Bose-Einstein condensates in radio-frequency-dressed potentials on an atom chip

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A The rf-dressed potential

We calculate the potential that a neutral atom with a magnetic moment experiences in a time-dependent magnetic field $\mathbf{B}(t)$. The hamiltonian is

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B}(t), \tag{A.1}$$

the same as in the static case. The magnetic dipole moment operator $\hat{\boldsymbol{\mu}}$ is proportional to the angular momentum operator $\hat{\mathbf{J}}$ with a proportionality factor μ_s

$$\hat{\boldsymbol{\mu}} = \mu_s \hat{\mathbf{J}}. \tag{A.2}$$

At any fixed point in space the total field $\mathbf{B}(\mathrm{t})$ can be divided in a static part $\mathbf{B}_\mathrm{s}$ and an oscillatory part $\mathbf{B}_\mathrm{rf}(t)$. The z axis of the coordinate system we align with $\mathbf{B}_\mathrm{s}$ and take this as the quantization direction. The oscillatory field we assume has linear polarization and a sinusoidal time dependence with frequency ω_rf. We align x along the oscillating field. We can thus write the field as

$$\mathbf{B} = \mathbf{B}_\mathrm{s} + \mathbf{B}_\mathrm{rf}(t) = \left(B_\mathrm{s} + B_{\mathrm{rf},z}(t)\right) \mathbf{e}_z + B_{\mathrm{rf},x}(t)\mathbf{e}_x. \tag{A.3}$$

The time dependence of the oscillatory field $\mathbf{B}_\mathrm{rf}(t) = \mathbf{b}_\mathrm{rf} \cos(\omega_\mathrm{rf} t)$ can also be written in terms of complex exponentials

$$\mathbf{B}_\mathrm{rf}(t) = \frac{1}{2}\mathbf{b}_\mathrm{rf}\left(e^{i\omega_\mathrm{rf} t} + e^{-i\omega_\mathrm{rf} t}\right). \tag{A.4}$$

We write the wave function as $|\psi\rangle = \sum_m c_m |m\rangle$ and take as orthonormal basis the eigenstates of $\hat{J}_z$: $|m\rangle$ with eigenvalues: $m = -J \dots J$. We write the Schrödinger equation in matrix form

$$i\hbar \dot{c}_n(t) = \sum_m H_{nm} c_m(t). \tag{A.5}$$

The matrix elements H_{nm} are

$$H_{nm} = \left[-\mu_s B_\mathrm{s} - \mu_s B_{\mathrm{rf},z}(t)\right] m\delta_{nm} - \mu_s B_{\mathrm{rf},x}(t)\langle n|\hat{J}_x|m\rangle, \tag{A.6}$$

with

$$\langle n|\hat{J}_x|m\rangle = \frac{1}{2}\sqrt{j(j+1) - m(m+1)}\,\delta_{n,m+1} + \frac{1}{2}\sqrt{j(j+1) - m(m-1)}\,\delta_{n,m-1}, \tag{A.7}$$

which directly follows from the ladder operators $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ which have the eigenvalues [167]:

$$\hat{J}_+|j\ m\rangle = \sqrt{j(j+1)-m(m+1)}|j\ m+1\rangle,$$
$$\hat{J}_-|j\ m\rangle = \sqrt{j(j+1)-m(m-1)}|j\ m-1\rangle.$$

In Eq. (A.6) and Eq. (A.7) δ_{nm} is the Kronecker delta symbol which equals 1 for $n = m$ and 0 in all other cases.

We now apply the transformation

$$\tilde{c}_k(t) \equiv c_k(t)e^{-ik\omega_{\mathrm{rf}}t}, \tag{A.8}$$

which can be interpreted as the transformation to the rotating frame of the precessing magnetic moment. Taking the derivative with respect to t and inserting Eq. (A.5) we find

$$i\hbar\dot{\tilde{c}}_k(t) = \sum_m \left(H_{km}e^{i(m-k)\omega_{\mathrm{rf}}t} + \delta_{km}m\hbar\omega_{\mathrm{rf}}\right)\tilde{c}_m(t), \tag{A.9}$$

which shows that the matrix elements in the rotating frame, $\tilde{H}_{km}$, are equal to

$$\tilde{H}_{km} = H_{km}e^{i(m-k)\omega_{\mathrm{rf}}t} + \delta_{km}m\hbar\omega_{\mathrm{rf}}. \tag{A.10}$$

