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The Many Faces of Interpolation

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Abstract We present a number of, somewhat unusual, ways of describing what Craig's interpolation theorem achieves, and use them to identify some open problems and further directions.

Keywords Interpolation, definability, cross-model relations, first-order fragments, meta-theory.

1 Introduction

Important results have many things to say. At first sight, the Interpolation Theorem of Craig 1957 seems a rather technical result for connoisseurs inside logical meta-theory. But over the past decades, its broader importance has become clear from many angles. In this paper, I discuss my own current favourite views of interpolation: no attempt is made at being fair or representative. First, I discuss the *entanglement of inference and vocabulary* that is crucial to interpolation. Next, I move to the role of interpolants in facilitating *generalized inference across different models*. Then I raise the perhaps surprising issue of 'what is the right formulation of Craig's Theorem?', high-lighting the existence of non-trivial *options in formulating meta-theorems*. This becomes important also when taking interpolation to an area which remains mysterious, as a guide toward the *fine-structure of fragments of first-order logic*. For weaker languages than first-order logic, interpolation theorems are scarce, but on the other hand, we sometimes find stronger results of 'uniform interpolation'. Finally, I discuss the 'end of history'. Craig's Theorem is about *the last significant property of first-order logic* that has come to light. Is there something deeper going on here, and can we prove it?

This paper is short, and in the nature of a survey – with an emphasis on ideas. My purpose is to open some doors to the reader, not to visit the palaces they give access to.

2 The interpolation theorem: inference and language intertwined

First recall the formulation of Craig's result for first-order logic (*FOL*, for short):

Interpolation Theorem For all first-order formulas φ, ψ, if $\varphi \models \psi$, then there is a first-order formula α with $Voc(\alpha) \subseteq Voc(\varphi) \cap Voc(\psi)$ and $\varphi \models \alpha \models \psi$.

This is a significant statement concerning derivability and definability in tandem. One contemporary of Craig's in the 1950s who felt that logic is in principle about both, expressive power of definition and deductive power of derivation, was E. W. Beth. Indeed, his Definability Theorem (Beth 1953) was a striking instance of this 'genre': for theories in *FOL*, implicit semantic definability is equivalent to explicit syntactic definability. Though this turned out a consequence of interpolation, it still stands out on its own. Beth viewed his result as a sort of expressive completeness theorem. [1]

Beth's work was inspired by his reading of Padoa's analysis of definability from 1900. Moving even further back in time, we arrive at a pioneer of modern logic whose achievements are still ill-understood, viz. Bernard Bolzano. The *Wissenschaftslehre* (Bolzano 1837) presents the following original analysis of logical consequence. Whether an inference is valid, depends on a prior choice of the role of its vocabulary: what is kept constant in meaning, and what variable. The logical consequences of today are valid if we fix the meanings of Boolean constants or quantifiers. But the account is broader. Consider the famous example of comparatives. Do *"John is taller than Mary, Mary is taller than Paul"* imply *"John is taller than Paul"*? Bolzano would say: not if we consider all grammatical parts as variable here; but yes, if we fix the meaning of *'taller'*, and even yes, if we just fix the meaning of the comparative suffix *'–er'*. Thus, Bolzano's notion of consequence involved vocabulary right from the start:

> Premises □ *imply* conclusion □ *in fixed vocabulary A* if each substitution of concrete expressions for variable parts making □ true also makes □ true. [2]

Bolzano's logic supplies valid principles involving explicit mention of the fixed part of the vocabulary *A*. Here are two examples (van Benthem 1985, 2003 have details):

(a) if □ $□_A$ □ and *A* □ *B*, then □ $□_B$ □

(b) □ $□_A$ □ and □ $□_B$ □ do not imply □ $□_{A□B}$ □:
p □ *q* $□_{\{□,\ \}}$ *p* *T*, *p* □ *q* $□_{\{T,\ \}}$ *p* *T*,
but not *p* □ *q* $□_{\{\ \}}$ *p* *T!*

Fact (b) also makes an independent point on the interplay of vocabulary and validity. There is not always a *unique most general schema* behind a given valid inference!

Given all this, Bolzano's work does not have the same rigid border line between 'logical' and 'non-logical' vocabulary sought by many logicians and philosophers. Instead, it explores various kinds of consequence, appropriate to different domains of reasoning, in a way reminiscent of 'logical pluralism' today. But the only point of our excursion has been to show how Craig's Theorem has a history in vocabulary-based forms of consequence – even though its standard formulation only involves the usual non-logical vocabulary. There are some interpolation theorems with respect to logical

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[1] Of course, there are other ways of stating that a logical vocabulary is 'complete' for its purpose – involving semantic invariances and Ehrenfeucht-Fraïssé games.

