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A Generic Basis Theorem for Cancellation Meadows

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Abstract

Let Q_0 denote the rational numbers expanded to a “meadow”, that is, after taking its zero-totalized form ($0^{-1} = 0$) as the preferred interpretation. In this paper we consider “cancellation meadows”, i.e., meadows without proper zero divisors, such as Q_0 . We prove a representation result and a generic completeness result. We apply these results to cancellation meadows extended with the sign function, and with floor and ceiling, respectively.

1 Introduction

This paper contributes to the algebraic specification theory of number systems. Advantages and disadvantages of algebraic specification of abstract data types have been amply discussed in the computer science literature and we do not wish to add anything to those matters here and refer to Wirsing [17], the seminal 1977-paper [13] of Goguen *et al.*, the overview in Bjørner and Henson [8], and the ASF+SDF meta-environment of Klint *et al.* [9].

Our focus will be on a particular loose algebraic specification for fields called *meadows*, using the terminology of Broy and Wirsing [10] who first wrote about loose specifications. The theory of algebraic specifications is based on theories of universal algebras. Some references to universal algebra are, e.g., Wechler [16] and Graetzer [14].

The equational specification of the variety of meadows has been proposed by Bergstra and Tucker [6] and it has subsequently been elaborated with more systematic detail in [2]. Starting from the signature of fields one obtains the signature of meadows by adding a unary inverse operator. At the basis of meadows, now, lies the design decision to turn the inverse (or division if one prefers a binary notation for pragmatic reasons) into a total operator by means of the assumption that $0^{-1} = 0$. By doing so the investigation of number systems as abstract data types can be carried out within the original framework of algebraic specifications without taking any precautions for partial functions or for empty sorts.

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Following [6] we write Q_0 for the rational numbers expanded to a meadow, that is after taking its zero-totalized form as the preferred interpretation. The main result of [6] consists of obtaining an equational initial algebra specification of Q_0 . The specification takes the form of a general loose specification, valid in all fields equipped with a totalized inverse, to which an equation L specifically designed for the case of rational numbers is taken in addition: the equation L is based on Lagrange's theorem that every natural number can be represented as the sum of four squares and reads

$$\frac{1 + x^2 + y^2 + z^2 + u^2}{1 + x^2 + y^2 + z^2 + u^2} = 1.$$

So L expresses that for a large collection of numbers q , it holds that $q \cdot q^{-1} = 1$ (in particular, those q which can be written as 1 plus the sum of four squares).

In [4] meadows without proper zero divisors are termed *cancellation meadows*. In [2] it is shown that the equational theory of cancellation meadows (there called zero-totalized fields) has a finite basis. In this paper we will extend that result to a generic form. This enables its application to extended signatures. In particular we will examine the case of signed meadows. The sign operator provides one of several mutually interchangeable ways in which the presence of an ordering can be equationally specified. We notice that the generic proof is an elaboration of the proof used for the case of closed terms that has been dealt with in [6]. The proof of the finite basis theorem in [2] uses the existence of maximal ideals. Although shorter and simpler, the proof via ideals seems not to generalize in the way our proof below does. The importance of the generalization follows from the fact that most uses of rational numbers in computer science theory exploit their ordering. An example is timed process algebra as treated in [1].

Bethke and Rodenburg [7] demonstrate that finite meadows are products of fields, thus strengthening the result in [2] (for the finite case) that establishes that each meadow can be embedded in a product of fields, a result which was named the *embedding theorem for meadows*. We notice that the basis theorem for meadows, but not its generic form, is an immediate consequence of the embedding theorem.

The paper is structured as follows: in the next section we recall the axioms for meadows and introduce a representation result. Then, in Section 3 we present our main result, the generic basis theorem. In Section 4 we discuss the extension of cancellation meadows with the sign function, and a further extension with floor and ceiling functions. We end the paper in Section 5 with some conclusions.

2 Meadows: preliminaries and representation

In this section we introduce cancellation meadows in detail and we discuss a representation result that will be used in Section 4.

In [6] *meadows* were defined as the members of a variety specified by 12 equations. However, in [2] it was established that the 10 equations in Table 1 imply those used in [6]. Summarizing, a meadow is a commutative ring with unit equipped with a total unary oper-

$$\begin{aligned}
(x + y) + z &= x + (y + z) \\
x + y &= y + x \\
x + 0 &= x \\
x + (-x) &= 0 \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z) \\
x \cdot y &= y \cdot x \\
1 \cdot x &= x \\
x \cdot (y + z) &= x \cdot y + x \cdot z \\
(x^{-1})^{-1} &= x \\
x \cdot (x \cdot x^{-1}) &= x
\end{aligned}$$

Table 1: The set Md of axioms for meadows

ation $(-)^{-1}$ named inverse that satisfies the two equations

$$\begin{aligned}
(x^{-1})^{-1} &= x, \\
x \cdot (x \cdot x^{-1}) &= x, \quad (RIL)
\end{aligned}$$

and in which $0^{-1} = 0$. Here *RIL* abbreviates *Restricted Inverse Law*. We write Md for the set of axioms in Table 1.

From the axioms in Md the following identities are derivable:

$$\begin{aligned}
(0)^{-1} &= 0, \\
(-x)^{-1} &= -(x^{-1}), \\
(x \cdot y)^{-1} &= x^{-1} \cdot y^{-1}, \\
0 \cdot x &= 0, \\
x \cdot -y &= -(x \cdot y), \\
-(-x) &= x.
\end{aligned}$$

The term *cancellation meadow* is introduced in [4] for a zero-totalized field that satisfies the so-called “cancellation axiom”

$$x \neq 0 \ \& \ x \cdot y = x \cdot z \ \longrightarrow \ y = z.$$

An equivalent version of the cancellation axiom that we shall further use in this paper is the *Inverse Law (IL)*, i.e., the conditional axiom

$$x \neq 0 \ \longrightarrow \ x \cdot x^{-1} = 1. \quad (IL)$$

So *IL* states that there are no proper zero divisors. (Another equivalent formulation of the cancellation property is $x \cdot y = 0 \ \longrightarrow \ x = 0$ or $y = 0$.)

