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Metalogic and the Overgeneration Argument

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A prominent objection against the logicality of second-order logic is the so-called Overgeneration Argument. However, it is far from clear how this argument is to be understood. In the first part of the article, we examine the argument and locate its main source, namely, the alleged entanglement of second-order logic and mathematics. We then identify various reasons why the entanglement may be thought to be problematic. In the second part of the article, we take a metatheoretic perspective on the matter. We prove a number of results establishing that the entanglement is sensitive to the kind of semantics used for second-order logic. These results provide evidence that by moving from the standard set-theoretic semantics for second-order logic to a semantics which makes use of higher-order resources, the entanglement either disappears or may no longer be in conflict with the logicality of second-order logic.

1. Introduction

A certain brand of nominalism bans properties and other entities thought to lack well-defined identity criteria. Quine (1956) went as far as calling such entities 'creatures of darkness'. Nominalistic scruples of this kind have been a source of concern about second-order logic, for second-order logic allows quantification into predicate position, and it is natural to read this type of quantification as quantification over properties.

In today's revival of metaphysics, these nominalist scruples have much less traction. However, a different Quinean objection to second-order logic remains high on the philosophical agenda. This objection targets certain principles of second-order logic, known as Comprehension Axioms, which assert the existence of second-order entities. According to the objection, what makes second-order logic

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problematic is not the *kind* of entities to which this logic is committed. Rather, it is the fact that there *are* entities to which it is committed. Whether the second-order quantifiers range over properties or extensional entities such as sets, these ontological commitments of second-order logic are claimed to be in tension with its status as pure logic.

Seminal work of George Boolos (1984, 1985) has paved the way for an interpretation of second-order logic in terms of plural quantification which promises to sidestep this objection. More recently, Agustín Rayo and Stephen Yablo (2001) have offered an alternative interpretation of second-order logic which also aims to undermine the objection. Both interpretations purport to offer a reading of second-order quantification, and hence of the Comprehension Axioms, that is free from ontological commitments.

Ontological concerns derive from the fact that in second-order logic there are existential validities, as witnessed by the Comprehension Axioms. But even if such validities need not signal ontological commitment, there might be validities of second-order logic, not necessarily of the form $\exists X \varphi(X)$, which are problematic for its status as logic. In particular, the possibility remains that the class of formal validities of second-order logic exceeds the class of logical truths.

This is the starting point of another objection to second-order logic, the so-called *Overgeneration Argument*. The argument is typically associated with John Etchemendy's discussion in *The Concept of Logical Consequence* (1990) and has received much attention in the literature (see, for example, Priest 1995; Ray 1996; Hanson 1997, 1999; Gómez-Torrente 1998/9). However, it is far from clear how this argument is to be understood, and recent attempts to reconstruct it have called its significance into question (Parsons 2013; Paseau 2013; Griffiths and Paseau 2016).

We shall offer an interpretation of the argument which locates its main source in the entanglement of second-order logic and mathematics. This interpretation vindicates the philosophical significance of the argument by bringing to light the conflict that lies at its heart, namely, that between the entanglement and the alleged neutrality of logic. Where does this leave defenders of second-order logic?

To address this question, we take a metatheoretic perspective on the matter. We prove a number of results establishing that the entanglement is sensitive to the kind of semantics used for second-order logic. These results provide evidence that by moving from the standard set-theoretic semantics for second-order logic to a semantics which makes use of higher-order resources, the entanglement either disappears or may no longer be in conflict with the neutrality of second-order logic.

2. Second-order logic and the Overgeneration Argument

One of the chief purposes of a formal system is to capture the relation of logical consequence for natural language arguments.[1] This typically involves a process of formalization in which every natural language sentence from a given target class is associated with a sentence in the language of the formal system. We use **S** to denote the formalization of a natural language sentence S. For simplicity, we assume the formalization to yield a bijection.[2] Thus when a formal sentence **S** is first introduced, its natural language reading will simply be S. We extend this notation to sets of sentences. Hence the set of sentences **T** will consist of the formalizations of the natural language sentences in T.

A formal system comes equipped with notions of *validity* and *entailment* specified in model-theoretic or proof-theoretic terms. These formal notions are meant to correspond to the informal notions of *logical truth* and *logical consequence* in the following sense. If the formal sentence **S** is declared valid by the system, then its natural language counterpart S should be a logical truth. In this case we say that the system is *sound with respect to logical truth*. Conversely, if S is a logical truth, then **S** should be declared valid by the system. In this case we say that the system is *complete with respect to logical truth*. Similar definitions can be given for logical consequence.

Friends of classical logic are agreed that first-order logic is sound with respect to logical truth and consequence. However, some of them have denied that it is also complete. To make up for this perceived limitation, extensions of first-order logic have been advocated. Second-order logic has a good claim to reduce this incompleteness (see, for example, Shapiro 1991; Higginbotham 1998).

[1] Here 'natural language' can be understood broadly, so as to include the semi-formal language of mathematics, and indeed even *interpreted* formal languages.

[2] This assumption is overkill. Note that both natural language sentences and formal sentences are naturally divided into equivalence classes induced by the relations of logical equivalence. Suppose that we have two operations, one mapping natural language sentences to formal sentences, the other going in the opposite direction. What matters for our discussion is that successive applications of these operations never take a sentence outside of its equivalence class.

As noted above, second-order logic extends first-order logic with quantification into predicate position. Following custom, we use capital letters to denote second-order variables. At the deductive level, we add rules for the second-order quantifiers and the Axiom Scheme of Comprehension, which states that to every open formula with parameters there corresponds a property or relation. In symbols:

$$\exists X^n \, \forall x_1, \ldots, x_n(X^n x_1, \ldots, x_n \;\leftrightarrow\; \varphi(x_1, \ldots, x_n))$$

where X^n is a second-order variable of nth degree. To simplify notation, we shall henceforth omit the superscript and let the context disambiguate.

The usual model-theoretic semantics for second-order logic is *set-theoretic* and has traditionally been developed in two ways, *standard* and *Henkin* (see Shapiro 1991, pp. 70–76). Advocates of second-order logic tend to restrict attention to the former. Standard semantics interprets monadic second-order variables as ranging over *all* subsets of the first-order domain, dyadic variables as ranging over *all* sets of ordered pairs of objects from the domain, and so on.

The expressive power of second-order logic with standard semantics goes well beyond that of first-order logic. Many notions that resist first-order characterization, such as finiteness, infinity and countability, can be captured semantically by second-order means. That is, there are second-order sentences that hold in all and only interpretations with, respectively, finite, infinite and countable domains. The price to be paid is that any sound proof system is incomplete for standard semantics: if a proof system doesn't prove too much, it doesn't prove enough. Less colourfully, for any effective proof system, if every provable sentence is true in all interpretations, then there are validities that cannot be proved in the system. For this reason, defenders of second-order logic normally focus on model-theoretic notions rather than proof-theoretic ones. We will follow suit in the remainder of this article.

The fact that second-order logic affords the means of characterizing important mathematical notions plays a role in its capacity to reduce the alleged incompleteness of first-order logic with respect to logical truth and consequence. Consider the following argument:

There is at least one thing.
There are at least two things.
There are at least three things.
$\vdots$
$\therefore$ There are infinitely many things.

Some (for example, McGee 2014) have taken the conclusion to be a logical consequence of the premisses (analogous arguments are given by Yi 2006 and Oliver and Smiley 2013). While this fact cannot be captured in the usual first-order formalizations, it can be captured in second-order logic (for instance, by regimenting the plural quantifier in the conclusion with a second-order quantifier in the style of Boolos 1984). The Overgeneration Argument aims to establish that this particular attempt to reduce the incompleteness of first-order logic with respect to logical truth and consequence goes too far. The greater expressive power of second-order logic makes it *unsound* with respect to logical truth (and hence, *a fortiori*, with respect to consequence).