[2] Also, Bolzano requires that the premises be consistent – but this is a detail here.

vocabulary – for instance, when the logical system has the 'sub-formula property' (Troelstra & Schwichtenberg 1996). But matters are tricky in general, especially for logical interpolation in weak logical languages (Hendriks 2000).

There is yet more to the entanglement of vocabulary and inference in Craig's Theorem. Many people claim that interpolation shows that a logic is in 'expressive harmony'. While I am in complete resonance with this sentiment, I find it hard to see exactly what is being stated by this slogan – and on this particular score, I will hand over the torch to Jouko Väänänen's article in this same issue, dealing with abstract model theory.

3 The surplus of interpolation, and inference across models

What good does the information do us that a given consequence is 'witnessed' by an interpolant? One answer is that this allows us to extend the scope of that consequence! Standard logical consequence elucidates what is true in one model under consideration: if the premises are true there, so is the conclusion. But other views of inference exist. In particular, Barwise and Perry (Barwise & Perry 1983) have emphasized the fact that we often infer facts about some situation of interest by observing facts about another. The same point occurs much earlier in history, in Indian logic, where inference was invoked to derive facts about situations which are impossible, or dangerous, to inspect, such as a dark room with a coiled object which may be either a piece of rope or a cobra. The general schema in this case becomes this (Barwise & van Benthem 1999):

Entailment along arbitrary relations R: If $\boldsymbol{M} \models \square$ and $\boldsymbol{M}\ R\ \boldsymbol{N}$, then $\boldsymbol{N} \models \square$.

Now we can state the semantic surplus in Craig's Theorem. The interpolant allows for 'model crossing' via the following specific relation known from abstract model theory:

Entailment along L–potential isomorphism: $\square\ \square^{\ p.i.(L)}\ \square$
If $\boldsymbol{M}, s \models \square$ and $\boldsymbol{M}, s$ *pot-isom* $_{L(\square)\square L(\square)}$ $\boldsymbol{N}, t$, then $\boldsymbol{N}, t \models \square$ #

This tells us that when the premise $\square$ holds in a model $\boldsymbol{M}$, and there is another model $\boldsymbol{N}$ 'much like' $\boldsymbol{M}$ as far as the shared vocabulary of the interpolant is concerned, then the conclusion $\square$ also 'spreads to' $\boldsymbol{N}$. For first-order logic, this assertion # is equivalent to the usual notion of logical consequence. [3] In general, it is stronger: cf. Section 4 below.

[3] Coding up the existence of a potential isomorphism as usual with a set of first-order formulas, any counter-example to # must already occur between countable models by the Löwenheim-Skolem theorem. But between countable models potential isomorphisms induce isomorphisms, and hence we have a contradiction with standard consequence.

More generally, what we see is a contact between two types of model-theoretic result: interpolation theorems and *preservation theorems* for formulas of special syntactic shapes that 'transfer' along model relations. Here is one example of how this works for inference from larger to smaller situations. The following observation restates the classical Łos–Tarski Theorem from model theory in the 1950s in a suggestive manner, in terms of interpolation for formulas of the right syntactic shape:

Theorem The following assertions are equivalent for first-order formulas □, □:

(a) □ entails □ along *sub-models,*

(b) there is a universal formula □ such that □ |= □ |= □.

While this result easily follows from just the standard model-theoretic proof, it shows something general. There are many results of this mixed kind (cf. also Section 5), and together, they make for a significant theory of *cross-model inference!* This conclusion is also in line with the development of abstract model theory. Lindström 1966 already studied abstract model-crossing inference, and judicious mixtures of interpolation and preservation are still used there to characterize logics (cf. Barwise & Feferman 1985).

These observations suggest again that book-keeping vocabulary is crucial to describing generalized inference in logical systems. But the *structural rules* valid in this setting need not be the usual ones! Thus, entailment along potential isomorphism satisfies:

(a) □□ ^{L} □ implies □□ $^{L□L'}$ □.

(b) □, □□ ^{L} □ does not imply □□ ^{L} □□ □.
But □, □□ ^{L} □ does imply □□ $^{L□Voc(□)}$ □□ □.

