Fine aspects of pluripotential theory

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Chapter 1

Introduction

In this introductory chapter we briefly review some properties of plurisubharmonic functions and pluripolar sets. Moreover we give a quite detailed description of the contents of this thesis in Section 1.5.

1.1 Subharmonic Functions

From a historical perspective subharmonic functions appeared first in the remarkable paper of Friedrich Hartogs [74]. However, Hartogs did not give a name to this new class of functions. It was F. Riesz [119] who first used the french term subharmonique and started a systematic study of these functions.

In this section we recall some basic properties of subharmonic functions in $\mathbb{R}^n$. For a complete treatment of the subject we refer to [2, 36, 76, 77, 82].

Definition 1.1.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $\varphi : \Omega \rightarrow [-\infty, +\infty]$ is said to be subharmonic in $\Omega$ if

a) $\varphi$ is upper semi-continuous, i.e, $\limsup_{x \to a} \varphi(x) \leq \varphi(a)$ for each point $a \in \Omega$;

b) for every point $a \in \Omega$ and every $r > 0$ such that $B(a, r) \subset \Omega$, the value $\varphi(a)$ does not exceed the average value of $\varphi$ on the sphere $\partial B(a, r)$:

$$\varphi(a) \leq \frac{1}{\sigma_n(\partial B(a, r))} \int_{\partial B(a, r)} \varphi(x) d\sigma_n,$$  \hspace{1cm} (1.1.1)

where $\sigma_n$ is the Lebesgue measure on the sphere $\partial B(a, r)$.

We shall let $\text{SH}(\Omega)$ designate the set of subharmonic functions on $\Omega$. Since we are mainly interested in the complex case, i.e, $\mathbb{C}^n$, $n \geq 1$, Examples of subharmonic functions are postponed to the next section. Also, in order not to say everything twice we will only provide here a brief discussion and move on to the next section.

Condition a) in the above definition amounts to $\{x \in \Omega : \varphi(x) < c\}$ being open for each $c \in \mathbb{R}$. Hence, by a simple covering argument, one can show that $\varphi$ attains its maximum on each compact subset $K \subset \Omega$. Moreover, the submean
value property (1.1.1) forces subharmonic functions to “attain” their maximum on the boundary. Explicitly, the following maximum principle holds.

**Theorem 1.1.2.** If $\Omega$ is a bounded connected open subset of $\mathbb{R}^n$, and if $\varphi \in \text{SH}(\Omega)$, then either $\varphi$ is constant or, for each $x \in \Omega$,

$$
\varphi(x) < \sup_{z \in \partial \Omega} \left\{ \limsup_{y \to z, y \in \Omega} \varphi(y) \right\}.
$$

(1.1.2)

The following properties follow immediately from the definition of subharmonic functions:
1) $\text{SH}(\Omega)$ is a convex cone, i.e $c_1 u_1 + c_2 u_2 \in \text{SH}(\Omega)$ for any $c_1, c_2 \geq 0$ and any $u_1, u_2 \in \text{SH}(\Omega)$.

2) If $u_1, \ldots, u_2 \in \text{SH}(\Omega)$, then the function $u(x) = \max(u_1(x), \ldots, u_2(x))$ belongs also to $\text{SH}(\Omega)$.

3) The limit of a uniformly convergent or monotonically decreasing sequence of subharmonic functions is subharmonic.

The upper semi-continuity in Definition 1.1.1 is the appropriate condition for certain key results (e.g property 3) above). Noteworthy is that although subharmonic function can be highly discontinuous in general, they possess some interesting continuity properties, see Theorem 2.3.16.