Indeed, some of the notions that resist first- but not second-order characterization are the main ingredients of the standard example of the overgeneration of second-order logic (see, for example, Shapiro 1991, pp. 102–109). This example concerns the logical status of the Continuum Hypothesis (henceforth CH), that is, the statement of ordinary mathematical language that there is no cardinality between that of the natural numbers and that of the real numbers. Although overgeneration is a broader phenomenon, there are good reasons for choosing CH. It is a concrete statement whose first-order formalization can be neither proved nor disproved in **ZFC**, the first-order formalization of standard Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). Indeed, this remains the case even when those axioms are supplemented with standard large cardinal hypotheses (Lévy and Solovay 1967; Hamkins and Woodin 2000).[3]

The fact that in second-order logic we can characterize the cardinality notions mentioned above provides the basis for further definitions. We can define what it means for a property to have size aleph-0, to have size aleph-1, and to have the size of the continuum (see the Appendix for details). So we can express that there is no size between that of the natural numbers (aleph-0) and that of the reals (continuum) by stating that any property has size aleph-1 if and only if it has the size of the continuum. In symbols:

$$\textbf{(CH2)} \quad \forall X(\textsc{aleph-1}(X) \leftrightarrow \textsc{continuum}(X))$$

CH2 is a pure sentence of second-order logic in that it does not contain any non-logical vocabulary. What is more, given the standard set-theoretic semantics for second-order logic and the associated

[3] This, of course, holds modulo relevant consistency assumptions. For ease of readability, we omit the qualification in the remainder of the paper.

definition of validity, the following biconditional is provable from the ZFC axioms: **CH2** is valid if and only if CH is true.[4] It follows from the soundness and completeness of second-order logic with respect to logical truth that CH is true if and only if CH2 is a logical truth.

Note that the validity of ¬**CH2** is not equivalent to the negation of CH. However, there is another pure sentence of second-order logic which is valid if and only if CH is false, namely:

(**NCH2**) $\forall X(\textsc{aleph-1}(X) \rightarrow \neg\textsc{continuum}(X))$

Again, it follows from the soundness and completeness of second-order logic with respect to logical truth that the negation of CH is true if and only if NCH2 is a logical truth.

The connection between the truth or falsity of CH and second-order validities usually provides the starting point of the Overgeneration Argument. But how does this connection yield an argument against the soundness of second-order logic with respect to logical truth?

3. The interpretative problem

In a recent article, Alexander Paseau considers five interpretations of the Overgeneration Argument (Paseau 2013). In four of them, the conclusion of the argument is that CH is logically true. Paseau takes this conclusion to be unacceptable, even on the assumption that CH is true. However, he argues that in each of these four interpretations either the conclusion does not follow from the premisses or one of the premisses is not true.

In the other interpretation considered by Paseau, the conclusion is not that CH is logically true, but that CH2 is. The argument goes as follows:

(P1) **CH2** is valid.
(P2) If **CH2** is valid, then CH2 is a logical truth.
(C) CH2 is a logical truth.

This argument is valid, and Paseau takes its first premiss to be true on the assumption that CH is true. The second premiss is just an instance of the soundness of second-order logic with respect to logical truth. A similar argument could be run from the assumption that CH is false using **NCH2** rather than **CH2**. Thus if the conclusion were

[4] A proof is provided in Appendix B; see Theorem 2.

unacceptable, the soundness of second-order logic with respect to logical truth would have to be surrendered.

According to Paseau, however, the conclusion that CH2 is a logical truth is not troubling. To buttress his point, he draws an analogy between CH2 and the sentence 'If there are two *F*s and three *G*s, and no *F* is *G*, then there are five things that are *F* or *G*'. The first-order formalization of this sentence is as follows:

$$\textbf{(AR)} \quad (\exists_2 xFx \wedge \exists_3 xGx \wedge \neg\exists x(Fx \wedge Gx)) \rightarrow \exists_5 x(Fx \vee Gx)$$

Now consider the following valid argument:

(P1*) **AR** is valid.
(P2*) If **AR** is valid, then AR is a logical truth.
(C*) AR is a logical truth.

As Paseau observes, this argument too has true premisses, and its conclusion should be accepted as true. However, one should note that in this case the truth of the conclusion can be established independently by means of a derivation from what are assumed here to be uncontentious logical principles. The conclusion that CH2 is a logical truth does not enjoy a similar justification. The analogy Paseau draws does not, therefore, support this conclusion.

Paseau offers some considerations as to why accepting CH2 as a logical truth might not be problematic. In particular, he claims that CH2 is ontologically neutral and topic-neutral. How are we to understand these conditions on logicality? A natural suggestion is the following. First, a sentence is *ontologically neutral* if it does not constrain how many objects there are, and second, it is *topic-neutral* if it is pure, lacking any non-logical vocabulary.

One might take issue with the claim that CH2 is indeed ontologically neutral and topic-neutral. On the one hand, several authors have argued that second-order quantifiers carry ontological commitment (see, for example, Resnik 1988; Parsons 2013; Hazen 1993; Linnebo 2003; Shapiro 1993; Florio and Linnebo 2016). On the other hand, there is the traditional Quinean view that non-logical notions are hidden behind second-order notation (Quine 1986). But these considerations are not specific to CH2, and concern second-order logic more generally. Thus relying on them would reduce the Overgeneration Argument to other traditional complaints against second-order logic. The interest of the Overgeneration Argument lies primarily in its promise not to rely on such complaints.

In the following sections, we are going to develop an interpretation of the Overgeneration Argument which incorporates the argument from (P1) to (C) as an important step. The remaining steps will be motivated by conditions on logicality other than those considered by Paseau. As a result, the interpretation will be independent of the traditional complaints against second-order logic just mentioned.

So what is the Overgeneration Argument? Because it concludes that CH2 is a logical truth, Paseau considers the argument from (P1) to (C) to be a poor interpretation. On his view, the Overgeneration Argument's conclusion is meant to be that CH is a logical truth, not that some other sentence is. In support of his view, Paseau cites several authors: Etchemendy (1990), Blanchette (2001), Shapiro (1998), Hanson (1997) and Priest (1995). Whilst some of these authors make remarks suggestive of Paseau's interpretation, it is not clear whether they are ultimately committed to it. There are admittedly interpretative issues here, but one common feature of these discussions is that they highlight the problematic character of the relationship between the logical status of CH2 and the truth of CH. This relationship is sanctioned by the biconditional stating that CH is true if and only if **CH2** is valid. Note that the right-to-left direction of this biconditional does play a role in the argument from (P1) to (C), as it is implicitly used to derive its first premiss (on the assumption that CH is true). By incorporating the argument from (P1) to (C), our interpretation will therefore make use of the right-to-left direction. Moreover, as we will see, the left-to-right direction will also be involved in our interpretation of the argument. This reliance on the biconditional vindicates its important role in the literature by locating the main source of the argument in the conflict between the biconditional and certain features of logicality. Ultimately, however, we are less interested in exegetical accuracy than in articulating a genuinely new challenge for the view that second-order logic is sound with respect to logical truth.

4. Entanglement and logicality

The problematic character of the assertion that CH is true if and only if **CH2** is valid was emphasized by Etchemendy in his original discussion of the Overgeneration Argument:

> The problem lies with our faulty account of the logical properties, which mistakenly equates the logical status of [CH2] with the ordinary truth or falsity of [the Continuum Hypothesis]. (Etchemendy 1990, p. 124)

More generally, the literature contains frequent claims to the effect that biconditionals of the following form are problematic:

φ is true if and only if ψ is valid

where φ and ψ are names of sentences. Besides the truth and falsity of CH, relevant examples concern, for instance, the Axiom of Infinity (Etchemendy 1990; Parsons 2013), the Axiom of Choice (Koellner, 2003), and the existence of various large cardinals (Shapiro 1991; Parsons 2013).

