f is completely pluripolar in \mathbb{C}^2 . It is easy to see that by construction this f is finely holomorphic on the fine interior of F . In [6] the main result can be phrased as follows: *If a finely holomorphic function on a fine domain D admits a finely holomorphic extension to a strictly larger fine domain D' , then the graph of f over D is not completely pluripolar.* Hence, if D is the fine interior of a Euclidean closed set, every fine component of D is a fine domain of existence. This is extended by our Theorem 3.9 because the fine interior of any finely closed set is regular.

In the next section we recall relevant results about the fine topology and fine holomorphy.

2 Preliminaries on the Fine Topology and Finely Holomorphic Functions

Recall that a set $E \subset \mathbb{C}$ is *thin* at $a \in \mathbb{C}$ if $a \notin \overline{E}$ or else if there exists a subharmonic function u on an open neighborhood of a such that

$$\limsup_{z \rightarrow a, z \in E \setminus \{a\}} u(z) < u(a).$$

The fine topology was introduced by H. Cartan in a letter to Brelot as the weakest topology that makes all subharmonic functions continuous. He pointed out that E is thin at a if and

only if a is not in the fine closure of E . The fine topology has the following known features. Finite sets are the only compact sets in the fine topology, which follows easily from the fact that every polar set $X \subset \mathbb{C}$ is discrete in the fine topology, it consists of finely isolated points. The fine topology is Hausdorff, completely regular, Baire, and quasi-Lindelöf, i.e. a union of finely open sets equals the union of a countable subfamily and a polar set, cf. e.g. [1, Lemma71.2]. For our purposes fine connectedness is important.

Recall the following

Theorem 2.1 (Fuglede) *The fine topology on \mathbb{C} is locally connected. That is, for any $a \in \mathbb{C}$ and any fine neighborhood U of a , there exists a finely connected finely open neighborhood V of a with $V \subset U$.*

See [8, p.92] (or see [7] for a proof using only elementary properties of subharmonic functions).

In fact, the fine topology is even *locally polygonally arcwise connected* as was shown in [8], but we need something stronger.

Definition 2.2 A *wedge* is a polygonal path consisting of two line segments $[a, w]$ and $[w, b]$ of equal length.

The result we need is as follows,

Theorem 2.3 ([9]) *Let U be a finely open set in \mathbb{C} and $\alpha > 1$. Then for every $w \in U$, there exists a fine neighborhood V of w such that any two distinct points $a, b \in V$ can be connected by a wedge contained in U of total length less than $\alpha|a - b|$.*

Lyons, see [15, p. 16] had already proven that a and b can be connected by a polygonal path consisting of two line segments of total length less than $\alpha|a - b|$ and his proof essentially contained Theorem 2.3. For an even stronger result see Gardiner, [13, Theorem A].

We will also need the following elementary lemma. Let $B(x, r)$ denote the open disc with radius r about $x \in \mathbb{C}$, $\bar{B}(x, r)$ its closure, and let $C(x, r)$ denote its boundary.

Lemma 2.4 *Let V be a fine neighborhood of $a \in \mathbb{C}$. Then there exists $C_0 > 1$ such that for $C > C_0$ and every $n \in \mathbb{N}$ there exists $t \in [C^{-n-1}, C^{-n}]$ with $C(a, t) \subset V$.*

Proof We can assume $a = 0$. A local basis at 0 for the fine topology consists of the sets $B(0, r, h) = \{z \in B(0, r) : h(z) > 0\}$, where $r > 0$, h is subharmonic on $B(0, r)$ and $h(0) = 1$. See [7, Lemma 3.1] for a proof. Thus for some h and r we have $V \supset W := \{z \in B(0, r) : h(z) \geq 1/2\}$ and $B(0, r) \setminus V$ is contained in $F := B(0, r) \setminus W$, which is Euclidean open, hence an F_σ , that is thin at 0, because W contains the finely open set $B(0, r, h)$ which contains 0. Theorem 5.4.2 in [17] states for given $r > 0$ that there is a constant $\Gamma > 0$ such that

$$\int_E \frac{1}{x} dx < \Gamma < \infty,$$

where

$$E := \{s : 0 < s < r \mid \exists \theta \text{ with } se^{i\theta} \in F\}.$$

Let $C_0 = \max\{e^\Gamma, 1/r\}$. For $C > C_0$ each interval of the form $[C^{-n-1}, C^{-n}]$ is contained in $(0, r)$, but can not be contained in E . Hence there exists $t \in [C^{-n-1}, C^{-n}]$ such that $C(0, t) \cap F = \emptyset$, that is, $C(0, t) \subset W \subset V$. □

For more information on the fine topology see [4, Part I, Chapter XI], [1, Chapter VII].

There are several equivalent definitions for finely holomorphic functions on a fine domain in $D \subset \mathbb{C}$, cf. [10–12, 15].

Definition 2.5 A function f on a fine domain D is called finely differentiable at $z_0 \in D$ with fine complex derivative $f'(z_0)$ if there exists $f'(z_0) \in \mathbb{C}$ such that for every $\varepsilon > 0$ there exists a fine neighborhood $V \subset D$ of z_0 such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{for all } z \in V.$$

In other words, the limit $(f(z) - f(z_0))/(z - z_0)$ exists as $z - z_0 \rightarrow 0$ finely.

The function f is finely holomorphic on D if and only if f is finely differentiable at every point of D and f' is finely continuous on D .

We will use the following characterizations of fine holomorphy.