It can be proved that replacing inequality (1.1.1) in the above definition by the following one, yields an equivalent definition

$$
\varphi(a) \leq \frac{1}{\lambda_n(B(a, r))} \int_{B(a, r)} \varphi(x) d\lambda_n,
$$

(1.1.3)

Here $d\lambda_n$ is the Lebesgue measure in $\mathbb{R}^n$. Observe that subharmonic functions are allowed to take the value $-\infty$. In fact, the sets of the form $\{ \varphi = -\infty \}$, where $\varphi \neq -\infty$, are called polar and they play an important role in potential theory and complex analysis. Observe that using (1.1.3), one can easily prove the following.

**Theorem 1.1.3.** If $\Omega \subseteq \mathbb{R}^n$ is connected and $\varphi \in \text{SH}(\Omega)$, then either $\varphi \equiv -\infty$ or $\varphi \in L^1_{\text{loc}}(\Omega)$.

Here $L^1_{\text{loc}}(\Omega)$ denotes the family of locally Lebesgue integrable functions in $\Omega$.

As a consequence of Theorem 1.1.3, polar sets have Lebesgue measure zero. A very particular class of polar subsets of $\mathbb{R}^n$, the so-called pluripolar sets, will be introduced and discussed in section 1.3. In fact, pluripolar sets will be a central topic in the next chapters.

The next theorem characterizes subharmonic functions in terms of the Laplace operator.

**Theorem 1.1.4.** If $\Omega \subseteq \mathbb{R}^n$ is open and $\varphi \in \text{SH}(\Omega)$ such that $\varphi \neq -\infty$, then $\Delta \varphi$ computed in the sense of distribution theory is a positive measure. Conversely, if $v \in L^1_{\text{loc}}(\Omega)$ is such that $\Delta v \geq 0$ in $\Omega$ in the sense of distributions, then there exists a function $\varphi \in \text{SH}(\Omega)$ such that $\varphi = v$ almost everywhere in $\Omega$. 
The function $\varphi$ in the second part of the above theorem is obtained as a limit of convolutions of $v$ with the standard smoothing kernels. We must also point out that $\varphi$ need not coincide everywhere with $v$: for example, if $v$ is the characteristic function of a compact subset $K$ of $\Omega$ of measure 0, then $\varphi \equiv 0$.

In potential theory one often has to take upper envelopes of subharmonic functions. This method, called balayage or sweeping out, goes back to H. Poincaré [111] and was systematically studied by Brelot [15]. The method of balayage has been widely applied and successfully used in different areas. For example, the harmonic measure as a "solution" of a particular Dirichlet problem is obtained via this method, and it serves as a decisive tool with applications ranging from complex analysis to probability theory. This method of balayage partly owes its success to the following key property of subharmonic functions known as the fundamental convergence theorem of Cartan:

**Theorem 1.1.5.** If $\{u_\alpha, \alpha \in A\} \subset \text{SH}(\Omega)$ is locally bounded above on $\Omega$, and if $U(x) = \sup_{\alpha \in A} u_\alpha(x)$, then the function

$$U^*(x) := \limsup_{y \to x} U(y)$$

is again subharmonic on $\Omega$. Moreover, the set $\{U < U^*\}$ is a polar subset of $\Omega$.

The function $U^*$ is called the upper semi-continuous regularization of $U$. We note in passing that in axiomatic potential theory the assertion of Theorem 1.1.5 is sometimes taken as an axiom [27, page 100].

### 1.2 Plurisubharmonic Functions

The aforementioned paper of Hartogs [74] from 1908 contains some fundamental results on the theory of holomorphic functions of several variables, including the striking fact that when $n > 1$ holomorphic functions can not have isolated singularities nor isolated zeros. More precisely, on $\mathbb{C}^n$ for $n > 1$, any analytic function $F$ defined on the complement of a compact set $K$ extends (necessarily uniquely) to an analytic function on $\mathbb{C}^n$. Thus, whereas every plane domain carries noncontinuous holomorphic function (i.e., a domain of holomorphy), there are domains in $\mathbb{C}^n$, $n > 1$, that do not enjoy this property. This Hartogs extension phenomenon has led to the question of characterizing the domains of holomorphy.