These examples are manifestations of a more general phenomenon, which we may call the *entanglement of logic and mathematics.* Those who find the entanglement problematic fall into two camps. Some hold that the entanglement is generally problematic. This is meant to undermine the model-theoretic account of validity regardless of whether the underlying logic is first- or second-order. A representative of this view is Etchemendy:

> The truth value of [some sentence] is guaranteed by the axiom of infinity, which, though certainly not a matter of logic, is nonetheless a far more comfortable assumption to make than either the continuum hypothesis or its negation. … But the difference here does not show that there is anything peculiar about the logic of second-order languages, or that, as it is sometimes put, second-order logic is really 'set-theory in disguise'. (Etchemendy 1990, p. 124)

Others hold that is that it is a matter of degree whether the entanglement is problematic. Therefore, the entanglement *per se* does not undermine the model-theoretic account of validity. For instance, Parsons writes:

> 'When is the entanglement of a proposed logic with mathematics problematic?' The answer is surely that being problematic is a matter of degree … One can hardly doubt that the entanglement of second-order logic with mathematics is more problematic than that of first-order logic. … Etchemendy seems to be demanding that if a sentence is not logically true, this has to be by virtue of statements that are logical truths …. I don't see how this demand can be satisfied … Minimizing the mathematical commitment of a metatheory of logic makes sense. Eliminating it altogether does not. (Parsons 2013, pp. 158–9)

To make progress, we need to determine why the entanglement might be thought to be problematic. Unfortunately, explicit considerations on the matter are hard to come by, even in the writings of those who accept the view. It seems to us that the problem is best viewed as a conflict between the entanglement and the *neutrality* of logic. In particular, some cases of entanglement are in tension with two putative features of logicality capturing the idea that logic should be neutral. These features can be broadly described as follows:

Dialectical neutrality. Logic should be able to serve as a neutral arbiter in disputes over non-logical matters.

Informational neutrality. Logic alone should not be a source of new information

In the next two sections, we explain the exact nature of the conflict between these two features of logicality and the entanglement of second-order logic and mathematics. Following many discussions in the literature, we will focus on the particular case of the biconditional that CH is true if and only if **CH2** is valid. As our discussion will make clear, what matters for the Overgeneration Argument is the provability of this biconditional in a given background theory. We will refer to this provability as the *entanglement of second-order logic with* CH.

5. Dialectical neutrality

Dialectical neutrality articulates the idea that logic should be a neutral arbiter in disputes, whether metaphysical, mathematical or scientific. In the context of his discussion of modal logic as metaphysics, Tim Williamson writes:

> A natural meta-metaphysical hope is that logic should be able to act as a neutral arbiter of metaphysical disputes, at least as a framework on which all parties can agree for eliciting the consequences of the rival metaphysical theories. (Williamson 2014, p. 212)

Although Williamson's conclusion is that this hope should be abandoned, the idea of logic as a neutral arbiter has a long and influential tradition (see MacFarlane 2000).

In the context of foundational disputes in mathematics, dialectical neutrality is endorsed by Peter Koellner:

> [W]hen [disputants] employ a logic to articulate their differences the logic should be such that each party agrees on (i) what implies what and (ii) the fact that logical validities are true. (Koellner 2010, p. 19)

The argument to be considered in this section aims to establish that, given the entanglement of second-order logic with CH, second-order logic cannot serve as a neutral arbiter in disputes over CH. The argument assumes the soundness and completeness of second-order logic with respect to logical truth. Thus, if dialectical neutrality is to be upheld, one of these two assumptions has to go. Let us spell out the argument.

Suppose that second-order logic is sound and complete with respect to logical truth. And suppose that we want to allow for the possibility of a dispute over CH in the context of ZFC. If second-order logic is to serve as a neutral arbiter, settling the question of whether a statement is a logical truth should not preclude that dispute. However, from the supposition that CH2 is a logical truth, it follows that CH is true. And from the supposition that CH2 is not a logical truth, it follows that CH is false. This can be shown as follows.

In ZFC, one can prove that CH is true if and only if **CH2** is valid, which sanctions the entanglement of second-order logic with CH and arises from the background set-theoretic semantics (see §2). Suppose that CH2 is a logical truth. If we deny CH, we may infer from the entanglement that **CH2** is not valid. By completeness of second-order logic with respect to logical truth, we may conclude that CH2 is not a logical truth, which contradicts our assumption. Thus we must accept CH. Suppose, on the other hand, that CH2 is *not* a logical truth. If we accept CH, we may follow the argument from (P1) to (C) of §3. That is, by soundness of second-order logic with respect to logical truth, we may conclude that CH2 is a logical truth, thereby again obtaining a contradiction. Thus we must deny CH. So whether or not we settle for CH2 as a logical truth, a dispute over CH is precluded: second-order logic cannot be a neutral arbiter in this dispute.

As is clear, both the soundness and completeness of second-order logic with respect to logical truth are used in the argument. To preserve dialectical neutrality, it seems we must reject one of those assumptions. Which one? If the argument is to establish that second-order logic overgenerates, the fault must lie with soundness. However, our reconstruction of the argument brings to light that there might be an alternative avenue of response, namely, denying completeness.

One might think that failure of completeness with respect to logical truth is not worrisome. After all, as we mentioned above, first-order logic may provide an example of a system that is sound but not complete with respect to logical truth. Why is incompleteness tolerable in this case? Consider the even simpler example of propositional logic, another system that is arguably sound but not complete with respect to logical truth. The reason for its incompleteness is obvious: propositional formalization is insensitive to logically relevant features of the target informal sentences (for instance, the presence of predicates or determiners). The same is true of first-order logic, or so the second-order logician contends. However, incompleteness becomes worrisome when we cannot point to logically salient features of the target sentences that are not captured by the formalization. In the case under consideration, these would have to be features of CH2, since the argument only makes use of the particular instance of completeness stating that if CH2 is a logical truth, then **CH2** is valid. Unless this challenge can be met, denying completeness is problematic.

Let us take stock. Dialectical neutrality appears to be in tension with the soundness and completeness of second-order logic with respect to logical truth. An important aspiration of second-order logic was to reduce the incompleteness with respect to logical truth of its first-order cousin. However, as the argument above shows, if logic is to be dialectically neutral, this effort appears to go too far, putting soundness with respect to logical truth in jeopardy. Failure to be sound with respect to logical truth is the sense in which second-order logic would overgenerate: there would be validities whose informal counterpart is not a logical truth. This is the aspect of the problem emphasized by the label 'Overgeneration Argument'. As we have stressed, the argument also relies on completeness with respect to logical truth, but its rejection does not offer the second-order logician an easy way out. Should one decide to give up completeness instead of soundness, the argument would become an *Undergeneration Argument*: there would be logical truths whose formalizations are not sanctioned as validities.

6. Informational neutrality

The second conflict between the entanglement of second-order logic with CH and the neutrality of logic involves *informational* neutrality. Recall the slogan for informational neutrality: logic alone should not be a source of new information. A version of the slogan features in the *Tractatus*, where Wittgenstein writes: 'If *p* follows from *q*, the sense of "*p*" is contained in that of "*q*"' (Wittgenstein 1922, 5.122). The idea behind the slogan is that the informational content of the logical consequences of non-logical principles should be contained in those principles. This is a common view about logical consequence, and is intended to be consistent with the possibility of epistemic gains obtained through deduction.[5]

[5] In recent years, the issue has been discussed with reference to what Hintikka (1970, p. 289) has called the 'scandal of deduction'. See, for example, Sequoiah-Grayson 2008; D'Agostino and Floridi 2009; Jago 2013.

A colourful rendering of informational neutrality is given by Carl Hempel, who also extends it to mathematics:

> Thus, in the establishment of empirical knowledge, mathematics (as well as logic) has, so to speak, the function of a theoretical juice extractor: the techniques of mathematical and logical theory can produce no more juice of factual information than is contained in the assumptions to which they are applied; but they may produce a great deal more juice of this kind than might have been anticipated upon a first intuitive inspection of those assumptions which form the raw material of the extractor. (Hempel 1945, p. 391)

A more recent example can be found in Panu Raatikainen's discussion of neo-logicism:

> [I]f we accept as little as Q+ [Robinson arithmetic augmented with basic second-order rules and a second-order induction axiom], the background logic flings us directly to the powerful PA2 [second-order Peano arithmetic].… This is indeed a huge leap, and it is somewhat problematic if it is allowed by the mere rules of the background logic. (Raatikainen forthcoming, p. 14)

The problematic nature of the 'huge leap' presumably lies in the fact that the application of basic second-order rules to relatively weak theories yields theories which appear to have far greater informational content. Thus, Raatikainen concludes, basic second-order rules cannot count as logical.

Let us now turn to the argument for the second conflict. The argument divides into two steps. The first step is simply the argument from (P1) to (C) of §3, which concludes that CH2 is a logical truth, on the assumption that CH is true. As noted before, this step implicitly uses the entanglement of second-order logic with CH.