We write $\Sigma_m = (0, 1, +, \cdot, -, {}^{-1})$ for the signature of (cancellation) meadows and we shall often write $1/t$ or

$$\frac{1}{t}$$

for t^{-1} , tu for $t \cdot u$, t/u for $t \cdot 1/u$, $t-u$ for $t+(-u)$, and freely use numerals and exponentiation with constant integer exponents. We shall further write

$$1_x \text{ for } \frac{x}{x} \quad \text{and} \quad 0_x \text{ for } 1 - 1_x,$$

so, $0_0 = 1_1 = 1$, $0_1 = 1_0 = 0$, and for all terms t ,

$$0_t + 1_t = 1.$$

With the axioms in Table 1 we derive a few more identities comprising 1_t and 0_t :

$$\begin{aligned} 1_t \cdot t &= t, & (\text{use } RIL) \\ 1_t \cdot 1/t &= 1/t, & (\text{use } RIL) \\ (1_t)^2 &= 1_t, & (\text{use } RIL) \\ 1_t \cdot 0_t &= 0, & (\text{by } 1_t \cdot 0_t = 1_t(1 - 1_t) = 1_t - 1_t = 0) \\ 0_t \cdot t &= 0, & (\text{by } (1 - 1_t)t = t - t = 0) \\ 0_t \cdot 1/t &= 0, & (\text{by } (1 - 1_t)1/t = 1/t - 1/t = 0) \\ (0_t)^2 &= 0_t. & (\text{by } (1 - 1_t)^2 = 1 - 2 \cdot 1_t + (1_t)^2 = 1 - 1_t = 0_t) \end{aligned}$$

We will use these identities quite a lot.

In the remainder of this section we discuss a particular standard representation for meadow terms. We will use this representation in Section 4 in order to prove an expressiveness result.

Definition 1. A term P over Σ_m is a **Standard Meadow Form (SMF)** if, for some $n \in \mathbb{N}$, P is an **SMF of level n** . SMFs of level n are defined as follows:

SMF of level 0 : each expression of the form s/t with s and t ranging over polynomials (i.e., expressions over Σ_m without inverse operator),

SMF of level $n + 1$: each expression of the form

$$0_t \cdot P + 1_t \cdot Q$$

with t ranging over polynomials and P and Q over SMFs of level n .

Observe that if P is an SMF of level n , then also of level $n + k$ for all $k \in \mathbb{N}$.

Lemma 1. If P and Q are SMFs, then in Md , $P + Q$, $P \cdot Q$, $-P$, and $1/P$ are provably equal to an SMF having the same variables.

Proof. By natural induction on level height n . We spell out the proof in which *RIL* is often used.

Case $n = 0$. Let s, t, u, v be polynomials and let $P = s/t$ and $Q = u/v$. Then

- For the case $P + Q$, first observe that $0_t \cdot s/t = 0_t \cdot 1/t \cdot s = 0$. We derive

$$\begin{aligned} P + Q &= 0_t \cdot (P + Q) + 1_t \cdot (P + Q) \\ &= 0_t \cdot (s/t + u/v) + 1_t \cdot (s/t + u/v) \\ &= 0_t \cdot u/v + 1_t \cdot (s/t + 1_t \cdot u/v) \end{aligned}$$

so it suffices to show that $s/t + 1_t \cdot u/v$ is equal to an SMF of level 0:

$$\begin{aligned} s/t + 1_t \cdot u/v &= 0_v \cdot (s/t + 1_t \cdot u/v) + 1_v \cdot (s/t + 1_t \cdot u/v) \\ &= 0_v \cdot s/t + 1_v \cdot (s/t \cdot 1_v + 1_t \cdot u/v) \\ &= 0_v \cdot s/t + 1_v \cdot \left(\frac{sv + tu}{tv} \right). \end{aligned}$$

- $P \cdot Q = su/tv$.
- $-P = -s/t$.
- $1/P = t/s$.

Case $n + 1$. Let $P = 0_t \cdot S + 1_t \cdot T$ and $Q = 0_s \cdot U + 1_s \cdot V$ with S, T, U, V all SMFs of level n . Then

- For the case $P + Q$ we derive

$$\begin{aligned} P + Q &= 0_t \cdot P + 1_t \cdot P + Q \\ &= 0_t \cdot (S + Q) + 1_t \cdot (T + Q) \\ &= 0_t \cdot (0_s \cdot (S + U) + 1_s \cdot (S + V)) + 1_t \cdot (0_s \cdot (T + U) + 1_s \cdot (T + V)) \end{aligned}$$

and by induction each of the pairwise sums of S, T, U, V equals some SMF.

- For $P \cdot Q$ we derive

$$\begin{aligned} P \cdot Q &= 0_s \cdot P \cdot U + 1_s \cdot P \cdot V \\ &= 0_s \cdot (0_t \cdot S \cdot U + 1_t \cdot T \cdot U) + 1_s \cdot (0_t \cdot S \cdot V + 1_t \cdot T \cdot V) \end{aligned}$$

and by induction each of the pairwise products of S, T, U, V equals some SMF.