Theorem 2.6 ([11]) *Let f be a complex valued function on a fine domain D . The following are equivalent*

- (1) *The function f is finely holomorphic on D .*
- (2) *Every point $z_0 \in D$ admits a fine neighborhood $V \subset D$ such that f is a uniform limit of rational functions on V .*
- (3) *The functions f and $z \mapsto zf(z)$ are both (complex valued) finely harmonic functions on D .*

In the following theorem we collect properties of finely holomorphic functions that indicate how much this theory resembles classical function theory.

Theorem 2.7 ([10]) *Let $f : D \rightarrow \mathbb{C}$ be finely holomorphic. Then*

- (1) *The function f has fine derivatives $f^{(k)}$ of all orders k , and these are finely holomorphic on D .*
- (2) *Every point $z_0 \in D$ admits a fine neighborhood $V \subset D$ such that for every $m = 0, 1, 2, \dots$*

$$\left| f(w) - \sum_{k=0}^{m-1} \frac{f^{(k)}(z)}{k!} (w - z)^k \right| / |w - z|^m \tag{2.1}$$

is bounded on $V \times V$ for $z \neq w$.

- (3) *At any point z of D the Taylor expansion (2.1) uniquely determines f on D . (If all coefficients equal 0, then f is identically 0.)*

We now introduce finely isolated singularities.

Definition 2.8 Let D be a fine domain, $a \in D$, and f finely holomorphic on $D \setminus \{a\}$.

- If f extends as a finely holomorphic function to all of D , then f has a removable singularity at a .

- If $f(z) \rightarrow \infty$ for $z \rightarrow a$ finely, f has a pole at a .
- If f has no pole at a nor a removable singularity, then f has an essential singularity at a .

Theorem 2.9 *Let D be a fine domain and $a \in D$. Let f be finely holomorphic on $D \setminus \{a\}$ and suppose that f is bounded on a fine neighborhood V of a . Then a is a removable singularity of f .*

Proof (As indicated in [12].) We apply Theorem 2.6, no. 3. Clearly f and $z \mapsto zf(z)$ are bounded, finely harmonic functions on $V \setminus \{a\}$. [8, Corollary 9.15] states that these functions extend to be finely harmonic on all of D and again by Theorem 2.6 f is finely holomorphic on D . □

3 Fine Domains of Existence

For a set A in \mathbb{C} , we denote its fine interior by $\text{int}_f A$, its fine closure by $\text{cl}_f A$ and its fine boundary by $\partial_f A$.

Definition 3.1 A fine domain U is called a *fine domain of existence* if there exists a finely holomorphic function f on U with the property that for every fine domain V that intersects $\partial_f U$ and for every fine component Ω of $V \cap U$, the restriction $f|_\Omega$ admits no finely holomorphic extension to V .

Definition 3.2 Let U be a fine domain in \mathbb{C} . If there exist compact sets $K_n \subset U$ such that $U = \bigcup_{n=1}^\infty K_n$ with $K_1 \subset \text{int}_f K_2 \subset K_2 \subset \text{int}_f K_3 \subset \dots$, then the sequence $\{K_n\}$ is called a *fine exhaustion* of U . If, moreover, the K_n have the property that every bounded component of $\mathbb{C} \setminus K_n$ contains a point from $\mathbb{C} \setminus U$, we call $\{K_n\}$ a *special fine exhaustion* of U .

We have the following lemma, which is a consequence of the Lusin–Menchov property of the fine topology.

Lemma 3.3 ([14, Corollary 13.92]) *Let U be a finely open set and let K be a compact subset of U . Then there exists a Borel finely open set V such that $K \subset V \subset \overline{V} \subset U$.*

Proposition 3.4 *Let U be a fine domain that is a Euclidean F_σ . Then U admits a special fine exhaustion.*

Proof Let $U = \bigcup_{n=1}^\infty F_n$ for compact sets $F_1 \subset F_2 \subset \dots$. Fix a strictly increasing sequence $(r_n)_{n \geq 1}$ tending to infinity such that $F_n \subset B(0, r_n)$ for all $n \geq 1$. We put $K_1 := F_1$. For the construction of K_2 , note that $F_2 \cup K_1 (= F_2)$ is a compact subset of U . By Lemma 3.3, there exists a finely open set V_2 such that $F_2 \cup K_1 \subset V_2 \subset \overline{V_2} \subset U$. Then the set $K_2 := \overline{V_2} \cap \overline{B(0, r_2)}$ is compact and

$$\text{int}_f K_2 = \text{int}_f \overline{V_2} \cap \text{int}_f \overline{B(0, r_2)} \supset V_2 \cap B(0, r_2) \supset F_2 \supset F_1 = K_1.$$

As induction hypothesis, suppose that for some $n \geq 2$, we have found compact sets K_1, \dots, K_n such that $K_j \subset \text{int}_f K_{j+1}$ for all $1 \leq j \leq n - 1$, $F_j \subset K_j$ for all $1 \leq j \leq n$

and $K_j \subset B(0, r_{j+1})$ for all $1 \leq j \leq n$ and that we have found finely open sets V_2, \dots, V_n such that $F_j \cup K_{j-1} \subset V_j \subset \overline{V_j} \subset U$ for all $2 \leq j \leq n$.