The second step is as follows. We start from the following key assumption: ZFC together with the thesis that second-order logic is complete with respect to logical truth does not informationally contain CH.[6] By informational neutrality, it follows that if any additional principle enables us to derive CH from ZFC plus completeness, the additional principle cannot be a logical truth, for such an increase in

[6] We intend our argument to be compatible with different notions of information. Indeed, for our purposes, any notion that makes the key assumption true would do. One such notion may be the following: a theory T informationally contains S if **T** entails **S** in second-order logic. On this notion, ZFC does not informationally contain CH, and it is plausible to think that neither does ZFC together with the thesis that second-order logic is complete with respect to logical truth.

informational content cannot be brought about by logical principles alone.

With the help of the additional principle that CH2 is a logical truth, we now derive CH from ZFC plus completeness. To this end, suppose that CH2 is a logical truth. Then by the completeness of second-order logic with respect to logical truth, it follows that **CH2** is valid. Since second-order logic is entangled with CH in ZFC given the background set-theoretic semantics, we conclude that CH is true. This completes the derivation.

Now, given the key assumption that ZFC plus completeness does not informationally contain CH, we can apply informational neutrality to the preceding derivation to conclude that the additional principle is not a logical truth. That is, we can conclude that the principle that CH2 is a logical truth is not *itself* a logical truth. Finally, we make the plausible assumption that logical truth is an S5 modality.[7] So the fact that it is not a logical truth that CH2 is a logical truth implies that CH2 is not a logical truth. But this contradicts the conclusion of the first step of the argument, namely, that CH2 is a logical truth.

A couple of remarks are in order. First, the argument assumes that CH is true. But this assumption can be dispensed with, since a parallel argument can be run from its negation, using **NCH2**. Second, unlike the argument from dialectical neutrality, the present argument invokes only soundness of second-order logic with respect to CH. Completeness features in a claim about containment, that is, the key assumption that CH is not informationally contained in ZFC plus completeness. Thus the argument from informational neutrality cannot be resisted by denying completeness. This justifies the rejection of soundness to the extent that the other assumptions are deemed plausible. The conclusion appears to be, again, that second-order logic overgenerates.

7. Higher-order semantics and entanglement

In the previous two sections, we saw that the neutrality of logic provides the basis for at least two arguments to the effect that second-order logic overgenerates. The defender of second-order logic might react by forgoing the neutrality of logic. This would be the response favoured by Shapiro (1991, 2012) and anti-exceptionalists about logic

[7] The fact that logical truth is an S5 modality has been defended by, for instance, John Burgess (1999). Note that we only need Axiom 4 for the purposes of the argument.

such as Williamson (2013, 2014). However, a different response might be available. Both arguments were framed in the context of a set-theoretic semantics for second-order logic, the kind of semantics assumed in discussions of overgeneration. We are now going to explore the important but overlooked question of what happens to the entanglement when second-order logic is given an alternative semantics. As we shall see, this opens up the possibility of reconciling the neutrality of logic with the soundness of second-order logic with respect to logical truth.

Higher-order semantics is an alternative and increasingly popular semantics that appeals to higher-order resources in the metatheory (Boolos 1984, 1985; Rayo and Uzquiano 1999; Rayo 2002; Rayo and Williamson 2003; Yi 2005, 2006; McKay 2006; Oliver and Smiley 2013). In this semantics, higher-order expressions are not interpreted as standing for set-theoretic entities. Rather, they are interpreted by means of higher-order expressions in the metalanguage standing for properties, pluralities or Fregean concepts. For current purposes, our metatheoretic framework will be cast in terms of properties, but pluralities or concepts would also do. Our basic metatheory consists of the standard principles of second-order logic, including second-order Comprehension and a second-order version of the Axiom of Choice, stating that there is a choice function corresponding to every relation. Additional metatheoretic principles will be considered below.

A delicate issue in higher-order semantics is how to characterize the notion of satisfaction or truth in a model. A natural way of proceeding is to introduce a primitive satisfaction predicate holding between an interpretation—now construed as a second-order entity—and a formula of the object language. In our case, however, this is not necessary, since we will be concerned only with logical *validity*. More specifically, as pointed out by Vann McGee (1997), validity and entailment for arguments with finitely many premisses can be defined using the second-order resources already available in our metatheory.

For simplicity, our object language will be $\mathscr{L}^2_\in$, the second-order language whose only non-logical symbol is $\in$, the membership predicate of set theory. Now, let φ^U be the restriction of φ's quantifiers to U. Then we say that φ is a *higher-order validity* if for every non-empty property U and every relation E, $\varphi[E/\in]^U$ holds (where $\varphi[E/\in]$ stands for the metalinguistic formula resulting from replacing all occurrences of $\in$ in φ with E). This, in effect, amounts to equating validity to truth with respect to any higher-order domain and any higher-order reinterpretation of the membership predicate.[8] On the basis of this definition, one can further define entailment for any argument with finitely many premisses, $\gamma_1, \ldots, \gamma_n \therefore \varphi$. That is, we say that the premisses *entail* the conclusion just in case $(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \varphi$ is a higher-order validity.

We now present an array of results that shed light on the entanglement of second-order logic with CH in the context of higher-order semantics. (Proofs are provided in the Appendix.) Analogous results hold for the negation of CH.

Our first set of results indicates that a higher-order semantics does not sanction the entanglement of second-order logic with CH in a second-order metatheory obtained by a *merely logical* expansion of the first-order set theory. Let us make this more precise. If T is a theory, the second-order closure of T (denoted by T^*) is the set of sentences derivable from T in our axiomatization of second-order logic. Our focus will be on the status of the entanglement when we adopt as a metatheory the second-order closure of ZFC (ZFC*). Note, however, that our results carry over to second-order closures of various extensions of ZFC (see the Appendix for details).

We begin with a useful lemma, provable in pure second-order logic:

Equivalence Lemma. CH2 *is true if and only if* **CH2** *is a higher-order validity.*

For the next two theorems, we work in ZFC together with the assumption that there is an ω-model of **ZFC**. We have:

First Negative Theorem. ZFC* *does not prove that if* **CH2**, *then* **CH**.

Second Negative Theorem. **ZFC*** *does not prove that if* **CH**, *then* **CH2**.

Thus neither direction of the equivalence of **CH** and **CH2** is provable in **ZFC***.

How about the informal counterpart of this result? That is, can the equivalence of CH and CH2 be proved in ZFC*? We think not. For were there to be an informal proof of the equivalence of CH and CH2 from ZFC*, such a proof could be turned into a formal proof of the equivalence of **CH** and **CH2** from **ZFC***. It seems to us that the situation is analogous to that arising from standard independence results and their assumed significance for mathematical practice. The

[8] For instance, the validity of $\exists x \exists y\, x \in y$ will be equivalent to the truth of the following statement: for every non-empty property U and every relation E, there are x and y having U such that Exy.

independence of **CH** from **ZFC** (Gödel 1939; Cohen 1963) is usually taken to show that CH is independent of ZFC.

But if there is no proof of the equivalence of CH and CH2 in ZFC*, then, given the lemma above, there is no proof of the equivalence of CH and the *validity* of **CH2** in ZFC*. That is to say, in the presence of minimal consistency assumptions, the entanglement of second-order logic with CH is independent of ZFC*.

It is worth clarifying the role played by higher-order semantics in our results. Crucial use of this semantics was made in establishing the Equivalence Lemma using pure second-order logic.[9] Note, on the other hand, that Negative Theorems are unprovability results. As such, they concern what can be *derived* from **ZFC***. In this respect, the semantics adopted for second-order logic is immaterial. The proofs of these theorems given in the Appendix take place in a meta-metalanguage where the second-order vocabulary is interpreted set-theoretically.

The upshot of the foregoing results is that higher-order semantics opens up the possibility of reconciling the dialectical and informational neutrality of second-order logic with its soundness and completeness with respect to logical truth, as both conflicts arose because second-order logic is entangled with CH in ZFC if a set-theoretic semantics is used. Since this is no longer the case in the new metatheory, the conflict appears to have been resolved.