- $-P = 0_t \cdot (-S) + 1_t \cdot (-T)$, which by induction is provably equal to an SMF.
- $1/P = 0_t \cdot (1/P) + 1_t \cdot (1/P)$, hence

$$\begin{aligned} 1/P &= 0_t \cdot \frac{1}{0_t \cdot S + 1_t \cdot T} + 1_t \cdot \frac{1}{0_t \cdot S + 1_t \cdot T} \\ &= 0_t \cdot \frac{0_t}{0_t \cdot (0_t \cdot S + 1_t \cdot T)} + 1_t \cdot \frac{1_t}{1_t \cdot (0_t \cdot S + 1_t \cdot T)} \\ &= 0_t \cdot \frac{0_t}{0_t \cdot S} + 1_t \cdot \frac{1_t}{1_t \cdot T} \\ &= 0_t \cdot 1/S + 1_t \cdot 1/T \end{aligned}$$

and by induction there exist SMFs S' and T' such that $S' = 1/S$ and $T' = 1/T$, hence $1/P = 0_t \cdot S' + 1_t \cdot T'$.

□

Theorem 1. *Each term over Σ_m can be represented by an SMF with the same variables.*

Proof. By structural induction. Let P be a term over Σ_m . If $P = 0$ or $P = 1$ or $P = x$, then $P = P/1$, and the latter is an SMF of level 0. The other cases follow immediately from Lemma 1. \square

3 A generic basis theorem

In this section we prove a finite basis result for the equational theory of cancellation meadows. This result is formulated in a generic way so that it can be used for any expansion of a meadow that satisfies the propagation properties defined below.

Definition 2. *Let Σ be an extension of $\Sigma_m = (0, 1, +, \cdot, -, ^{-1})$, the signature of meadows. Let $E \supseteq Md$ (with Md the set of axioms for meadows given in Table 1).*

1. (Σ, E) has the **propagation property for pseudo units** if for each pair of Σ -terms t, r and context $C[\]$,

$$E \vdash 1_t \cdot C[r] = 1_t \cdot C[1_t \cdot r].$$

2. (Σ, E) has the **propagation property for pseudo zeros** if for each pair of Σ -terms t, r and context $C[\]$,

$$E \vdash 0_t \cdot C[r] = 0_t \cdot C[0_t \cdot r].$$

Preservation of these propagation properties admits the following nice result:

Theorem 2 (Generic Basis Theorem for Cancellation Meadows). *If $\Sigma \supseteq \Sigma_m$, $E \supseteq Md$ and (Σ, E) has the pseudo unit propagation property and the pseudo zero propagation property, then E is a basis (a complete axiomatisation) of $Mod_\Sigma(E \cup IL)$.*

The structure of our proof of this theorem is as follows: let $r = r(\bar{x})$ and $s = s(\bar{x})$ be Σ -terms and let \bar{c} be a series of fresh constants. We write $\Sigma(\bar{c})$ for the signature extended with these constants. Then

$$E \cup IL \models r = s \quad \text{in } \Sigma \quad \iff \quad E \cup ILC \models r(\bar{c}) = s(\bar{c}) \quad \text{in } \Sigma(\bar{c}) \quad (1)$$

$$\iff \quad E \vdash_{IR} r(\bar{c}) = s(\bar{c}) \quad (2)$$

$$\iff \quad E \vdash r = s \quad \text{in } \Sigma. \quad (3)$$

Here provability (\vdash) refers to equational logic; the notation further used means this:

- *ILC*, the *Inverse Law for Closed terms* is the set $\{t = 0 \vee 1_t = 1 \mid t \in T(\Sigma(\bar{c}))\}$, where $T(\Sigma(\bar{c}))$ denotes the set of closed terms over $\Sigma(\bar{c})$.
- *IR* is the *Inverse Rule*: $E \vdash_{IR} r = s$ means that $\exists k \in \mathbb{N}$ s.t. $E \vdash_{IR}^k r = s$, and $E \vdash_{IR}^k r = s$ means that $E \vdash r = s$ provided that the rule

$$IR \quad \frac{E \cup \{t = 0\} \vdash r = s \quad E \cup \{1_t = 1\} \vdash r = s}{E \vdash r = s}$$

with t ranging over $T(\Sigma(\bar{c}))$ may be used k times.

Before we prove Theorem 2 — i.e., equivalences (1)–(3) — we establish the following preliminary result:

Proposition 1. *Assume $\Sigma \supseteq \Sigma_m$, $E \supseteq Md$ and (Σ, E) has the propagation property for pseudo units and for pseudo zeros. Then for $t, r, s \in T(\Sigma)$,*

$$E \cup \{t = 0\} \vdash_{IR} r = s \implies E \vdash 0_t \cdot r = 0_t \cdot s, \quad (4)$$

$$E \cup \{1_t = 1\} \vdash_{IR} r = s \implies E \vdash 1_t \cdot r = 1_t \cdot s. \quad (5)$$

Proof. We prove

$$E \cup \{t = 0\} \vdash_{IR}^k r = s \implies E \vdash 0_t \cdot r = 0_t \cdot s \quad \text{and} \quad (6)$$

$$E \cup \{1_t = 1\} \vdash_{IR}^k r = s \implies E \vdash 1_t \cdot r = 1_t \cdot s \quad (7)$$

simultaneously by induction on k . We use the symbol \equiv to denote syntactic equivalence.

Case $k = 0$. By induction on proof lengths. For (6) the only interesting case is $(r = s) \equiv (t = 0)$, so we have to show that $E \vdash 0_t \cdot t = 0_t \cdot 0$. This follows directly from $E \supseteq Md$.

For (7) the only interesting case is $(r = s) \equiv (1_t = 1)$, and also $E \vdash (1_t)^2 = 1_t \cdot 1$ follows directly from $E \supseteq Md$.