We now prove the induction step. Note that the set $F_{n+1} \cup K_n$ is a compact subset of U . By the previous lemma, there exists a finely open set V_{n+1} such that $F_{n+1} \cup K_n \subset V_{n+1} \subset \overline{V_{n+1}} \subset U$. The set $K_{n+1} := \overline{V_{n+1}} \cap \overline{B(0, r_{n+1})}$ is compact, contained in U and in $B(0, r_{n+2})$ and

$$\text{int}_f K_{n+1} = \text{int}_f \overline{V_{n+1}} \cap \text{int}_f \overline{B(0, r_{n+1})} \supset V_{n+1} \cap B(0, r_{n+1}) \supset F_{n+1} \cup K_n$$

and therefore $\text{int}_f K_{n+1} \supset K_n$ and $K_{n+1} \supset F_{n+1}$. This proves the induction step.

Consequently, we can find compact subsets K_1, K_2, \dots of U such that $K_1 \subset \text{int}_f K_2 \subset K_2 \subset \text{int}_f K_3 \subset \dots$ and $F_n \subset K_n$ for all $n \geq 1$. It follows that $U = \bigcup_{n=1}^\infty K_n$, which proves that U admits a fine exhaustion.

We will next adapt this fine exhaustion so that every bounded component of $\mathbb{C} \setminus K_n$ contains a point from $\mathbb{C} \setminus U$. Observe that $K_n = \mathbb{C} \setminus \bigcup_j D_j^n$, where for each n the D_j^n are the countably, possibly infinitely, many mutually disjoint open components of $\mathbb{C} \setminus K_n$. Let D_0^n be the unbounded component. Then for any n and any finite or infinite sequence $1 \leq j_1 \leq j_2 \leq \dots$ the set $K_n \cup \bigcup_k D_{j_k}^n$ is compact too. For every n we set $K_n^* = K_n \cup \bigcup_k D_{j_k}^n$, where the $D_{j_k}^n$ are those components, if any, of $\mathbb{C} \setminus K_n$ that are completely contained in U . We claim that $K_n^* \subset \text{int}_f(K_{n+1}^*)$.

Indeed, if $x \in K_n$ then $x \in \text{int}_f K_{n+1} \subset \text{int}_f(K_{n+1}^*)$. Now let $x \in D_{j_k}^n (\subset U)$. For the proof that $x \in K_{n+1}^*$ we may suppose that $x \notin K_{n+1}$. Then x belongs to a (necessarily bounded) component of $\mathbb{C} \setminus K_{n+1}$ that is completely contained in $D_{j_k}^n \subset U$, hence $x \in K_{n+1}^*$. It follows that $D_{j_k}^n \subset K_{n+1}^*$ and as $D_{j_k}^n$ is Euclidean open, $D_{j_k}^n \subset \text{int}_f(K_{n+1}^*)$. This proves the claim. □

We will need the following lemma. Fig. 1 illustrates its content.

Lemma 3.5 *Let D be an open disc, U a finely open subset of D , and let $a \in C \cap U$ where C is an open circular arc contained in D with the property that $D \setminus C$ has two components.*

Then there exists a sequence $(r_j)_j$ of positive numbers decreasing to 0, such that for every r_j there exists a compact set $C' (= C'_j) \subset U$, which is the union of four arcs, such that

- (1) $D \setminus C'$ is connected;
- (2) $C' \setminus \overline{B(a, r_j)} = C \setminus \overline{B(a, r_j)}$;
- (3) Every wedge $\ell = [p, b] \cup [b, q]$ with $|p - b| = |b - q| > r_j$ that meets C also meets C' .

Proof By Theorem 2.3 there exists a fine neighborhood $V \subset U$ of a such that any two distinct points $p, q \in V$ can be connected by a wedge $L \subset U$ of length less than $\sqrt{2}|p - q|$. Lemma 2.4 provides us with a sequence $(r_j)_j$ such that $C(a, r_j) \subset V$.

After scaling and rotating we may assume that $C = C(-i, 1) \cap D$ and that $a = 0$. We can also assume all r_j less than 1 and so small that $B(0, r_j) \subset D$. Now we fix $r = r_j$ and set

$$C_1 = (C \setminus (B(0, r) \cap \{\text{Re } z < 0\})) \cup \left(\{r e^{i\theta} : \pi/6 \leq \theta \leq 11\pi/6\} \right).$$

C_1 is contained in V along with $C(0, r)$, therefore ir , respectively $-ir$, (both on C_1) can be connected to $0 (= a) \in V \subset U$ by a wedge W_1 , respectively W_2 , of length less than $\sqrt{2}r$ contained inside U . Observe that the wedges W_1, W_2 are contained in $\overline{B(0, r)}$, and in fact W_1 is contained in the closed square S_1 with diagonal $[0, ir]$ and similarly W_2 is contained in the closed square S_2 with diagonal $[0, -ir]$, see Fig. 1.