This is a significant conclusion. It shows that higher-order semantics provides a framework in which the supporter of second-order logic can overcome the two ways of spelling out the Overgeneration Argument identified above. However, this assessment is dependent upon the particular metatheory on which we have been focusing, namely, the second-order closure of ZFC.

Our next two theorems show that if one allows the second-order resources to feature in the set-theoretic axioms of the metatheory, second-order logic is again entangled with CH. In particular, the entanglement is sanctioned in a metatheory consisting of ZFC2—the second-order theory obtained from ZFC by replacing the Axiom Schema of Replacement with the corresponding second-order axiom.

In ZFC2, we can prove:

[9] Indeed, when validity is construed according to the set-theoretic semantics, the Equivalence Lemma is not even provable in ZFC*. To see this, recall that ZFC, and hence ZFC*, proves that CH is true if and only if **CH2** is set-theoretically valid. So if ZFC* proved that CH2 is true if and only if **CH2** is set-theoretically valid, it would also prove that CH is true if and only if CH2 is true, contradicting our Negative Theorems.

First Positive Theorem. *If* CH2, *then* CH.

Second Positive Theorem. *If* CH, *then* CH2.

Given the Equivalence Lemma, it follows that in ZFC2, second-order logic is entangled with CH. That is, ZFC2 proves that CH is true if and only if **CH2** is a higher-order validity. What does this mean for the supporter of second-order logic who embraces higher-order semantics?

In answering this question, the crucial observation is that **ZFC2**—the formalization of ZFC2—remains *quasi-categorical* when moving from a set-theoretic to a higher-order semantics. Let us elaborate. First, observe that, since **ZFC2** consists of finitely many axioms, we can take the theory to be the conjunction of those axioms. Next, say that F is a *quasi-isomorphism between* $\langle U_1, E_1\rangle$ *and* $\langle U_2, E_2\rangle$ if F is a bijection between U_1 and a subproperty of U_2 or between U_2 and a subproperty of U_1 such that F preserves the relations E_1 and E_2; that is, if Fxy and Fuw, then E_1xu if and only if E_2yw. Then the following is provable in pure second-order logic:[10]

Pure Internal Categoricity Theorem. *For every* U_1, E_1, U_2, E_2, *if* $\mathbf{ZFC2}[E_1/\in]^{U_1}$ *and* $\mathbf{ZFC2}[E_2/\in]^{U_2}$, *then there is a quasi-isomorphism between* $\langle U_1, E_1\rangle$ *and* $\langle U_2, E_2\rangle$.

From this theorem one can infer that **ZFC2** decides **CH** according to higher-order semantics. Let us sketch the argument (for details, see Button and Walsh 2018, ch. 11). To start with, note that **ZFC2** proves by a simple existential introduction the existence of some U and E such that $\mathbf{ZFC2}[E/\in]^U$. Now, one can reformulate the set-theoretic notion of a level V_α of the cumulative hierarchy in terms of properties. Then one can show that levels so defined can be well-ordered. It follows from **ZFC2** that there must be a least such level U' such that, for some E', $\mathbf{ZFC2}[E'/\in]^{U'}$. The Pure Internal Categoricity Theorem implies that for any U and E such that $\mathbf{ZFC2}[E/\in]^U$, there is a quasi-isomorphism between $\langle U, E\rangle$ and $\langle U', E'\rangle$. Thus, for any sentence S concerning the hierarchy below U', this holds: if $\mathbf{ZFC2}[E_1/\in]^{U_1}$ and $\mathbf{ZFC2}[E_2/\in]^{U_2}$, then $\mathbf{S}[E_1/\in]^{U_1}$ if and only if $\mathbf{S}[E_2/\in]^{U_2}$. But since CH can be formulated as a sentence concerning a level below U' (in particular, the level corresponding to $V_{\omega+2}$), we

[10] The result has been proved by Stewart Shapiro (unpublished manuscript) and follows from a theorem of Väänänen and Wang (2015) together with the fact that for every U_1, E_1, U_2, E_2, if $\mathbf{ZFC2}[E_1/\in]^{U_1}$ and $\mathbf{ZFC2}[E_2/\in]^{U_2}$, then there is a quasi-isomorphism between $\langle \mathrm{Ord}_1, E_1\rangle$ and $\langle \mathrm{Ord}_2, E_2\rangle$, where Ord_n is the property of being an ordinal in U_n with respect to E_n.

have that if $\mathbf{ZFC2}[E_1/\in]^{U_1}$ and $\mathbf{ZFC2}[E_2/\in]^{U_2}$, then $\mathbf{CH}[E_1/\in]^{U_1}$ if and only if $\mathbf{CH}[E_2/\in]^{U_2}$. On the basis of this fact, the higher-order definition of entailment yields the following corollary.

Corollary. *Either* **ZFC2** *entails* **CH** *or it entails* ¬**CH**.

With this observation in mind, we can proceed to reassess the arguments from dialectical neutrality and informational neutrality. Let us start with the argument from dialectical neutrality. One could run the previous version of the argument almost verbatim. Suppose that second-order logic is sound and complete with respect to logical truth. And suppose that we want to allow for the possibility of a dispute over CH in the context of ZFC2. If second-order logic is to serve as a neutral arbiter, settling the question of whether a statement is a logical truth should not preclude that dispute. However, in ZFC2, one can prove that CH is true if and only if **CH2** is a higher-order validity. We now face a dilemma. Either CH2 is a logical truth or it is not. Assume that it is. If we deny CH, we may infer from the entanglement that **CH2** is not a higher-order validity. By the completeness of second-order logic with respect to logical truth, we may conclude that CH2 is not a logical truth, which contradicts our assumption. Thus we must accept CH. Assume, on the other hand, that CH2 is not a logical truth. If we accept CH, by the soundness of second-order logic with respect to logical truth, we may conclude that CH2 is a logical truth, thereby again obtaining a contradiction. Thus, whether or not we settle for CH2 as a logical truth, a dispute over CH is precluded.

A central assumption of the argument is that we want to allow for the possibility of a dispute over CH in the context of ZFC2. The corresponding assumption in the context of ZFC owes its plausibility to the fact that **ZFC**, if consistent, entails neither **CH** nor ¬**CH**. However, since **ZFC2** entails either **CH** or its negation, one of $\mathbf{ZFC2} \rightarrow \mathbf{CH}$ and $\mathbf{ZFC2} \rightarrow \neg\mathbf{CH}$ is a higher-order validity. This means, in particular, that two parties who subscribe to ZFC2 cannot agree on the logic while disagreeing on CH: once they agree on one of these higher-order validities, they can settle the dispute by a simple *modus ponens*. Thus dialectical neutrality appears to be violated quite independently of the entanglement of second-order logic with CH. It could be pointed out that **ZFC2**, if consistent, proves neither **CH** nor ¬**CH** (Weston 1977). But the genuine notion of consequence for our defender of second-order logic is semantic. To insist on the syntactic notion of consequence is to dismiss at the outset the conception of second-order logic that gives rise to the Overgeneration Argument. Therefore the plausibility of the assumption that we should allow for the possibility of a dispute over CH in the context of ZFC2 is undermined. So the argument from dialectical neutrality does not get off the ground if the entanglement is established using ZFC2.

Let us now turn to the argument from informational neutrality. Given the entanglement of second-order logic with CH in ZFC2, one could try to run an analogue of the previous version of the argument using ZFC2. Whereas the previous version of the argument relied on the assumption that CH is not implicitly contained in ZFC, the new version would rely on the assumption that CH is not implicitly contained in ZFC2. But the two assumptions are not on a par. The first is supported by the undecidability of **CH** from **ZFC**, even if supplemented with standard large cardinal hypotheses. In contrast, it follows from the Pure Internal Categoricity Theorem that **ZFC2** semantically decides **CH**. Thus one cannot simply assume that CH is not implicitly contained in ZFC2.[11] This blocks the argument from informational neutrality when the metatheory encompasses ZFC2.

8. Conclusion

We have advanced two novel reconstructions of the Overgeneration Argument. These reconstructions vindicate the view that the Overgeneration Argument poses a significant challenge, and therefore deserves a place among the standard objections to second-order logic. Just as in the case of the other objections, the Overgenertion Argument makes use of the assumption that logic has certain features. In our reconstructions of the argument, the relevant features are dialectical and informational neutrality.