Case $k + 1$. By induction on the length of the proofs of $E \cup \{t = 0\} \vdash_{IR}^{k+1} r = s$ and $E \cup \{1_t = 1\} \vdash_{IR}^{k+1} r = s$. There are 3 interesting cases for each of (6) and (7):

1. The \vdash_{IR}^{k+1} results follow from the assumption $(r = s) \equiv (t = 0)$ or $(r = s) \equiv (1_t = 1)$, respectively. These results follow in the same way as above.
2. The \vdash_{IR}^{k+1} results follow from the context rule, so $r \equiv C[v]$, $s \equiv C[w]$ and
 - (6) $E \cup \{t = 0\} \vdash_{IR}^{k+1} v = w$. By induction, $E \vdash 0_t \cdot v = 0_t \cdot w$. Hence, $E \vdash 0_t \cdot C[0_t \cdot v] = 0_t \cdot C[0_t \cdot w]$, and by (Σ, E) having the propagation property for pseudo zeros, $E \vdash 0_t \cdot C[v] = 0_t \cdot C[w]$.
 - (7) $E \cup \{1_t = 1\} \vdash_{IR}^{k+1} v = w$. By induction, $E \vdash 1_t \cdot v = 1_t \cdot w$. Hence, $E \vdash 1_t \cdot C[1_t \cdot v] = 1_t \cdot C[1_t \cdot w]$, and by (Σ, E) having the propagation property for pseudo units, $E \vdash 1_t \cdot C[v] = 1_t \cdot C[w]$.
3. The \vdash_{IR}^{k+1} results follow from the *IR* rule, that is
 - (6) $E \cup \{t = 0\} \cup \{h = 0\} \vdash_{IR}^k r = s$ and $E \cup \{t = 0\} \cup \{1_h = 1\} \vdash_{IR}^k r = s$. By induction, $E \cup \{h = 0\} \vdash 0_t \cdot r = 0_t \cdot s$ and $E \cup \{1_h = 1\} \vdash 0_t \cdot r = 0_t \cdot s$. Again applying induction (\vdash derivability implies \vdash_{IR}^k derivability) yields

$$E \vdash 0_h \cdot 0_t \cdot r = 0_h \cdot 0_t \cdot s,$$

$$E \vdash 1_h \cdot 0_t \cdot r = 1_h \cdot 0_t \cdot s.$$

We derive $0_t \cdot r = (0_h + 1_h) \cdot 0_t \cdot r = 0_h \cdot 0_t \cdot r + 1_h \cdot 0_t \cdot r = 0_h \cdot 0_t \cdot s + 1_h \cdot 0_t \cdot s = 0_t \cdot s$.

- (7) $E \cup \{1_t = 1\} \cup \{h = 0\} \vdash_{IR}^k r = s$ and $E \cup \{1_t = 1\} \cup \{1_h = 1\} \vdash_{IR}^k r = s$. Similar.

□

Proof of Theorem 2. We now give a detailed proof of equivalences (1)–(3), using Proposition 1. For model theoretic details we refer to [12].

- (1) (\implies) Assume $E \cup IL \models r = s$. Let \mathbb{M} be a model of $E \cup ILC$ (over $\Sigma(\bar{c})$). Then $\mathbb{M} \models r(\bar{c}) = s(\bar{c})$ if and only if $\mathbb{M}' \models r(\bar{c}) = s(\bar{c})$ for \mathbb{M}' the minimal submodel of \mathbb{M} . Now \mathbb{M}' is also a model for IL because ILC concerns all closed terms and each value in the domain of \mathbb{M}' is the interpretation of a closed term. So, by assumption $\mathbb{M}' \models r = s$, and, in particular (by substitution), $\mathbb{M}' \models r(\bar{c}) = s(\bar{c})$.

(\impliedby) Assume $E \cup ILC \models r(\bar{c}) = s(\bar{c})$. Let \mathbb{M} be a model of $E \cup IL$ (over Σ). We have to show that $\mathbb{M} \models r(\bar{x}) = s(\bar{x})$, or, stated differently, that for $\bar{a} = a_1, \dots, a_n$ a series of values from \mathbb{M} 's domain, $(\mathbb{M}, x_i \mapsto a_i) \models r = s$ where $x_i \mapsto a_i$ represents the assignment of a_i to x_i . Extend Σ with a fresh constant c_i for each a_i and let $\mathbb{M}(\bar{c})$ be the expansion of \mathbb{M} in which each constant c_i is interpreted as a_i . Then $\mathbb{M}(\bar{c})$ satisfies ILC because \mathbb{M} satisfies IL , so by assumption $\mathbb{M}(\bar{c}) \models r(\bar{c}) = s(\bar{c})$, and therefore $(\mathbb{M}(\bar{c}), x_i \mapsto a_i) \models r = s$ and thus also $(\mathbb{M}, x_i \mapsto a_i) \models r = s$, as was to be shown.

- (2) (\implies) Let E^C be the set of all closed instances over the extended signature $\Sigma(\bar{c})$, then

$$E^C \cup ILC \models r(\bar{c}) = s(\bar{c}).$$

By compactness there is a finite set $F \subseteq E^C \cup ILC$ such that $F \models r(\bar{c}) = s(\bar{c})$.

Now apply induction on the number of elements from ILC in F , say k .

Case $k = 0$. By completeness we find $E \vdash r(\bar{c}) = s(\bar{c})$, and thus $E \vdash_{IR} r(\bar{c}) = s(\bar{c})$.

Case $k + 1$. Assume $(t = 0 \vee 1_t = 1) \in F$ and let $F' = F \setminus \{t = 0 \vee 1_t = 1\}$. Reasoning in sentential logic we find

$$F' \models (t = 0 \vee 1_t = 1) \rightarrow r(\bar{c}) = s(\bar{c})$$

and thus $F' \models (t = 0 \rightarrow r(\bar{c}) = s(\bar{c})) \wedge (1_t = 1 \rightarrow r(\bar{c}) = s(\bar{c}))$, which in turn is equivalent with

$$F' \cup \{t = 0\} \models r(\bar{c}) = s(\bar{c}) \quad \text{and} \quad F' \cup \{1_t = 1\} \models r(\bar{c}) = s(\bar{c}).$$

By induction, $E \cup \{t = 0\} \vdash_{IR} r(\bar{c}) = s(\bar{c})$ and $E \cup \{1_t = 1\} \vdash_{IR} r(\bar{c}) = s(\bar{c})$, and thus by IR ,

$$E \vdash_{IR} r(\bar{c}) = s(\bar{c}).$$

(\impliedby) This follows from the soundness of IR with respect to ILC . That is, if $E \vdash u = v$ because $E \cup \{t = 0\} \vdash u = v$ and $E \cup \{1_t = 1\} \vdash u = v$, then $E \cup \{t = 0 \vee 1_t = 1\} \models u = v$, so $E \cup ILC \models u = v$.