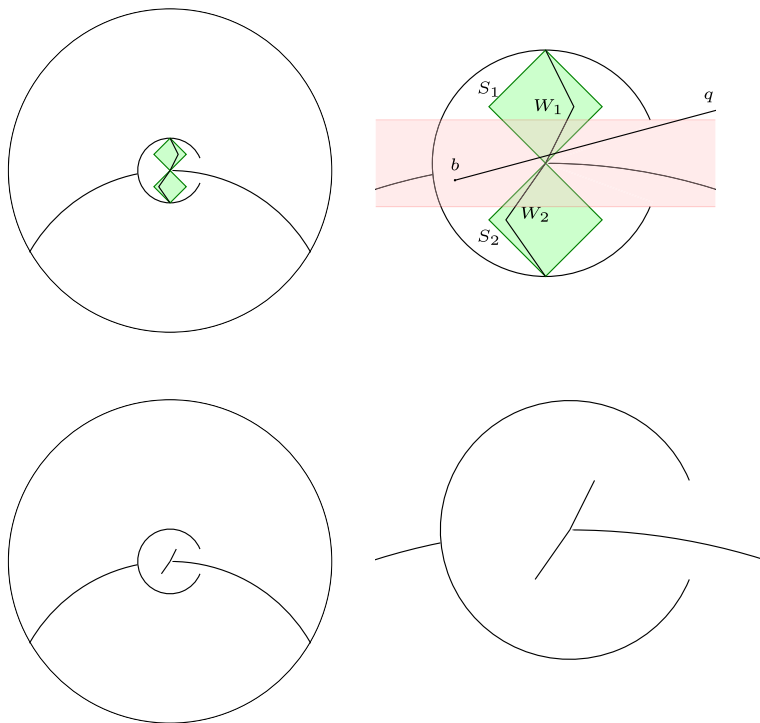


Fig. 1 Replacing a small circular arc

Let $\ell = [p, b] \cup [b, q]$ be any wedge with $|p - b| = |b - q| > 2r$ that meets C and assume that ℓ does not meet C_1 . Then one of its constituting segments, say $[b, q]$, meets $C \cap B(0, r) \cap \{\operatorname{Re} z < 0\}$ and because $|q - b| > 2r$, it meets $C(0, r)$ in a point $re^{i\theta_0}$ with $-\pi/6 < \theta_0 < \pi/6$. Therefore, as $r < 1$, $[b, q] \cap (S_1 \cup S_2)$ is contained in $\{z : |\operatorname{Im} z| < r \sin(\pi/6) = r/2\} \cap (S_1 \cup S_2)$, as indicated in Fig. 1.

Now we define $L_j = W_j \cap \{z : |\operatorname{Im} z| \leq r/2\} \subset U$, $j = 1, 2$ and $C' = C_1 \cup L_1 \cup L_2 \subset U$ is the the union of three circular arcs and one arc consisting of two straight segments, all contained in U . Clearly ℓ meets C' and $D \setminus C'$ is connected. □

Proposition 3.6 *Let $U \subsetneq \mathbb{C}$ be a nonempty fine domain that is a Euclidean F_σ and let $\emptyset \neq F_1 \subset F_2 \subset \dots$ be an increasing sequence of compact sets in $\mathbb{C} \setminus U$. Then there exist sequences $(K_n)_{n \geq 1}$ and $(L_n)_{n \geq 1}$ of nonempty compact sets such that*

- (a) $K_1 \subset \operatorname{int}_f K_2 \subset K_2 \subset \operatorname{int}_f K_3 \subset \dots$ and $\bigcup_{n=1}^\infty K_n = U$,
- (b) $d(K_n, L_n) > 0$ for all $n \geq 1$,
- (c) $L_n \subset K_{n+1}$ for all $n \geq 1$,
- (d) for any $n \geq 1$, every two points $w, z \in \mathbb{C} \setminus (K_n \cup L_n)$ in the same component of $\mathbb{C} \setminus K_n$ also lie in the same component of $\mathbb{C} \setminus (K_n \cup L_n)$,
- (e) every bounded component of $\mathbb{C} \setminus (K_n \cup L_n)$ contains a point from $\mathbb{C} \setminus U$,
- (f) every wedge of length at least $1/n$ that meets F_n and K_n , also meets L_n .

Moreover, there exists a sequence $(f_\nu)_{\nu \geq 1}$ of rational functions with poles in $\mathbb{C} \setminus U$ such that

- (1) the sup-norm $\|f_n - f_{n-1}\|_{K_n} < 1/2^n$ for all $n \geq 2$ and
- (2) $|f_n| > n$ on L_n for all $n \geq 1$.

The sequence $(f_v)_v$ converges uniformly on any K_n to a function f that is finely holomorphic on U and has the property that $|f| > n - 1/2^n$ on L_n for any $n \geq 1$.

Proof For (K_n) we take a special fine exhaustion of U , $U = \bigcup_{n=1}^\infty K_n$, which exists in view of Proposition 3.4. We next construct a sequence $(L_n)_{n \geq 1}$ of compact sets such that $d(L_n, K_n) > 0$ and $L_n \subset K_{n+1}$ for all $n \geq 1$.

Let $n \geq 1$. The set $F_n \subset \mathbb{C} \setminus U$ is compact and is therefore covered (uniquely) by those finitely many components of $\mathbb{C} \setminus K_n$ that meet F_n . We denote these components by $D_1^n, D_2^n, \dots, D_{k_n}^n$. Since K_n and F_n are compact and $K_n \cap F_n = \emptyset$, it follows that $\delta_n := d(K_n, F_n) > 0$. Consequently,

$$\{z \in \mathbb{C} : d(z, K_n) < \delta_n\} \cap F_n = \emptyset. \tag{3.1}$$

Since $K_n \subset \text{int}_f K_{n+1}$, by Lemma 2.4 each $z \in \partial K_n$ is the center of a circle $C(z, \varepsilon_z)$ contained in $\text{int}_f K_{n+1}$ of radius ε_z less than δ_n . We will now construct L_n by constructing the intersections $L_n \cap D_k^n$ (with some abuse of notation) in each of the components $D_1^n, D_2^n, \dots, D_{k_n}^n$ and taking the union of these sets.