A staunch defender of second-order logic could take issue with the assumption that logic has these features. Our results establish that she need not do so. Instead, she can regard the Overgeneration Argument as an argument for adopting a higher-order semantics. This is because in this semantics the entanglement of second-order logic with CH either disappears or becomes unproblematic. On the one hand,

[11] Indeed, according to the notion of information mentioned above in footnote 6, ZFC2 informationally contains CH.

there is no entanglement if the background set theory is essentially first-order. On the other hand, the entanglement is no longer problematic in the context of a stronger background set theory which makes full use of second-order resources. For the arguments from dialectical and informational neutrality assume, respectively, that a dispute over CH is legitimate and that CH is not informally contained in the background theory. But the stronger background theory semantically decides CH, which undermines both assumptions. Thus neither argument is available.

Our focus in this article has been on the paradigm example of the entanglement of second-order logic and mathematics, namely, the one arising from CH. Other cases of entanglement may be thought to be problematic, such as those involving the Axiom of Choice or the existence of large cardinals. An interesting question emerging from our discussion concerns their status when second-order logic is given a higher-order semantics. The generality of the techniques used here suggests that analogues of our results might be obtained for those cases too. This will be investigated in future work.

Although the use of higher-order semantics has received much attention in the recent philosophical literature, this has typically been in the context of attempts to provide second-order logic with its intended interpretation. However, the consequences of adopting a higher-order semantics for second-order logic are far-reaching, and remain to be fully explored. Our findings show that, perhaps surprisingly, this semantics can also help vindicate the neutrality of second-order logic.[12]

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9. Appendix

9.1 Notation and definitions

A property B is a *subproperty* of another A (in symbols: $B \sqsubseteq A$) if, for every x, Bx only if Ax. If the containment is proper, we write $B \sqsubset A$.

The set-theoretic formula stating that the set f is a bijection between a and b is abbreviated as $a \cong_f b$. We write $a \cong b$ when $a \cong_f b$ for some f. Likewise, the second-order formula asserting that the second-order relation F is a bijection between A and B is abbreviated as $A \cong_F B$. By analogy with the set-theoretic case, we write $A \cong B$ when $A \cong_F B$ for some F.

For any property X, $\{X\}$ denotes the set whose elements have X. That is,

$$\forall y\,(y \in \{X\} \leftrightarrow Xy)$$

Note that $\{X\}$ need not exist. This is the case, for instance, when X is a universal property.

Conversely, for any set a, let X_a be the property X such that

$$\forall y\,(Xy \leftrightarrow y \in a)$$

Second-order Comprehension implies that X_a exists whenever a does.

There are two types of cardinality notions. The first type concerns the set-theoretic definition of notions such as finiteness, countable infinity (aleph-0), aleph-1, and continuum. The second type concerns the second-order rendering of the same notions. We use lower-case letters to stand for the set-theoretic notions and small capital letters to stand for the second-order ones. All these notions, and the associated notation, will feature in both the object language and the metalanguage.

We define infinity according to the Dedekind characterization. A set a is *infinite* if there is some $b \subset a$ such that $b \cong a$, otherwise it is *finite*. A set a is *countably infinite* if a is infinite and any $b \subseteq a$ is either finite or $b \cong a$. A set is *aleph*-0 if it has the cardinality of countably infinite sets. Moreover, a set a is *aleph*-1 if a is neither finite nor aleph-0, and any $b \subseteq a$ is finite, aleph-0, or $b \cong a$. Finally, a set a is the *size of the continuum* (or *continuum* for short) if there is a set b such that b is aleph-0 and $\mathscr{P}(b) \cong a$.

Parallel definitions can be given in the language of pure second-order logic. A property A is INFINITE if there is some $B \sqsubset A$ such that $B \cong A$, FINITE otherwise. A property A is COUNTABLY INFINITE if A is INFINITE and any $B \sqsubseteq A$ is either FINITE or $B \cong A$. A property is ALEPH-0 if it has the cardinality of COUNTABLY INFINITE properties. Moreover, a property A is ALEPH-1 if A is neither FINITE nor ALEPH-0, and any $B \sqsubseteq A$ is FINITE, ALEPH-0, or $B \cong A$.

In order to define the second-order counterpart of being continuum, we need to define an auxiliary notion. Say that a relation R codes a bijection between A and the subproperties of B just in case the following conditions are met:

(i) For every $C \sqsubseteq B$, there is some x such that Ax and for every y Rxy if and only if Cy.

(ii) Whenever x and y have A, the fact that for every z Rxz if and only if Ryz implies that $x = y$.

If R codes a bijection between A and the subproperties of B, we say that x is the *code* of $C \sqsubseteq B$ (relative to R) if the following condition is met: for every y, Rxy if and only if Cy.

We define a property A to be the SIZE OF THE CONTINUUM (or CONTINUUM for short) if there is some B such that B is ALEPH-0 and there is a relation R coding a bijection between A and the subproperties of B.

We are now in a position 1 to define **CH**, **CH2**, and **NCH2**.

(CH) $\forall x(\text{aleph-1}(x) \leftrightarrow \text{continuum}(x))$
(CH2) $\forall X(\text{ALEPH-1}(X) \leftrightarrow \text{CONTINUUM}(X))$
(NCH2) $\forall X(\text{ALEPH-1}(X) \rightarrow \neg\text{CONTINUUM}(X))$

Note that in the presence of **ZFC**, ¬**CH** is equivalent to the first-order counterpart of **NCH2**, namely,

$$\forall x(\text{aleph-1}(x) \rightarrow \neg\text{continuum}(x))$$

Finally, we need some notation to distinguish between the two notions of validity relevant to our discussion. By Val_{ST} we denote the usual set-theoretic definition of validity as truth in all set-theoretic models. By Val_{HO} we denote the higher-order definition of validity given in §7.

9.2 Proofs

We now proceed to prove the results discussed in the main body of the article. We begin by showing that the entanglement of second-order logic with CH can be established in ZFC when the semantics is set-theoretic. To this end, we first prove a lemma showing that, in the set-theoretic semantics, second-order cardinality notions mirror the corresponding set-theoretic ones. As usual, set-theoretic models of second-order logic will be ordered pairs consisting of a domain and a valuation function.

Lemma 1. *Assume* ZFC. *Let $m = \langle d, v\rangle$ be a model of second-order logic. Then $m \models$* INFINITE(X) *if and only if $v(X)$ is infinite. The same holds of the other notions defined above:* FINITE, ALEPH-0, ALEPH-1, *and* CONTINUUM.

Proof. Suppose that $m \models$ INFINITE(X). Note that INFINITE(X) abbreviates the following formula:

$$\exists F \exists Y (Y \sqsubset X \ \wedge \ X \cong_F Y)$$

By the definition of satisfaction in set-theoretic semantics, it follows that there are $f \subseteq d \times d$ and $y \subset v(X)$ such that $v(X) \cong_f y$. Thus $v(X)$ is infinite.

For the other direction, suppose that $v(X) \subseteq d$ is infinite. Then there are f and $y \subset v(X)$ such that $v(X) \cong_f y$. Since $v(X) \subseteq d$, $y \subseteq d$. Therefore $f \subseteq d \times d$. It follows that, relative to m, f and y witness the existential quantifiers in the formula displayed above. Thus $m \models$ INFINITE(X).

The proofs for the other notions are similar. □

Theorem 2. *Assume* ZFC. *Then* CH *is true if and only if* $\mathrm{Val}_{\mathrm{ST}}(\mathbf{CH2})$ *holds.*

Proof. For the left-to-right direction, suppose that CH is true but $\mathrm{Val}_{\mathrm{ST}}(\mathbf{CH2})$ does not hold. This implies that there is a model $m = \langle d, v\rangle$ such that $m \not\models$ **CH2**. In turn, this implies that there is some $a \subseteq d$ witnessing the falsity of **CH2**. By Lemma 1, it follows that a is aleph-1 but not continuum, which contradicts CH.