- (3) (\implies) By induction on the length of the proof, using Proposition 1: if $E \vdash_{IR} r(\bar{c}) = s(\bar{c})$ follows from IR (the only interesting case), then

$$E \cup \{t = 0\} \vdash_{IR} r(\bar{c}) = s(\bar{c}) \quad \text{and} \quad E \cup \{1_t = 1\} \vdash_{IR} r(\bar{c}) = s(\bar{c})$$

so $E \vdash 0_t \cdot r(\bar{c}) = 0_t \cdot s(\bar{c})$ by (4) and $E \vdash 1_t \cdot r(\bar{c}) = 1_t \cdot s(\bar{c})$ by (5). Thus

$$E \vdash r(\bar{c}) = (0_t + 1_t) \cdot r(\bar{c}) = 0_t \cdot r(\bar{c}) + 1_t \cdot r(\bar{c}) = 0_t \cdot s(\bar{c}) + 1_t \cdot s(\bar{c}) = s(\bar{c}).$$

A similar proof result is obtained by replacing $r(\bar{c})$ by r and $s(\bar{c})$ by s .

(\Leftarrow) Trivial: if $E \vdash r = s$, then $E \vdash r(\bar{c}) = s(\bar{c})$ in the extended signature $\Sigma(\bar{c})$. So, $E \vdash_{IR} r(\bar{c}) = s(\bar{c})$.

□

A first application of Theorem 2 concerns the equational theory of cancellation meadows:

Corollary 1. *The set of axioms Md (see Table 1) is a finite basis (a complete axiomatisation) of $Mod_{\Sigma_m}(Md \cup IL)$.*

Proof. It remains to be shown that the propagation properties for pseudo units and for pseudo zeros hold in Md . This follows easily by case distinction on the forms that $C[r]$ may take and the various identities on 1_t and 0_t . As an example consider the case $C[-] \equiv - + u$. Then

$$1_t \cdot C[r] = 1_t \cdot (r + u) = 1_t \cdot r + 1_t \cdot u = 1_t \cdot 1_t \cdot r + 1_t \cdot u = 1_t \cdot C[1_t \cdot r].$$

The remaining cases can be proved in a similar way.

□

4 Signed meadows

In this section we consider *signed meadows*: we extend the signature $\Sigma_m = (0, 1, +, \cdot, -, {}^{-1})$ of meadows with the unary sign (or signum) function $\mathbf{s}(x)$. We write Σ_{ms} for this extended signature, so $\Sigma_{ms} = (0, 1, +, \cdot, -, {}^{-1}, \mathbf{s})$. The sign function $\mathbf{s}(x)$ presupposes an ordering on its domain and is defined by

$$\mathbf{s}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We define the sign function in an equational manner by the set *Signs* of axioms given in Table 2. First, notice that by Md and axiom (8) (or axiom (9)) we find

$$\mathbf{s}(0) = 0 \quad \text{and} \quad \mathbf{s}(1) = 1.$$

Then, observe that in combination with the inverse law IL , axiom (13) is an equational representation of the conditional equational axiom

$$\mathbf{s}(x) = \mathbf{s}(y) \longrightarrow \mathbf{s}(x + y) = \mathbf{s}(x).$$

From Md and axioms (10)–(13) one can easily derive $\mathbf{s}(t)$ for any closed term t .

$$\begin{aligned} \mathbf{s}(1_x) &= 1_x & (8) \\ \mathbf{s}(0_x) &= 0_x & (9) \\ \mathbf{s}(-1) &= -1 & (10) \\ \mathbf{s}(x^{-1}) &= \mathbf{s}(x) & (11) \\ \mathbf{s}(x \cdot y) &= \mathbf{s}(x) \cdot \mathbf{s}(y) & (12) \\ 0_{\mathbf{s}(x) - \mathbf{s}(y)} \cdot (\mathbf{s}(x + y) - \mathbf{s}(x)) &= 0 & (13) \end{aligned}$$

Table 2: The set *Signs* of axioms for the sign function

Some more consequences of the $Md \cup Signs$ axioms are these:

$$\mathbf{s}(x^2) = 1_x \text{ because } \mathbf{s}(x^2) = \mathbf{s}(x) \cdot \mathbf{s}(x) = \mathbf{s}(x) \cdot \mathbf{s}(x^{-1}) = \mathbf{s}(1_x) = 1_x, \quad (14)$$

$$\mathbf{s}(x^3) = \mathbf{s}(x) \text{ because } \mathbf{s}(x^3) = \mathbf{s}(x) \cdot \mathbf{s}(x) \cdot \mathbf{s}(x^{-1}) = \mathbf{s}(x \cdot (x \cdot x^{-1})) = \mathbf{s}(x), \quad (15)$$

$$1_x \cdot \mathbf{s}(x) = \mathbf{s}(x) \text{ because } 1_x \cdot \mathbf{s}(x) = \mathbf{s}(x^2) \cdot \mathbf{s}(x) = \mathbf{s}(x^3) = \mathbf{s}(x), \quad (16)$$

$$\begin{aligned} \mathbf{s}(x)^{-1} &= \mathbf{s}(x) \text{ because } \mathbf{s}(x)^{-1} = (\mathbf{s}(x)^2 \cdot \mathbf{s}(x)^{-1})^{-1} = (\mathbf{s}(x^2) \cdot \mathbf{s}(x)^{-1})^{-1} & (17) \\ &= (1_x \cdot \mathbf{s}(x)^{-1})^{-1} = 1_x \cdot \mathbf{s}(x) = \mathbf{s}(x). \end{aligned}$$