Fix a component D_k^n . Since $\partial D_k^n \subset \partial K_n$ is compact and is covered by the open disks $B(z, \varepsilon_z)$ where $z \in \partial D_k^n$, it follows that there are finitely many points $z_1, \dots, z_m \in \partial D_k^n$ such that $\partial D_k^n \subset \bigcup_{j=1}^m B(z_j, \varepsilon_{z_j})$. Note that $(\partial \bigcup_{j=1}^m B(z_j, \varepsilon_{z_j})) \cap D_k^n$ is a finite union of closed circular arcs \mathcal{C}_l –and possibly finitely many points disjoint from these arcs– of the components of $\bigcup_{j=1}^m \partial B(z_j, \varepsilon_{z_j}) \setminus \bigcup_{k=1}^m B(z_k, \varepsilon_{z_k})$. We define a compact set $L_{n,k}^* := \bigcup_{l=1}^M \mathcal{C}_l \subset D_k^n$. Also note that every path γ , in particular every wedge, that connects a point $z \in F_n \cap D_k^n$ and a point $w \in K_n$ must intersect one of the arcs \mathcal{C}_l in $L_{n,k}^*$, because $d(z, K_n) \geq \delta_n > \varepsilon_z$ in view of Eq. 3.1, and $w \in K_n \subset \mathbb{C} \setminus D_k^n$. The open set $\mathbb{C} \setminus L_{n,k}^*$ has only finitely many, say $N \geq 2$, components as each of the \mathcal{C}_l belongs to the boundary of two of these components. We will replace $L_{n,k}^*$ by $L_n \cap D_k^n$ in the following way to achieve that $\mathbb{C} \setminus (L_n \cap D_k^n)$ is connected and the properties (a), (b) and (c) are kept.

To do this, note that \mathcal{C}_1 belongs to the boundary of two components of $\mathbb{C} \setminus L_{n,k}^*$. Pick a point a of \mathcal{C}_1 that is not an endpoint of this closed arc. Since $a \in \text{int}_f K_{n+1} \setminus K_n$, clearly $a \notin F_{n+1} \supset F_n$, and again since a lies in the relative interior of the arc \mathcal{C}_1 , we see that for all $r > 0$ small enough, $\overline{B(a, r)} \cap K_n = \emptyset = \overline{B(a, r)} \cap F_n$ and $\overline{B(a, r)} \cap L_{n,k}^*$ is contained in the relative interior of \mathcal{C}_1 .

We apply Lemma 3.5 with $D = B(a, r)$, $U = \text{int}_f K_{n+1} \cap B(a, r)$, $r_j < 1/n$ and obtain a compact \mathcal{C}'_1 . Then $\mathbb{C} \setminus (\mathcal{C}'_1 \cup \bigcup_{l=2}^M \mathcal{C}_l)$ has $N - 1$ components. If \mathcal{C}_2 belongs to the boundary of two different components of $\mathbb{C} \setminus (\mathcal{C}'_1 \cup \bigcup_{l=2}^M \mathcal{C}_l)$ we replace it by \mathcal{C}'_2 likewise, reducing the number of components of the complement by one again. If \mathcal{C}_2 belongs to the boundary of only one component, we just put $\mathcal{C}'_2 = \mathcal{C}_2$. Proceeding in this way, we end with $L_n \cap D_k^n := \bigcup_{l=1}^M \mathcal{C}'_l$, the complement of which is connected. By setting $L_n := \bigcup_{k=1}^{k_n} (L_n \cap D_k^n)$, we now find that any two points $w, z \in \mathbb{C} \setminus (K_n \cup L_n)$ that are in the same component of $\mathbb{C} \setminus K_n$ are in the same component of $\mathbb{C} \setminus (K_n \cup L_n)$ as well, while Lemma 3.5 guarantees that (f) is satisfied. Also, note that $L_n \subset \text{int}_f K_{n+1} \setminus K_n$, so that $K_n \cap L_n = \emptyset$. It follows that $d(K_n, L_n) > 0$ for every $n \geq 1$. We have obtained sequences $(K_n)_{n \geq 1}$ and $(L_n)_{n \geq 1}$ of compact sets that satisfy the properties (a) – (f).

Because of property (b), we can apply Runge’s theorem recursively for each n to the holomorphic function g_n that equals 0 on an open set containing K_n and is equal to

$\sum_{j=1}^{n-1} \max_{z \in L_n} \{|R_j(z)|\} + n + 1$ on an open set containing L_n . Thus there exists for each n a rational function R_n with

$$|R_n| < 1/2^n, \quad \text{on } K_n; \tag{3.2}$$

$$|R_n| > \sum_{j=1}^{n-1} \max_{z \in L_n} \{|R_j(z)|\} + n, \quad \text{on } L_n, \tag{3.3}$$

while R_n has at most one pole in a preassigned point in each of the bounded components of $\mathbb{C} \setminus (K_n \cup L_n)$. Because of property (e) these poles may be taken from $\mathbb{C} \setminus U$. Set

$$f_k = \sum_{j=1}^k R_j.$$

Then f_k clearly satisfies (1) and (2). It now follows from (1) and (a) that there exists a function f on U such that $f_k \rightarrow f$ uniformly on any K_n . Note that f is finely holomorphic on U , because if $z \in U$, then $z \in K_n$ for some $n \geq 1$ and, by property (a), K_{n+1} is a compact fine neighborhood of z on which $f_k \rightarrow f$ uniformly. Furthermore, it follows from (2) and (c) above that for any $n \geq 1$ and $z \in L_n$

$$|f_n(z)| \geq |R_n(z)| - \sum_{j=1}^{n-1} |R_j(z)| > n.$$

Finally, $|f| > n - 1/2^n$ follows immediately from (1) and (2). □

We shall prove in Theorem 3.9 that under suitable conditions on the fine domain, the function that we have just constructed admits no finely holomorphic extension outside the domain. We need some facts about (regular) fine domains.