(c) □□ ^{L1} □ and □□ ^{L2} □ do not imply □□ $^{L1□L2}$ □?

We leave verifications of these facts to the reader: they will give a more concrete sense of how model-crossing consequence works.

Once we understand these notions, we can also consider others, changing the □□ quantifier pattern in definition # above. Here is an example.

Digression Other forms of cross-model inference.
Barwise & Perry 1983 also proposed the following alternative form of first-order consequence, to account for inferences of the type 'smoke implies fire':

□□□: every model for □ *has an extension to* a model for □.

With this quantifier combination □□, the complexity changes drastically. It is easily seen that the above notion # is recursively enumerable for first-order sentences. But a simple argument shows how things change here: □ □ □ iff □ is *conservative over* □ w.r.t. universal *L(□)*-sentences. It is also easy to see the following:

Fact Interpolation fails for existential situation-theoretic inference.

For instance, □*xAx* □ □*xBx,* but there is no interpolant in the shared empty vocabulary.

Finally, here is another way of extending the calculus of cross-model inference (cf. van Benthem 2007 for details). Just exploring structural rules for one fixed relation *R* may not be very revealing. We just lose classical structural rules, such as Transitivity: □ □ ^{R} □, □ □ ^{R} □ do not imply □ □ ^{R} □. To get more positive principles, one wants to consider several relations at the same time. To do so, add an algebra of cross-model relations, and then explore analogies with *modal logic*. For instance, writing

□ □ *[R]*□

for the above entailment along a relation *R*, instead of the failed Transitivity, we now get perhaps even more informative valid principles such as:

□ □ *[R]*□, □ □ *[S]*□ imply □ □ *[R;S]*□

Here the relational composition *;* shows what is needed for the correct inference. Likewise, there is no general Contraposition principle that □ □ ^{R} □ implies ¬□ □ ^{R} ¬□. But we do have a more general validity when we add an explicit converse operator $^□$:

□ □ *[R]*□ implies ¬□ □ *[R$^□$]*¬□.

4 What ís the Interpolation Theorem really?

Interpolation is a fragile good. Many logical systems extending first-order logic lose it. One well-known example is the infinitary language $L_{inf,\,□}$ (Barwise 1975). Interpolation fails there because we can define each ordinal up to isomorphism, and then observe

definition of *omega-o(<)* |= ¬ definition of *omega-1 (<'))*,

whereas no formula in the shared vocabulary *{=}* can be an interpolant, as all infinite models satisfy the same sentences of $L_{inf,\,□}$ in this vocabulary.

But now comes a question which is not usually asked. Does this counter-example mean that 'interpolation has been refuted', or just that *some version of* interpolation has been refuted, maybe not even the most appropriate one for generalization? Indeed, failure or success of meta-theorems may depend on their formulation! Recall an earlier observation. The following result also has a claim to being 'the interpolation theorem' for first-order logic, by high-lighting the special model-crossing role of interpolants:

Interpolation Theorem*

The following assertions are equivalent for first-order formulas □, □:

(a) □ entails □ along potential isomorphisms in language *L*,

(b) there is an *L*-formula □ of $L_{inf,\,\square}$ such that □ |= □ |= □.

As we saw in Footnote 3, by the Löwenheim-Skolem Theorem (and Cantor's zig-zag method for countable models) this notion coincides with standard consequence in first-order logic. But with this reformulation, the potential for generalization to other logical systems changes. Here is the main positive result in Barwise & van Benthem 1999:

Theorem The Interpolation* Theorem does hold for $L_{inf,\,\square}$.

Similar points have been made in the meantime about 'failures' of *FOL* meta-theorems in finite model theory (Alechina and Gurevich 1997). These, too, can be formulation-dependent. And so, we are left with a surprising technical and philosophical question:

What ís the interpolation theorem really – and is there a best version?

5 Looking down: interpolation is scarce for fragments of first-order logic

Let us now leave interpolation as it is, but rather change our direction where to use it.

The dominant trend in logical model theory has been to extend the expressive power of first-order logic, to deal with mathematical notions beyond its reach, all the way up to infinitary languages or higher-order logic. And indeed, in this direction, there are still some technical challenges that may not be widely known. Just one example is the widely used system of *propositional dynamic logic PDL*. The interpolation theorem has been claimed for it several times since the 1970s, including published papers in the Journal of Symbolic Logic, but so far, no proof has stood up to scrutiny. The same question may be open for the full first-order fixed-point logic *LFP(FO)* (Ebbinghaus & Flum 1995, van Benthem 2005), though we suspect the answer is negative. Given the fact that these fixed-point logics are very natural and elegant in many ways, these observations also raises an issue whether interpolation is the right sort of thing to demand, or again: whether we are demanding the right sort of interpolation!