So, $0 = \mathbf{s}(x) - \mathbf{s}(x) = \mathbf{s}(x) - \mathbf{s}(x)^3 = \mathbf{s}(x)(1 - \mathbf{s}(x)^2)$ and hence

$$\mathbf{s}(x) \cdot (1 - \mathbf{s}(x)) \cdot (1 + \mathbf{s}(x)) = 0. \quad (18)$$

Identity (18) implies with *IL* that for any closed term t , $\mathbf{s}(t) \in \{-1, 0, 1\}$, and thus also that $\mathbf{s}(\mathbf{s}(t)) = \mathbf{s}(t)$. However, with some effort we can derive $\mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$, which of course is an interesting consequence.

Proposition 2. $Md \cup Signs \vdash \mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$.

Before giving a proof of the idempotency of $\mathbf{s}(x)$ we explain how we found one, as there seems not to be an obvious proof for this identity — at the same time this explanation illustrates the proof of Theorem 2. Consider a fresh constant c , then:

$$\begin{aligned} Md \cup Signs \cup \{\mathbf{s}(c) = 0\} &\vdash_{IR} \mathbf{s}(\mathbf{s}(c)) = \mathbf{s}(c), \\ Md \cup Signs \cup \{\mathbf{1}_{\mathbf{s}(c)} = 1, 1 - \mathbf{s}(c) = 0\} &\vdash_{IR} \mathbf{s}(\mathbf{s}(c)) = \mathbf{s}(c), \\ Md \cup Signs \cup \{\mathbf{1}_{\mathbf{s}(c)} = 1, \mathbf{1}_{1 - \mathbf{s}(c)} = 1\} &\vdash_{IR} \mathbf{s}(\mathbf{s}(c)) = \mathbf{s}(c). \end{aligned}$$

The first two derivabilities are trivial, the third one is obtained from (18) after multiplication with $1/\mathbf{s}(c) \cdot 1/(1 - \mathbf{s}(c))$ (thus yielding $\mathbf{s}(c) = -1 = \mathbf{s}(\mathbf{s}(c))$). The proof transformations that underly the proof of Theorem 2 dictate how to eliminate the *IR* rule in this particular case. The proof below shows the slightly polished result.

Proof of Proposition 2. Recall $0_t + 1_t = 1$. The result $\mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$ follows from

$$\mathbf{s}(\mathbf{s}(x)) = (0_{\mathbf{s}(x)} + \mathbf{1}_{\mathbf{s}(x)}) \cdot \mathbf{s}(\mathbf{s}(x)), \quad \mathbf{s}(x) = (0_{\mathbf{s}(x)} + \mathbf{1}_{\mathbf{s}(x)}) \cdot \mathbf{s}(x),$$

and (19) and (20):

$$0_{\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 0_{\mathbf{s}(x)} \cdot \mathbf{s}(x), \quad (19)$$

$$1_{\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot \mathbf{s}(x). \quad (20)$$

Identity (19) follows from $0 = 0_{\mathbf{s}(x)} \cdot \mathbf{s}(x)$ by $0 = \mathbf{s}(0) = \mathbf{s}(0_{\mathbf{s}(x)} \cdot \mathbf{s}(x)) = 0_{\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x))$, and (20) follows from combining (21) and (22):

$$1_{\mathbf{s}(x)} \cdot 0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot 0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x), \quad (21)$$

$$1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x). \quad (22)$$

Identity (21) follows simply: $0_{1-\mathbf{s}(x)} \cdot (1 - \mathbf{s}(x)) = 0$, so $0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x) = 0_{1-\mathbf{s}(x)}$ and thus

$$0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x)) = \mathbf{s}(0_{1-\mathbf{s}(x)}) = 0_{1-\mathbf{s}(x)} = 0_{1-\mathbf{s}(x)} \mathbf{s}(x).$$

Identity (22) can be derived as follows: from (18) infer

$$1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot (1 + \mathbf{s}(x)) = 0,$$

thus $1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x) = 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot -1$, and thus with $\mathbf{s}(-1) = -1$,

$$1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot -1 = 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x). \quad \square$$

Next we establish the expected corollary of Theorem 2:

Corollary 2. *The set of axioms $Md \cup Signs$ (see Tables 1 and 2) is a finite basis (a complete axiomatisation) of $Mod_{\Sigma_{m,s}}(Md \cup Signs \cup IL)$.*

Proof. It suffices to show that the propagation properties are satisfied for $\mathbf{s}(_)$.

$$\text{Pseudo units: } 1_x \cdot \mathbf{s}(y) = (1_x)^2 \cdot \mathbf{s}(y) = 1_x \cdot \mathbf{s}(1_x) \cdot \mathbf{s}(y) = 1_x \cdot \mathbf{s}(1_x \cdot y).$$

$$\text{Pseudo zeros: } 0_x \cdot \mathbf{s}(y) = (0_x)^2 \cdot \mathbf{s}(y) = 0_x \cdot \mathbf{s}(0_x) \cdot \mathbf{s}(y) = 0_x \cdot \mathbf{s}(0_x \cdot y). \quad \square$$

We notice that the initial algebra of $Md \cup Signs$ equals Q_0 as introduced in [6] expanded with the sign function (a proof follows immediately from the techniques used in that paper). It remains to be shown that the *Signs* axioms (in combination with those of *Md*) are independent. We leave this as an open question.