It is known that every regular fine domain U is an F_σ because $\mathbb{C} \setminus U$ is a base in the sense of Brelot, and hence a G_δ . Moreover, we need the following Lemma, which is (iv) of [1, Theorem 7.3.11].

Lemma 3.7 *Every finely closed set $A \subset \mathbb{C}$ is the disjoint union of a (Euclidean) F_σ and a polar set.*

Lemma 3.8 *Let $\emptyset \neq U \subset V$ be fine domains and let E be a polar subset of $\mathbb{C} \setminus U$. Then either there exists a point $a \in (\partial_f U \cap V) \setminus E$, or $V = U \cup e$ for the polar set $e := (V \setminus U) \cap E$, whereby e is empty and hence $U = V$ in case U is a regular fine domain.*

Proof If $(\partial_f U \cap V) \setminus E$ is empty then $(\partial_f U \cap V) \subset E \cap (V \setminus U) = e$, hence the fine boundary $\partial_f U \cap (V \setminus e)$ of U relative to $V \setminus e$ is empty. Since $V \setminus e$ is finely open and finely connected along with V , by [8, Theorem 12.2], and since $U \neq \emptyset$ this means that $U = V \setminus e$, that is, $V = U \cup e$ as claimed. If U is regular then e is empty because e and U are disjoint and V is finely open. □

Theorem 3.9 *Let $U \subset \mathbb{C}$ be a non-empty fine domain that is either (1) both Euclidean F_σ and Euclidean G_δ , or (2) a regular fine domain. Then U is a fine domain of existence.*

Proof In both cases U is an F_σ , and we can write $U = \bigcup_{n=1}^\infty K_n$ with $K_j \subset K_{j+1}$ compact subsets of U . Because of Lemma 3.7, $\mathbb{C} \setminus U = F \cup E$, where $F = \bigcup_{n=1}^\infty F_n$ with $F_j \subset F_{j+1}$

compact in $\mathbb{C} \setminus U$, is a Euclidean F_σ and E is a polar set which we can assume to be empty in Case (1). Let $L_j \subset K_{j+1}$ be compact sets and f a finely holomorphic function on U as constructed in Proposition 3.6.

Let V be a fine domain that meets $\partial_f U$ and let Ω be a fine component of $U \cap V$. Then $\partial_f \Omega \cap V \subset \partial_f U \cap V$, because if $x \in \partial_f \Omega \cap V \cap U$ there would be a finely connected fine neighborhood of x contained in $V \cap U$, contradicting that Ω is a fine component of $V \cap U$. Suppose that $f|_\Omega$ admits a finely holomorphic extension \tilde{f} defined on V . By Lemma 3.8 applied to Ω in place of U , there is a point $z_0 \in (\partial_f \Omega \cap V) \setminus E$, whereby in Case (1) $E = \emptyset$ and hence $z_0 \in (\partial_f \Omega \cap V) \subset \partial_f U \cap V \subset F$. In Case (2), if $V = \Omega \cup e$, then $U \cup V = U \cup e$, which is impossible because U is a regular domain. Hence $z_0 \in (\partial_f \Omega \cap V) \setminus E = (\partial_f U \cap V) \setminus E \subset F$.

Again by Proposition 3.6 there is a compact fine neighborhood V_0 of z_0 in V on which \tilde{f} is the uniform limit of rational functions with poles off V_0 . In particular, \tilde{f} is continuous and bounded on V_0 , say $|\tilde{f}| \leq M$ on V_0 . Choose a fine domain $V_1 \subset V_0$ containing z_0 and such that, for given $\alpha > 1$, any two distinct points a, b of V_1 can be joined by a wedge of total length less than $\alpha|a - b|$ contained in V_0 , see Theorem 2.3.

Choose $z_1 \in \Omega \setminus \{z_0\}$. By the above and by Lemma 2.4, there exists for every $\varepsilon > 0$ an $0 < r_0 < \varepsilon$ with $r_0 < |z_1 - z_0|$ such that $C(z_0, r_0) \subset V_1$. Choose $z_2 \in \Omega$ with $|z_2 - z_0| < r_0$ (possible since $z_0 \in \partial_f \Omega \subset \partial \Omega$). There is a path $\gamma \subset \Omega$ joining z_1 and z_2 and meeting $C(z_0, r_0)$ at a point w_0 of $\Omega \cap V_1$. Thus $C(z_0, r_0) \cap \Omega \neq \emptyset$. Recall that also $z_0 \in V_1$. Let $l_0 = [w_0, b_0] \cup [b_0, z_0]$ with $|w_0 - b_0| = |b_0 - z_0|$ be a wedge of total length less than $\alpha|w_0 - z_0|$ contained in V_0 . Since $w_0 \in U$, we have $w_0 \in K_n$ for all n large enough. Also, $z_0 \in F$, hence $z_0 \in F_n$ for all n large enough. Therefore we can take an $n \geq 1$ large enough such that $w_0 \in K_n$ and $z_0 \in F_n$ and such that $n - 1/2^{n-1} > M$ and $4/n < |w_0 - b_0| + |b_0 - z_0|$ hold simultaneously. Then by Proposition 3.6 (f), l_0 meets $L_n \cap \Omega$.