But more interesting is the following change in direction of logical system building. Under the influence of computer science, there is a growing interest in *weaker languages* 'below' *FOL*, as these may strike a better balance between expressive power and computational complexity. [4] A typical case is modal logic, whose formulas

[4] Van Benthem, ten Cate & Väänänen 2007 explore Lindström Theorems for weak logics, and show how the lack of *FOL* expressive power calls for very different proof techniques.

can only distinguish models up to bisimulation, but whose expressive power fits many applications, and whose validity problem is decidable. In fragments of first-order logic, interpolation is not an obvious feature. For instance, when one modal formula implies another, there must indeed be a first-order interpolant by Craig's Theorem. But what guarantees that such an interpolant is itself a modal formula inside the fragment? As it happens, interpolation does hold for basic modal logic, but the argument is non-trivial, and it does not extend very widely. Here is a telling result from Ten Cate 2005:

Theorem The only languages containing modal logic plus a universal modality and nominals which have an Interpolation Theorem are the Bounded Fragment (invariant for end extensions, Feferman & Kreisel 1968) and *FOL* itself.

Indeed, the *Guarded Fragment*, perhaps the most natural decidable first-order generalization of modal logic today (Andréka, van Benthem & Németi 1998) lacks an interpolation theorem, as was shown in Hoogland, Marx & Otto 2000. [5] It seems too early to tell what all this means, but again we might explore alternative formulations in the spirit of Section 4. For instance, here is the model-crossing variant (cf. Section 3) of the standard characterization theorem for the modal fragment of *FOL*:

Theorem The following assertions are equivalent for first-order formulas □, □:

(a) □ entails □ along bisimulations (w.r.t. some fixed binary relation *R*)

(b) for some modal □ with *UnPred(□)* □ *UnPred(□)*□*Unpred(□)*, □ |= □ |= □.

Finally, the two directions in logic design that we have mentioned often interact. One can first retreat from *FOL* to some weaker sub-language, and then add non-first-order gadgets. These may not have the same disastrous complexity effects then which they would have on a first-order base. A beautiful example is the *modal □–calculus*, which adds fixed-point operators to the basic modal language (Bradfield & Stirling 2006). While the full first-order fixed-point language *LFP(FO)* is a highly complex fragment of second-order logic, the □–calculus is bisimulation-invariant and decidable, and also, it does satisfy interpolation. Indeed, it does better than this, as we shall see now.

6 'Stronger than Craig': uniform interpolation

The interpolant in Craig's theorem (cf. Section 2) depends on two inputs, viz. the antecedent and the consequent formula. What is not true in *FOL* is the stronger version where the interpolant would only depend on the antecedent. And yet, such features have been found in other areas, especially with intuitionistic and modal logic:

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[5] By contrast, the Guarded Fragment does have the Beth Definability Theorem though.

Uniform interpolation: For all □ and *L*, there is a formula □ such that, for all □, if *L(□)□L(□) □ L, then* □ |= □ iff □ |= □.

Here are two surprising results from the 1990s:

Theorem Basic modal logic has uniform interpolation (Pitt 1992, Visser 1996).

Theorem The □–calculus has uniform interpolation (d'Agostino & Hollenberg 2000).

Uniform interpolation is proved by thorough syntactic or automata-theoretic inspection of modal (fixed-point) formulas. Moreover, it is tied up with the existence of *strongest consequences* of given formulas with respect to parts of their vocabulary. Such strongest consequences need not always exist for first-order formulas. [6] Surprisingly, what the audience learnt at the Craig Symposium at Berkeley was that Craig himself had started in this way in the 1950s, with what he called the 'Consequence Problem'! His eventual interpolation theorem was a retreat to the best version that he could prove.