In the following we show that the sign function is not definable in Q_0 , the zero-totalized field of rational numbers as discussed in [6]. We say that $q, q' \in T(Q_0)$ are *different* if $1_{q-q'} = 1$. Let $r = r(x)$ and $s = s(x)$ and let $T(Q_0[x])$ be the set of terms that are either closed or have x as the only variable, so $r, s \in T(Q_0[x])$. We define

$$\begin{aligned} r \equiv_{\infty} s &\iff r(q) = s(q) \quad \text{for infinitely many different } q \text{ in } T(Q_0), \\ r \equiv_{ae} s &\iff r(q) \neq s(q) \quad \text{for finitely many different } q \text{ in } T(Q_0). \end{aligned}$$

We call these relations *infinite equivalence* and *almost equivalence*, respectively. Observe that both these relations are congruences over $T(Q_0[x])$.

Theorem 3. *Let $r = r(x)$ and $s = s(x)$. If $r \equiv_{\infty} s$ then $r \equiv_{ae} s$.*

Proof. By Theorem 1 it suffices to prove this for SMFs, say $P = P(x)$ and $Q = Q(x)$. Because $P - Q$ is then provably equal to an SMF, we further assume without loss of generality that $Q = 0$.

So, let $P \equiv_{\infty} 0$. We prove $P \equiv_{ae} 0$ by induction on the level n of P .

Case $n = 0$. Then $P = s/t$ for polynomials $s = s(x)$ and $t = t(x)$. Because $P \equiv_{\infty} 0$, at least one of $s \equiv_{\infty} 0$ and $t \equiv_{\infty} 0$ holds. Because polynomials always have a finite number of zero points, at least one of $s \equiv_{ae} 0$ and $t \equiv_{ae} 0$ holds. Thus $P \equiv_{ae} 0$.

Case $n + 1$. Then $P = 0_t \cdot S + 1_t \cdot T$.

- If $t \equiv_{ae} 0$ then $0_t \equiv_{ae} 1$ and $1_t \cdot T \equiv_{ae} 0$, so $S \equiv_{\infty} 0$. By induction, $S \equiv_{ae} 0$, and thus $0_t \cdot S \equiv_{ae} 0$ and hence $P \equiv_{ae} 0$.
- If $t \not\equiv_{ae} 0$ then $1_t \equiv_{\infty} 1$, so $1_t \equiv_{ae} 1$ and $0_t \cdot S \equiv_{ae} 0$, so $T \equiv_{\infty} 0$. By induction, $T \equiv_{ae} 0$, and thus $1_t \cdot T \equiv_{ae} 0$ and hence $P \equiv_{ae} 0$.

□

An immediate consequence of Theorem 3 is:

Corollary 3. *The sign function is not definable in Q_0 .*

Proof. Suppose otherwise. Then there is a term $t \in T(Q_0[x])$ with $\mathbf{s}(x) = t(x)$. So

$$t(x) \equiv_{\infty} 1$$

(because of all positive rationals). But then $t(x) \equiv_{ae} 1$ by Theorem 3, which contradicts $t(x) = -1$ for all negative rationals. □

Finally, we notice that with the sign function $\mathbf{s}(x)$, the functions $\max(x, y)$ and $\min(x, y)$ have a simple equational specification:

$$\begin{aligned} \max(x, y) &= \max(x - y, 0) + y, \\ \max(x, 0) &= (\mathbf{s}(x) + 1) \cdot x/2, \end{aligned}$$

and, of course, $\min(x, y) = -\max(-x, -y)$.

Signed Meadows with floor and ceiling. We end this section with a short discussion on the extension of signed meadows with the *floor* function $\lfloor x \rfloor$ and the *ceiling* function $\lceil x \rceil$. These functions are defined by

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$$

and

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}.$$

$$1_x \cdot \lfloor y \rfloor = 1_x \cdot \lfloor 1_x \cdot y \rfloor \quad (23)$$

$$0_x \cdot \lfloor y \rfloor = 0_x \cdot \lfloor 0_x \cdot y \rfloor \quad (24)$$

$$\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1 \quad (25)$$

$$\lfloor x + 1 \rfloor = \lfloor x \rfloor + 1 \quad (26)$$

$$\lfloor 0 \rfloor = 0 \quad (27)$$

$$(0_{1-\mathbf{s}(x)} \cdot 0_{1-\mathbf{s}(1-x)}) \cdot \lfloor x \rfloor = 0 \quad (28)$$

$$\lceil x \rceil = -\lfloor -x \rfloor \quad (29)$$

Table 3: The set FC of axioms for the floor and ceiling functions

We define these functions in an equational manner by the axioms in Table 3.

Some comments on these axioms: first, (23) and (24) guarantee the propagation properties. Then, consider $0_{1-\mathbf{s}(x)} \cdot 0_{1-\mathbf{s}(1-x)}$, which equals 1 if both $x > 0$ and $1 - x > 0$, and 0 otherwise. So, axiom (28) states that $\lfloor x \rfloor = 0$ whenever $0 < x < 1$. With (25)–(27) this is sufficient to compute $\lfloor t \rfloor$ for any closed t . Axiom (29), defining the ceiling function $\lceil x \rceil$ is totally standard.

Let Σ_{msfc} be the signature of this extension. As before, we have an immediate corollary of Theorem 2.

Corollary 4. *The set of axioms $Md \cup Signs \cup FC$ (see Tables 1, 2 and 3) is a finite basis (a complete axiomatisation) of $Mod_{\Sigma_{msfc}}(Md \cup Signs \cup FC \cup IL)$.*

Proof. For floor, the propagation properties for pseudo units and for pseudo zeros are directly axiomatized by axioms (23) and (24), and those for ceiling follow easily. So, the corollary follows immediately from Theorem 2 and the proof of Corollary 2. \square

We notice that the initial algebra of $Md \cup Signs \cup FC$ is Q_0 extended with the sign function $\mathbf{s}(x)$ and the floor and ceiling functions $\lfloor x \rfloor$ and $\lceil x \rceil$. It remains to be shown that the FC axioms (in combination with those of $Md \cup Signs$) are independent. We leave this as an open question.