Let $z \in l_0 \cap L_n \cap \Omega$. Since $z \in l_0 \cap \Omega \subset V_0$, we have $|\tilde{f}(z)| \leq M$. On the other hand, since $z \in L_n \cap \Omega$, we have $\tilde{f}(z) = f(z)$ and by (2) in Proposition 3.6 $|\tilde{f}(z)| = |f(z)| \geq n - 1/2^{n-1} > M$, which is a contradiction. We conclude that f does not admit a finely holomorphic extension and that U is a fine domain of existence. \square

In the rest of this section we extend the above theorem a little.

Lemma 3.10 *Let U_1 and U_2 be fine domains of existence, and suppose that $(\partial_f U_1) \cap (\partial_f U_2) = \emptyset$. Then every fine component of $U_1 \cap U_2$ is a fine domain of existence.*

Proof The assertion amounts to $U_1 \cap U_2$ being a “finely open set of existence” (if nonvoid) in the obvious sense. Because $(\partial_f U_1) \cap (\partial_f U_2) = \emptyset$ we have, writing $U_1 \cap U_2 = U_0$,

$$\partial_f U_0 = (U_1 \cap \partial_f U_2) \cup (U_2 \cap \partial_f U_1).$$

By hypothesis there exists for $i = 1, 2$ a finely holomorphic function h_i on U_i such that for any fine domain V that intersects $\partial_f U_i$ and any fine component Ω_i of $U_i \cap V$, $h_i|_{\Omega_i}$ does not extend finely holomorphically to V .

Let V be any fine domain that intersects $\partial_f U_0$ and let Ω be a fine component of $V \cap U_0$. Without loss of generality we may assume that V intersects $U_2 \cap \partial_f U_1$, and by shrinking V that $V \subset U_2$. Then Ω is a fine component of $U_1 \cap V = U_1 \cap (U_2 \cap V) = U_0 \cap V$.

The function $h_0 := h_1|_{U_0} + h_2|_{U_0}$ is then finely holomorphic on U_0 . Because h_2 is finely holomorphic on $V \subset U_2$, the function $h_0|_\Omega$ is extendible to V if and only if $h_1|_\Omega$ is extendible over V , and that is not the case. Therefore U_0 is a finely open set of existence. \square

Theorem 3.11 *Every fine domain $U \subset \mathbb{C}$ such that the set I of irregular fine boundary points for U is both an F_σ and a G_δ is a fine domain of existence.*

Proof In the above lemma take $U_1 = U_r$ (the regularization of U) and $U_2 = \mathbb{C} \setminus I$. Since I is polar, U_2 is a fine domain along with \mathbb{C} . Since U is finely connected so is U_r . In fact, $U_r = U \cup E$, where E denotes the polar set of finely isolated points of $\mathbb{C} \setminus U$. If $U_r = V_1 \cup V_2$ with V_1, V_2 finely open and disjoint then

$$U = U_r \setminus E = (V_1 \setminus E) \cup (V_2 \setminus E)$$

with $V_1 \setminus E$ and $V_2 \setminus E$ finely open and disjoint. It follows that for example $V_1 \setminus E = \emptyset$ and hence $V_1 = \emptyset$, showing that U_r indeed is finely connected. By hypothesis, the complement of U_2 and hence U_2 itself is an F_σ and a G_δ . Then by Theorem 3.9 U_1 and U_2 are fine domains of existence. By Theorem 3.9, $U_1 \cap U_2 = U_r \setminus I = U$ is indeed a fine domain of existence. \square

4 Fine Domains that are not Domains of Existence

Proposition 4.1 *Let (a_n) be a sequence in \mathbb{C} that converges to a . Suppose that V_n is a fine neighborhood of a_n of the form*

$$V_n = V_n(a_n, h_n, r) = \{z \in B(a_n, r) : h_n(z) > 0\},$$

where h_n is a subharmonic function on $B(a_n, r)$ such that $h_n(a_n) = 1/2$ and $h_n < 1$ on $B(a_n, r)$. Then $\bigcup_{n=1}^\infty V_n$ is a (possibly deleted) fine neighborhood of a .

Proof Let $r_1 < r$. Then there exists $n_0 > 0$ such that for $n \geq n_0$ the function h_n is defined on $B(a, r_1)$. Let

$$h = (\sup_{n \geq n_0} \{h_n|_{B(a, r_1)} : n \geq n_0\}),$$

on $B(a, r_1)$ and let h^* denote its upper semi-continuous regularization. Then h^* is subharmonic, $h^* \leq 1$, and $h^*(a) \geq \limsup_{n \rightarrow \infty} h(a_n) \geq 1/2$. Let $V_0 = \{z \in B(a, r_1) : h^*(z) > 0\}$. Then V_0 is a fine neighborhood of a , since h^* is finely continuous. Because the set $X = \{h < h^*\} \setminus \{a\}$ is polar, it is finely closed, therefore the set $V = V_0 \setminus X$ is a fine neighborhood of a .