In line with the language-design spirit of Section 3 above, modern authors have also considered *adding* (modal) operators to languages defining strongest consequences. An elegant example from Hollenberg 1998 again involves 'model-crossing':

***M**, s* |= *<R>*□ iff there is a model ***N**, t R **M**, s* with ***N**, t* |= □.

For instance, taking the relation *R* to be 'bisimulation for the whole language minus some specified proposition letter *p'*, this defines strongest consequences in a semantic sense. This new formalism provides an interesting new fixed-point free description of the modal □–calculus as '*PDL* + bisimulation quantifiers'. Cf. d'Agostino & Lenzi 2005 for more uses of these devices, whose scope remains to be explored. [7]

7 The End of History: was Craig's result the last great meta-theorem?

It seems fair to say that after Craig's theorem, no further significant properties of *FOL* have been discovered. Maybe this is just lack of imagination in the profession – but also, there may be a deeper historical fact here calling for explanation!

One way to go here are *Lindström's Theorems* (Lindström 1969), which say that first-order logic is maximal in having the Compactness and Löwenheim-Skolem properties

[6] It would be of interest to determine just which first-order formulas do have this 'Subconsequence Property'. Monadic formulas do, but there are many others.

[7] Van Benthem 1997 proposed such a device for Second-Order Logic, weakening existential quantifiers □*P* to an operator *<p.i., L>*□*:* "□ is true in some model ***N*** related to ***M*** by *L*–potential isomorphism", where predicate *P* does not occur in *L*.

(or Compactness and Invariance for Potential Isomorphism, or other combinations). But these results, fundamental as they are, reveal no special role for Interpolation. [8]

Thus, here is another take, which might be called 'Concrete Abstract Model Theory'. One might say that 'the elementary meta-theory of first-order logic' is just the concrete first-order theory of *one particular structure*, viz. the set of all first-order formulas, endowed with suitable structure needed to formulate significant meta-properties. And maybe these theories can be simple and axiomatizable, with – who knows – Craig's Theorem playing some crucial role? The first step in this enterprise looks promising. Consider just the theory of pure consequence over a countable non-logical vocabulary:

Fact The first-order theory of *(FORM,* $|=$*)* is axiomatizable, and indeed decidable.

The reason is that this structure is essentially a countable atomless Boolean algebra, whose complete first-order theory is categorical and decidable.

Next, consider the minimal extension needed to formalize the Interpolation Theorem. We now get a two-sorted structure with both formulas and vocabulary items, and binary relations of 'consequence' between formulas, and 'occurrence' between predicate symbols and formulas. In the corresponding language, interpolation is then a straightforward statement in $\Box^0{}_2$ form. Van Benthem asked in the early 1980s whether this model has an axiomatizable theory. But hopes were dashed in Mason 1985:

Theorem The complete first-order theory of the two-sorted first-order model *(FORM,* $|=$*, VOC,* □*)* is effectively equivalent with True Arithmetic.

This result does not say that no more positive results are to be had. But it does day that the interpolation theorem lives among a family of mathematical (combinatorial) properties of first-order logic which already has very high complexity.

The good news, of course, is that the elementary meta-theory of *FOL* is just as complex as True Arithmetic, so we can hope for many more surprising meta-theorems. Once again, none of these have been found so far – unless once counts Uniform Interpolation. But it is worth noting that, right around the time of Lindström's Theorem, the first significant *probabilistic* meta-properties of *FOL* were discovered, viz. the *Zero-One Laws* of Glebskii, Kogan, Liogonkii & Talanov 1969, Fagin 1976. We are far from a complete understanding of the meta-theory of first-order logic at this richer probabilistic level, and who knows, interpolation may make sense even there!

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[8] Interpolation properties do play a crucial role in Abstract Model Theory, however. Cf. again Jouko Väänänen's article in this issue of *Synthese*.

8 Conclusion

Craig's Interpolation Theorem is very much alive, and pondering its properties quickly leads into fundamental questions about what logic is, and where it might still expand.

9 References

G. d'Agostino, 1998, *Modal Logic and Non-Well-founded Set Theory: bisimulation, translation, and interpolation*, dissertation, Institute for Logic, Language and Computation, University of Amsterdam.

G. d'Agostino & M. Hollenberg, 2000, 'Logical Questions concerning the μ-Calculus', *Journal of Symbolic Logic* 65, 310–332.

G. d'Agostino & G. Lenzi, 2005, 'An Axiomatization of Bisimulation Quantifiers via the μ-Calculus',Theoretical Computer Science 338: 1-3, 64–95 .

N. Alechina & Y. Gurevich, 1997, 'Syntax vs. Semantics on Finite Structures', *Structures in Logic and Computer Science* 1997, 14–33.

H. Andréka, J. van Benthem & I. Németi, 1998, 'Modal Logics and Bounded Fragments of Predicate Logic', *Journal of Philosophical Logic* 27:3, 217–274.