We end this section by proving that in $Q_0(\mathbf{s})$, i.e., the rational numbers viewed as a signed meadow, a definition of ceiling and floor cannot be given. To this end, we first prove a general property of unary functions definable in $Q_0(\mathbf{s})$.

Theorem 4. *For any function $h(x)$ definable in $Q_0(\mathbf{s})$ there exist $r \in T(Q_0)$ and a function $g(x)$ definable in $Q_0[x]$ such that*

$$x > r \implies h(x) = g(x).$$

Proof. By structural induction on the form that $h(x)$ may take.

If $h(x) \in \{0, 1, x\}$, we're done. For $h(x) = -f(x)$ or $h(x) = 1/f(x)$ or $h(x) = f_1(x) + f_2(x)$ or $h(x) = f_1(x) \cdot f_2(x)$, the result also follows immediately (in the latter cases take $r = \max(r_1, r_2)$ for r_i satisfying the property for $f_i(x)$).

In the remaining case, $h(x) = \mathbf{s}(f(x))$. Let $g(x) \in T(Q_0[x])$ be such that $f(x) = g(x)$ for $x > r$. By induction on the form that $g(x)$ may take, it follows that an r' exists such that for $x > r'$, $\mathbf{s}(g(x))$ is constant. This proves that for $x > \max(r, r')$, $h(x) = \mathbf{s}(f(x)) = \mathbf{s}(g(x))$ is constant. \square

Corollary 5. *The floor function $\lfloor x \rfloor$ is not definable in $Q_0(\mathbf{s})$.*

Proof. Consider

$$h(x) = \frac{x - \lfloor x \rfloor}{x - \lfloor x \rfloor}.$$

If $h(x)$ were definable in $Q_0(\mathbf{s})$, then by the preceding result there exist r and a function $g(x)$ definable in $Q_0[x]$ such that $h(x) = g(x)$ for $x > r$. But then $g(x) \equiv_{\infty} 0$ (for all integers above r) and $g(x) \equiv_{\infty} 1$ (for all non-integers above r), and this contradicts Theorem 3. \square

5 Conclusions

The main result of this paper is a generic basis theorem for cancellation meadows. Its main contribution might be just this generic feature: as stated before, most uses of rational numbers in computer science exploit their ordering. We include this ordering by extending the initial algebraic specification of Q_0 with an equational specification of the sign function, resulting in a finite basis for what we called $Q_0(\mathbf{s})$ and we provided a non-trivial proof of the idempotency of the sign function in $Q_0(\mathbf{s})$. However, the question whether our particular axioms for $\mathbf{s}(x)$ are independent is left open.

As a further example we added the floor function $\lfloor x \rfloor$ and the ceiling function $\lceil x \rceil$ to $Q_0(\mathbf{s})$ and showed that the resulting equational specification is a finite basis. However, we did not investigate the independency of these axioms. We here notice that for $t(x)$ some term one can add this induction rule:

$$\frac{\begin{array}{l} t(0) = 0, \\ 0_{1-\mathbf{s}(x)} \cdot 0_{t(\lfloor x \rfloor)} \cdot t(\lfloor x \rfloor + 1) = 0, \\ 0_{1+\mathbf{s}(x)} \cdot 0_{t(\lceil x \rceil)} \cdot t(\lceil x \rceil - 1) = 0 \end{array}}{t(\lfloor x \rfloor) = 0, \quad t(\lceil x \rceil) = 0} \quad \text{thus} \quad \frac{\begin{array}{l} t(0) = 0, \\ (x > 0 \ \& \ t(\lfloor x \rfloor) = 0) \longrightarrow t(\lfloor x \rfloor + 1) = 0, \\ (x < 0 \ \& \ t(\lceil x \rceil) = 0) \longrightarrow t(\lceil x \rceil - 1) = 0 \end{array}}{t(\lfloor x \rfloor) = 0, \quad t(\lceil x \rceil) = 0}.$$

With this particular induction rule, the idempotency of $\lfloor x \rfloor$ can be easily proved (take $t(x) = x - \lfloor x \rfloor$), as well as the idempotency of ceiling. With a little more effort one can prove $\lfloor x - \lfloor x \rfloor \rfloor = 0$: first prove $\lfloor -\lfloor x \rfloor \rfloor = -\lfloor x \rfloor$ by induction on x , and then $\lfloor x + \lfloor y \rfloor \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ by induction on y . As a consequence, $\lfloor x - \lfloor x \rfloor \rfloor = \lfloor x \rfloor + \lfloor -\lfloor x \rfloor \rfloor = \lfloor x \rfloor + -\lfloor x \rfloor = 0$. In general, if using *IL* the premises can be proved (from some extension of *Md* that satisfies the propagation properties), then this can also be proved without *IL*, and therefore this also is the case for the conclusion.

We end the paper with some general remarks and questions. In [5] it is shown that computable algebras can be specified by means of a complete term rewrite system, provided auxiliary functions can be used. Useful candidates for auxiliary operators in the case of rational numbers can be found in Moss [15] and Calkin and Wilf [11]. In [6] the existence of an equational specification of Q_0 which is confluent and terminating as a rewrite system has been formulated as an open question. To that question we now add the corresponding question in the presence of the sign operator. We add that to the best of our knowledge the corresponding and probably simpler question which arises if only positive rationals are considered and zero as well as subtraction are omitted (thus validating the equation $x \cdot x^{-1} = 1$) is currently undecided as well.

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