We claim that $V \setminus \{a\} \subset \bigcup_{n \geq n_0} V_n$. Indeed, if $z \in V \setminus \{a\}$ then $h(z) = h^*(z) > 0$ hence $h_n(z) > 0$ for some $n \geq n_0$, and $z \in V_n$. \square

Proposition 4.2 *Let E be a non-empty polar set in \mathbb{C} and suppose that E is of the first Baire category in its Euclidean closure K . Suppose that f is finely holomorphic on a finely open set V such that $K \setminus E \subset V$. Then there exist a Euclidean open ball B that meets K , and a finely open fine neighborhood V_1 of $K \cap B$ such that f is bounded on $V_1 \setminus E$.*

Proof Let $x \in V \setminus E$. Denote by U_x a finely open subset of V containing x and having the property stated in Theorem 2.3 applied to V . By shrinking we may arrange that U_x has the form

$$U_x = U_x(x, h_x, r_x) = \{z \in B(x, r_x) : h_x(z) > 0\},$$

where h_x is a subharmonic function on $B(x, r_x)$ such that $h_x(x) = 1/2$ and $h_x < 1$ on $B(x, r_x)$.

Let X_j be the set of $x \in K \setminus E$ such that $r_x > 1/j$, and $|f| \leq j$ on U_x and that U_x satisfies Lemma 2.4 with $C_0 = j$. Then $K \setminus E = \bigcup_j X_j$, hence by the Baire category theorem, there exist j_0 and an open Euclidean ball which we may assume to be the unit disc \mathbb{D} , such that $\emptyset \neq K \cap \mathbb{D} \subset \overline{X_{j_0}}$ (the Euclidean closure of X_{j_0}).

Let $w \in K \cap \mathbb{D}$. Then $w = \lim_{n \rightarrow \infty} x_n$ for a sequence $(x_n)_n$ in X_{j_0} . As $r_{x_n} > 1/j_0$, Proposition 4.1 gives us that $\bigcup_n U_{x_n}$ is a (possibly deleted) fine neighborhood of w on which $|f| \leq j_0$ and $W_w := \{w\} \cup \bigcup_n U_{x_n}$ is a fine neighborhood of w . If $w \in (K \setminus E) \cap \mathbb{D}$, $|f| \leq j_0$ on W_w . In fact, f is finely holomorphic on V , hence $|f|$ is finely continuous on the finely open set $V \cap W_w$ which contains w because $V \supset K \setminus E$. If $|f(w)| > j_0$ then $|f| > j_0$ on some fine neighborhood Z of w , which contradicts that Z meets $W_w \setminus \{w\}$ because \mathbb{C} has no finely isolated points.

The set $V_1 = \bigcup_{w \in K \cap \mathbb{D}} W_w$ is a finely open fine neighborhood of $K \cap \mathbb{D}$ with the property that $|f| \leq j_0$ on $V_1 \setminus E$. □

Remark 4.3 The reader should be aware that the condition E is of the first Baire category in its Euclidean closure K prevents sets like $E = \{1/n, n = 1, 2, \dots\}$. Indeed, E can not be written as countable union of nowhere dense subsets in its Euclidean closure $K = E \cup \{0\}$, because this set E is relatively open in K .

Theorem 4.4 *Suppose that E and K are as in Proposition 4.2, and that V is a fine domain such that $V \cap E = \emptyset$ and $K \setminus E \subset V$. Then (1) $E \subset \partial_f V$ and (2) V is not a fine domain of existence.*

Proof For (1) observe that $\partial_f V = \partial V$, because V is connected and therefore not thin at any of its Euclidean boundary points. Thus it suffices to show that $E \subset \partial V$. We have $E = \bigcup_{n=1}^\infty F_n$, a countable union of nowhere dense subsets of K , which is of the second category in itself. Therefore, E is contained in the closure of $K \setminus E$. If not, suppose to reach a contradiction, that O were an open set in K not meeting $K \setminus E$, then $O = \bigcup_{n=1}^\infty (F_n \cap O)$, a countable union of nowhere dense sets, contradicting that K is of the second category. As $K \setminus E$ is contained in the domain V , we find that E must be contained in the Euclidean closure of V and hence in $\partial_f V = \partial V$ because $V \cap E = \emptyset$.

For (2) let f be finely holomorphic on V . By Proposition 4.2 there exist an open ball B that meets K (and hence E) and a finely open set V_1 containing $K \cap B$ such that f is bounded on $V_1 \cap V \cap B \subset V_1 \setminus E$. Because E is polar, $V_1 \cap V$ is a deleted finely open fine neighborhood of every point in $E \cap B$, hence each point a of $E \cap B$ is a finely isolated singularity of f . Theorem 2.9 implies that f extends over each $a \in E \cap B$ and hence has a finely holomorphic continuation to the finely open set $V \cup [(V \cup E) \cap B] = V \cup (E \cap B)$, which contains V properly because $(E \cap B) \subset \partial_f V$. □

Example 4.5 Let $V = \mathbb{C} \setminus \mathbb{Q}$, $E = \mathbb{Q}$, and $K = \mathbb{R}$. Then V , E , and K satisfy the conditions of Theorem 4.4, therefore $\mathbb{C} \setminus \mathbb{Q}$ is not a domain of existence for finely holomorphic functions.

Similarly $\mathbb{C} \setminus (\mathbb{Q} \oplus i\mathbb{Q})$ is not a domain of existence for finely holomorphic functions.

Question 4.6 From Theorem 4.4 and Example 4.5 it is clear that the requirement that U be an F_σ in Theorem 3.9 can not be dropped. But is it perhaps true that every fine domain that is a Euclidean F_σ is a fine domain of existence? It would suffice to prove this for $U = \mathbb{C} \setminus I$ with I a polar G_δ , see the proof of Theorem 3.11.

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