For the right-to-left direction, suppose that $\mathrm{Val}_{\mathrm{ST}}(\mathbf{CH2})$ holds. Then for every model m, $m \models$**CH2**. In particular, $m^* \models$**CH2**, where $m^* = \langle \mathscr{P}(\mathbb{N}), v\rangle$ and v is any valuation function. It follows from Lemma 1 that for every $a \subseteq \mathscr{P}(\mathbb{N})$, a is aleph-1 if and only if a is continuum. That is, CH is true. □

Theorem 3. *Assume* ZFC. *Then* CH *fails if and only if* $\mathrm{Val}_{\mathrm{ST}}(\mathbf{NCH2})$.

Proof. Note that, by Cardinal Comparability, we have the following cardinality facts. For any sets a and b, if a and b are both aleph-1, then there is a bijection between a and b. The same holds if a and b are both continuum.

For the left-to-right direction, suppose that CH and $\mathrm{Val}_{\mathrm{ST}}(\mathbf{NCH2})$ both fail. By the set-theoretic semantics, this means that there is a set a and a model $m = \langle d, v \rangle$ with $a = v(X) \subseteq d$ such that $m \models$ ALEPH-1$(X) \wedge$ CONTINUUM(X). By Lemma 1, a is both aleph-1 and continuum. Now, since CH fails, there is a set b that is either aleph-1 but not continuum or continuum but not aleph-1. But given the cardinality facts mentioned above, this cannot be.

For the other direction, suppose that $\mathrm{Val}_{\mathrm{ST}}(\mathbf{NCH2})$ holds. Let $m = \langle \mathscr{P}(\mathbb{N}), v \rangle$, where $v(X) = \mathscr{P}(\mathbb{N})$. Since $\mathrm{Val}_{\mathrm{ST}}(\mathbf{NCH2})$ holds, $m \models$ ALEPH-1$(X) \rightarrow \neg$CONTINUUM(X). That is, $m \models$ CONTINUUM$(X) \rightarrow \neg$ALEPH-1(X). By Lemma 1, it follows that if $\mathscr{P}(\mathbb{N})$ is continuum, it is not aleph-1. Now it is easy to show that $\mathscr{P}(\mathbb{N})$ is continuum. Therefore $\mathscr{P}(\mathbb{N})$ is not aleph-1. But the existence of a set, such as $\mathscr{P}(\mathbb{N})$, that is continuum but not aleph-1 is a counterexample to CH. □

We now turn to a higher-order interpretation of logical validity.

Equivalence Lemma. CH2 *is true if and only if* $\mathrm{Val}_{\mathrm{HO}}(\mathbf{CH2})$ *holds.*

Proof. The right-to-left direction is straightforward. For the other direction, suppose that CH2 is true, and suppose for contradiction that $\mathrm{Val}_{\mathrm{HO}}(\mathbf{CH2})$ does not hold. This means that there are U and E such that $\mathrm{CH2}[E/\in]^U$ is not true. But since CH2 is a pure sentence of second-order logic, $\mathrm{CH2}[E/\in]^U$ is equivalent to $\mathrm{CH2}^U$. Thus, $\mathrm{CH2}^U$ must be false, which means that there is a domain U and a subproperty F of U such that it is not the case that ALEPH-1$(F)^U$ if and only if CONTINUUM$(F)^U$.

Now since CH2 holds, then it holds a fortiori with respect to $F \sqsubseteq U$. But note that if ALEPH-1(F), then ALEPH-1$(F)^U$. This is because $F \sqsubseteq U$ and all quantifiers in ALEPH-1(F) range over subproperties of F and relations whose field is F. Similarly, if CONTINUUM(F), then CONTINUUM$(F)^U$. To prove this conditional, suppose that CONTINUUM(F). By definition, this means that there is a property G and a relation R such that ALEPH-0(G) and R codes a bijection between F and the subproperties of G. To show that CONTINUUM$(F)^U$, it is enough to show that there is a bijection between F and an ALEPH-1 subproperty of F such that the field of the bijection is F. To this end, let H be a property which applies to all and only the elements of F that, relative to R, code a singleton subproperty of G. So $H \sqsubset F$ and $H \cong G$, hence ALEPH-0(H). Moreover, since $H \cong G$ and R codes a bijection between F and the subproperties of G, it follows that there is some R' which codes a bijection between F and the subproperties of H. Given that $H \sqsubset F \sqsubseteq U$ and the field of R' is $F \sqsubseteq U$, we can conclude that $\textsc{continuum}(F)^U$.

Since by assumption $\textsc{aleph-1}(F)$ if and only if $\textsc{continuum}(F)$, we have that $\textsc{aleph-1}(F)^U$ if and only if $\textsc{continuum}(F)^U$, which contradicts the supposition that $\mathrm{CH2}^U$ is false. □

We can now prove the two Negative Theorems. Together with the Equivalence Lemma, they show that if we switch to higher-order semantics, the entanglement of second-order logic with CH can no longer be established in the presence of the set-theoretic resources available in ZFC. We work in ZFC together with the assumption that there is an ω-model of **ZFC**. This is a model whose natural numbers are isomorphic to $\mathbb{N}$.

First Negative Theorem. **ZFC*** *does not prove that if* **CH2**, *then* **CH**.

Proof. Suppose that there is a (set-theoretic) ω-model of **ZFC**. This is also a model of **ZFC** + Con(**ZFC**). By forcing, we obtain an ω-model m_1 of **ZFC** + Con(**ZFC**) + **CH**. Since m_1 satisfies Con(**ZFC**), it thinks—again via forcing—that there is a model $m_2 = \langle d_2, v_2 \rangle$ of **ZFC** + ¬**CH**.

Now, from the perspective of m_1, there is a model $m_3 = \langle d_2, v_3 \rangle$ of $\mathscr{L}^2_{\in}$, where v_3 is any valuation function which agrees with v_2 on the first-order fragment of the language. Since m_3 is a full model (from m_1's perspective) and agrees with v_2 on the membership relation, m_3 is a model of **ZFC*** + ¬**CH**. Note that since m_1 satisfies **ZFC** and **CH**, it thinks by Theorem 2 that every full model of $\mathscr{L}^2_{\in}$ satisfies **CH2**. Thus, in particular, m_1 thinks that m_3 satisfies **CH2**. So m_1, an ω-model, thinks that the theory **ZFC*** + ¬**CH** + **CH2** is consistent. But consistency facts are arithmetical facts. Thus the theory **ZFC*** + ¬**CH** + **CH2** is consistent. □

Second Negative Theorem. **ZFC*** *does not prove that if* **CH**, *then* **CH2**.

Proof. Suppose that there is an ω-model of **ZFC**. This is also a model of **ZFC** + Con(**ZFC**). By forcing, we obtain an ω-model m_1 of **ZFC** + Con(**ZFC**) + ¬**CH**. Since m_1 satisfies Con(**ZFC**), it thinks—again via forcing—that there is a model $m_2 = \langle d_2, v_2 \rangle$ of **ZFC** + **CH**.

Now, from the perspective of m_1, there is a full model $m_3 = \langle d_2, v_3 \rangle$ of $\mathscr{L}^2_{\in}$, where v_3 is any valuation function that agrees with v_2 on the first-order fragment of the language. It follows that m_3 satisfies

ZFC* + **CH**. Note that, in virtue of the Upward Löwenheim-Skolem Theorem, we can assume that d_2 is uncountable from m_1's perspective.

Since m_1 satisfies **ZFC** and ¬**CH**, it thinks by Theorem 3 that every full model of $\mathscr{L}^2_\in$ satisfies **NCH2**. So m_1 thinks that m_3 satisfies **NCH2**. This implies that m_3 satisfies ¬**CH2**, as shown by the following reasoning. Suppose for contradiction that m_3 satisfies **CH2**. Because m_3 also satisfies **NCH2**, it must satisfy $\neg\exists X$ ALEPH-1(X). However, m_1 thinks that d_2 is uncountable and hence that its powerset contains an aleph-1 set. By Lemma 1, this contradicts the fact that m_3 satisfies $\neg\exists X$ ALEPH-1(X).

Thus m_1, an ω-model, thinks that the theory **ZFC*** + **CH** + ¬**CH2** is consistent, as witnessed by m_3. But again, consistency facts are arithmetical facts. Hence the theory **ZFC*** + **CH** + ¬**CH2** is consistent. □

The Negative Theorems can be generalized to the second-order closure of any first-order theory which has an ω-model over which we can force **CH** and its negation. In particular, they apply to the second-order closures of von Neumann-Bernays-Gödel and Morse-Kelley set theory.