We now calculate the matrix elements after transformation by inserting Eq. (A.6) into Eq. (A.10). For the time dependence of the oscillating field we use Eq. (A.4).

$$\begin{aligned}\tilde{H}_{km} =& \mu_s m\left[\frac{\hbar\omega_{\mathrm{rf}}}{\mu_s} - B_{\mathrm{s}} + \frac{b_{\mathrm{rf},z}}{2}\left(e^{i\omega_{\mathrm{rf}}t}+e^{-i\omega_{\mathrm{rf}}t}\right)\right]\delta_{km} \\ &- \mu_s\frac{b_{\mathrm{rf},x}}{2}\left[\frac{1}{2}\sqrt{j(j+1)-m(m+1)}\left(1+e^{i2\omega_{\mathrm{rf}}t}\right)\delta_{k,m+1}\right. \\ &\left.+\frac{1}{2}\sqrt{j(j+1)-m(m-1)}\left(1+e^{-i2\omega_{\mathrm{rf}}t}\right)\delta_{k,m-1}\right].\end{aligned} \tag{A.11}$$

Observe that this expression has static terms and terms that are oscillating at $\pm\omega_{\mathrm{rf}}t$ and at $\pm 2\omega_{\mathrm{rf}}t$. We now apply the rotating wave approximation [108] which amounts to neglecting all time-dependent terms in Eq. (A.11), yielding

$$\begin{aligned}\tilde{H}_{km} = \mu_s m\left(\frac{\hbar\omega_{\mathrm{rf}}}{\mu_s} - B_{\mathrm{s}}\right)\delta_{km} - \mu_s\frac{b_{\mathrm{rf},x}}{2}&\left(\frac{1}{2}\sqrt{j(j+1)-m(m+1)}\delta_{k,m+1}\right. \\ &\left.+\frac{1}{2}\sqrt{j(j+1)-m(m-1)}\delta_{k,m-1}\right).\end{aligned} \tag{A.12}$$

From comparison of Eq. (A.6) and Eq. (A.12) we see that Eq. (A.12) corresponds to a magnetic moment in a magnetic field

$$\tilde{\mathbf{B}} = \left(B_{\mathrm{s}} - \frac{\hbar\omega_{\mathrm{rf}}}{\mu_s}\right)\mathbf{e}_z + \frac{1}{2}b_{\mathrm{rf},x}\mathbf{e}_x, \tag{A.13}$$

and hence

$$|\tilde{\mathbf{B}}| = \sqrt{\left(B_{\mathrm{s}} - \frac{\hbar\omega_{\mathrm{rf}}}{\mu_s}\right)^2 + \left(\frac{b_{\mathrm{rf},x}}{2}\right)^2}. \tag{A.14}$$

Since $\tilde{\mathbf{B}}$ is time independent it is easy to see that the magnetic moment in this field has eigenvalues $\tilde{m}\mu_s|\tilde{\mathbf{B}}|$ with $\tilde{m} = -J \ldots J$. If the atom moves slowly through the spatially-varying magnetic field, such that the adiabaticity criterion is satisfied we can write the position-dependent potential as

$$U = \tilde{m}\hbar\sqrt{\Delta^2 + \Omega^2}, \tag{A.15}$$

where Δ^2 is called the resonance term with Δ the local detuning of the rf frequency with respect to the Larmor frequency

$$\Delta = \omega_{\mathrm{rf}} - \omega_L = \omega_{\mathrm{rf}} - \frac{|g_F\mu_B|}{\hbar}|\mathbf{B}_{\mathrm{s}}(\mathbf{r})|. \tag{A.16}$$

The other term, Ω^2, is referred to as the coupling term. The position-dependent Rabi frequency, Ω, is given by the circularly polarized rf-field component referenced to the local direction of the static magnetic field. For a linearly polarized rf field the above shows that

$$\Omega = \frac{|g_F\mu_B|}{\hbar}\frac{|\mathbf{b}_{\mathrm{rf}} \times \mathbf{B}|}{2|\mathbf{B}|}. \tag{A.17}$$