J. Barwise, 1975, *Admissible Sets and Structures*, Springer, New York.

J. Barwise & J. van Benthem, 1999, 'Interpolation, Preservation & Pebble Games', *Journal of Symbolic Logic* 64, 881-903.

J. Barwise & S. Feferman, eds., 1985, *Model-Theoretic Logics*, Springer, New York.

J. Barwise & J. Perry, 1983, *Situations and Attitudes*, Bradford Books/ The MIT Press, Cambridge Mass.

J. van Benthem, 1985, 'The Variety of Logical Consequence, According to Bolzano', *Studia Logica* 44:4, 389-403.

J. van Benthem, 1997, 'Dynamic Bits and Pieces', Report LP-97-01, Institute for Logic, Language and Computation, University of Amsterdam.

J. van Benthem, 2003, 'Is there still Logic in Bolzano's Key?', in E. Morscher, ed., *Bernard Bolzano's Leistungen in Logik, Mathematik und Physik*, Academia Verlag, Sankt Augustin, 11-34.

J. van Benthem, 2005, 'Minimal Predicates, Fixed-Points, and Definability', *Journal of Symbolic Logic* 70:3, 696–712.

J. van Benthem, 2007, 'Inference in Action', *Mathematics Review*, Serbian Academy of Sciences, Belgrade, to appear.

J. van Benthem, B. ten Cate & J. Väänänen, 2007, 'Lindström Theorems for Fragments of First-Order Logic', *Proceedings LICS 2007.*

E. W. Beth, 1953, 'On Padoa's Method in the Theory of Definition', *Indagationes Mathematicae* 15, 330–339.

B. Bolzano, 1837, *Wissenschaftslehre*, Seidelsche Buchhandlung, Sulzbach. English translation, 1973, *Theory of Science* (J. Berg and B. Terrell), Reidel, Dordrecht and Boston.

J. Bradfield & C. Stirling, 2006, 'Modal □-Calculi', in P. Blackburn, J. van Benthem & F. Wolter, eds., 2006, *Handbook of Modal Logic*, Elsevier, Amsterdam, 721–756.

B. ten Cate, 2005, *Model Theory for Extended Modal Languages*, Ph.D. thesis, ILLC Amsterdam.

W. Craig, 1957, 'Linear Reasoning. A New Form of the Herbrand-Gentzen Theorem'. *Journal of Symbolic Logic* 22:3, 250–268.

H-D. Ebbinghaus & J. Flum, 1995, *Finite Model Theory*, Springer, Berlin.

R. Fagin, 1976, 'Probabilities on Finite Models', *Journal of Symbolic Logic* 41, 50–58.

S. Feferman & G. Kreisel, 1968, 'Persistent and Invariant Formulas for Outer Extensions', *Compositio Mathematicae* 20, 29–52.

Y. V. Glebskii, D. I. Kogan, M. I. Liogonkii, and V. A. Talanov, 1969, 'Range and Degree of Realizability of Formulas in the Restricted Predicate Calculus', *Cybernetics* 5, 142–154.

L. Hendriks, 2000, 'Doing Logic by Computer: Interpolation in Fragments of Intuitionistic Propositional Logic', *Annals of Pure and Applied Logic* 104, 97–112.

M. Hollenberg, 1998, *Logic and Bisimulation*, Ph.D. thesis, University of Utrecht.

E. Hoogland, M. Marx & M. Otto, 1999, 'Beth Definability for the Guarded Fragment', *Proceedings of LPA R'99*, Springer, Berlin, 273–285.

P. Lindström, 1966, ' First-order Predicate Logic with Generalized Quantifiers', *Theoria* 32, 165–171.

P. Lindström, 1969, 'On Extensions of Elementary Logic', *Theoria* 35, 1–11.

I. Mason, 1985, 'Undecidability of the Meta-Theory of the Propositional Calculus', *Journal of Symbolic Logic* 50, 451 – 457.

A. Pitts, 1992, 'On an Interpretation of Second Order Quantification in First-Order Intuitionistic Propositional Logic', *Journal of Symbolic Logic* 57, 33–52.

H. Schwichtenberg & A. Troelstra, 1996, *Basic Proof Theory*, Cambridge University Press, Cambridge U.K.

A. Visser, 1996, ' Uniform Interpolation and Layered Bisimulation', *Gödel '96 Brno*, Springer, Berlin, 139–164.