We now consider what happens when set theory and higher-order resources interact in the metatheory. In particular, we show that CH2 and CH are equivalent over ZFC2. To establish these results, we need a number of preliminary lemmas. Note that the following results are all syntactic. We work in ZFC2.

The first set of lemmas shows that if a set a has one of the cardinality properties with which we are concerned, then so does X_a.

Lemma 4. *If a is infinite, then X_a is* INFINITE. *Moreover, if a is finite, then X_a is* FINITE.

Proof. Suppose a is infinite. So there is some $b \subset a$ and f such that $b \cong_f a$. By second-order Comprehension using f as parameter, it follows that there is some F such that $X_b \cong_F X_a$, where of course $X_b \sqsubset X_a$. That is, X_a is INFINITE.

For the second part of the lemma, suppose for contradiction that a is finite but X_a is INFINITE. Since X_a is INFINITE, there is some $Z \sqsubset X_a$ and F such that $Z \cong_F X_a$. By second-order Separation, which follows from the Replacement Axiom, Z forms a set and $\{Z\} \subset a$. By second-order Separation again, there is a function f corresponding to F. That is:

$$\forall x \forall y (\langle x, y\rangle \in f \ \leftrightarrow \ Fxy)$$

Thus $\{Z\} \cong_f a$, which contradicts the supposition that a is finite. □

Lemma 5.

(a) *If a is aleph-*0*, then X_a is* ALEPH-0.

(b) *If a is aleph-*1*, then X_a is* ALEPH-1.

(c) *If a is continuum, then X_a is* CONTINUUM.

Proof. For (a), suppose that a is aleph-0. This means that a is infinite and every $b \subseteq a$ is either finite or there is some f such that $b \cong_f a$. By Lemma 4, X_a is infinite. Now consider any $Z \sqsubseteq X_a$. By second-order Separation, $\{Z\}$ forms a set and $\{Z\} \subseteq a$. If $\{Z\}$ is finite, then Z is finite, by Lemma 4. If there is some f such that $\{Z\} \cong_f a$, then by second-order Comprehension there is some F such that $Z \cong_F X_a$. Therefore, X_a is ALEPH-0.

Similar reasoning establishes (b).

For (c), suppose that a is continuum. Then there is some f and b such that $a \cong_f \mathscr{P}(b)$, where b is aleph-0. By (a), X_b is ALEPH-0. Let R be defined by the following condition:

$$\forall x \forall y (Rxy \;\leftrightarrow\; y \in f(x))$$

By second-order Comprehension, R exists. It is easy to verify that R codes a bijection between X_a and the subproperties of X_b. Thus since X_b is ALEPH-0, X_a is CONTINUUM. □

The second set of lemmas shows that if X has one of the cardinality properties with which we are concerned, then so does $\{X\}$ (if it exists).

Lemma 6. *Assume that $\{X\}$ exists. If X is* INFINITE, *then $\{X\}$ is infinite. If X is* FINITE, *then $\{X\}$ is finite.*

Proof. Immediate from Lemma 4. □

Lemma 7. *Assume that $\{X\}$ exists. Then:*

(a) *If X is* ALEPH-0, *then $\{X\}$ is aleph-*0.

(b) *If X is* ALEPH-1, *then $\{X\}$ is aleph-*1.

(c) *If X is* CONTINUUM, *then $\{X\}$ is continuum.*

Proof. For (a), suppose that X is ALEPH-0. Since X is INFINITE, by Lemma 6, $\{X\}$ is infinite. Let $b \subseteq \{X\}$. Since X is ALEPH-0 and $X_b \subseteq X$, then either X_b is FINITE or there is some F such that $X_b \cong_F X$. If X_b is FINITE, then by Lemma 6, b is finite. If there is some F such that $X_b \cong_F X$, then it follows from second-order Separation that there is some f such that $b \cong_f \{X\}$. Thus $\{X\}$ is aleph-0.

Similar reasoning establishes (b).

For (c), suppose that X is CONTINUUM. Then there is a relation R that codes a bijection between X and the subproperties of B, where B is ALEPH-0. Since $\{X\}$ exists, it follows from second-order Comprehension, Separation and Replacement that $\{B\}$ exists. For, using R as a parameter in second-order Comprehension, we can define a relation R' restricting R to the codes of subproperties of B satisfied by a single object. By second-order Separation, the domain of R' forms a set $c \subseteq \{X\}$. By second-order Replacement, the range of R' forms a set coextensive with B. Thus $\{B\}$ exists. Since B is ALEPH-0, it follows from Lemma 7 that $\{B\}$ is aleph-0. Now, second-order Separation implies that each subproperty of B forms a set and that there is some f such that $\{X\} \cong_f \mathscr{P}(\{B\})$. So $\{X\}$ is continuum. □

First Positive Theorem. *If* CH2, *then* CH.

Proof. Suppose CH2, that is, for every X, X is ALEPH-1 if and only if X is CONTINUUM. Let a be an arbitrary set. To establish CH, it suffices to show that a is aleph-1 if and only if a is continuum.

Assume that a is aleph-1. By Lemma 5, X_a is ALEPH-1. It follows from CH2 that X_a is CONTINUUM. Lemma 7 then implies that a is continuum. The other direction is proved similarly. □

In the proof of the second Positive Theorem, we invoke the following Lemma, which can be easily established using appropriate instances of second-order Comprehension.

Lemma 8. *If A and B are both* ALEPH-0, *then there is a bijection between A and B. The same holds if A and B are both* ALEPH-1 *or* CONTINUUM.

The proof of this lemma uses *Property Comparability*, that is, the statement that for any two properties A and B, there is either an injection of A into B or an injection of B into A. Note that this is a higher-order statement and does not follow from its set-theoretic counterpart, Cardinal Comparability. However, it *is* equivalent over the axioms of ZFC2 to the second-order version of the Axiom of Choice.[13]

[13] In the context of ZFC, three principles are famously known to be equivalent: the Axiom of Choice, the Well-Ordering Principle, and Cardinal Comparability. With regard to their second-order counterparts, the situation is as follows. In the context of pure second-order logic, the Global Well-Ordering Principle, that is, the second-order statement that the universe can be well-ordered, implies Property Comparability, but it is not known whether the converse

Second Positive Theorem. *If* CH, *then* CH2.

Proof. Suppose CH. ZFC proves that there is a set a that is aleph-0. Now, $\mathscr{P}(a)$ is continuum and hence, by CH, aleph-1. By Lemma 5, we have that X_a is ALEPH-0, and $X_{\mathscr{P}(a)}$ is ALEPH-1 and CONTINUUM.

Let X be an arbitrary property. We want to show that X is ALEPH-1 if and only if it is CONTINUUM. For the left-to-right direction, suppose that X is ALEPH-1. Given that X and $X_{\mathscr{P}(a)}$ are both ALEPH-1, it follows from Lemma 8 that there is some F such that $X \cong_F X_{\mathscr{P}(a)}$. This implies that X is CONTINUUM, given that $X_{\mathscr{P}(a)}$ is CONTINUUM.

The right-to-left direction is proved similarly. □

Counterparts of the above theorems hold for **NCH2** and the negation of **CH**. The proofs of the two Negative Theorems can be easily adapted. As before, we work in ZFC with the assumption that there is an ω-model of ZFC. Then we have:

Theorem 9. ZFC* *does not prove that if* **NCH2**, *then* **¬CH**.

Theorem 10. ZFC* *does not prove that if* **¬CH**, *then* **NCH2**.

Similarly, reasoning in ZFC2, one can prove counterparts of the the two Positive Theorems.

Theorem 11. *If* NCH2, *then* ¬CH.

Theorem 12. *If* ¬CH, *then* NCH2.

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holds. However, we do know that the second-order version of the Axiom of Choice does not imply the Global Well-Ordering Principle (Shapiro, 1991, pp. 107–8). Therefore, it cannot be the case both that the second-order version of the Axiom of Choice implies Property Comparability and that Property Comparability implies Global Well-Ordering. But these three second-order principles are equivalent over the ZFC2 axioms.

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