E. E. Levi discovered around 1912 that a domain of holomorphy must satisfy a certain convexity condition known as pseudoconvexity. He then asked if this condition characterizes domains of holomorphy. This question turned out to be very difficult and became known as the Levi problem. See Kiselman’s survey [79].

In the course of studying the Levi problem Kiyoshi Oka [108] introduced the notion of plurisubharmonic function who then cracked the problem in 1942, first in dimension two and later in all dimensions, see [79]. Lelong [84] independently introduced plurisubharmonic functions in the same year, and subsequently studied them in great detail. Nowadays, the theory of plurisubharmonic functions, labelled
pluripotential theory, has become a major field of research with a wide variety of applications. See e.g., [90] and [103]. Klimek’s monograph [80] provides a good introduction to the theory of plurisubharmonic functions. See also [77].

**Definition 1.2.1.** Let \( \Omega \subset \mathbb{C}^n \) be a domain. A function \( \varphi : \Omega \rightarrow [-\infty, +\infty] \) is said to be plurisubharmonic (psh in short) in \( \Omega \) if 

a) \( \varphi \) is upper semi-continuous

b) for every complex line \( L \subset \mathbb{C}^n \), \( \varphi|_{\Omega \cap L} \) is subharmonic on \( \Omega \cap L \)

If \( \varphi \) and \( -\varphi \) are plurisubharmonic, we say that \( \varphi \) is pluriharmonic in \( \Omega \). The set of plurisubharmonic functions on \( \Omega \) is denoted by \( \text{PSH}(\Omega) \). Note that if \( \Omega \) is a plane domain, i.e., \( n = 1 \), then \( \text{PSH}(\Omega) = \text{SH}(\Omega) \).

An equivalent way of stating property b) is: for all \( a \in \Omega \), \( b \in \mathbb{C}^n \), such that 

\[ \{ a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1 \} \subset \Omega, \]

we have

\[ \varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + e^{i\theta}b) d\theta. \]  

(1.2.1)

An integration of (1.2.1) over \( b \in \partial B(a,r) \) shows that (1.1.1) in Definition 1.1.1 holds. Hence \( \text{PSH}(\Omega) \subset \text{SH}(\Omega) \subset L^1_{\text{loc}}(\Omega) \).

It is immediate that the properties 1), 2) and 3) from Section 1.1 for subharmonic functions are valid also for plurisubharmonic ones.

If \( \varphi \) is twice continuously differentiable in \( \Omega \), then \( \varphi \) is psh if and only if for each \( z \in \Omega \) and \( b \in \mathbb{C}^n \), the Laplacian of \( \varphi \) \( \varphi(z + \lambda b) \) is nonnegative at \( \lambda = 0 \) (cf. Theorem 1.1.4), i.e.,

\[ \frac{\partial^2 \varphi}{\partial \lambda \partial \bar{\lambda}}(z + \lambda b) = \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z + \lambda b)b_j \bar{b}_k \geq 0, \quad \forall z \in \Omega, \forall b \in \mathbb{C}^n. \]  

(1.2.2)

Therefore, \( \varphi \) is plurisubharmonic on \( \Omega \) if and only if the complex Hessian

\[ \left[ \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z) \right] \]  

(1.2.3)

of \( \varphi \) is positive semidefinite at every point \( z \in \Omega \). We should mention that this characterization is still valid for arbitrary psh functions, namely, the analogue of Theorem 1.1.4 with the Laplacian replaced by the “distributional” Hessian matrix holds.

It is well known that convex functions in \( \mathbb{R}^{2n} \) that are twice continuously differentiable are characterized by the positivity of the real Hessian. Thus plurisubharmonic functions may be viewed as a generalization of the former. In fact we have the following.

**Example 1.2.2.** Any continuous convex function \( \varphi \) in \( \Omega \subset \mathbb{C}^n \) (with respect to the underlying real variables) is plurisubharmonic in \( \Omega \). Indeed, it suffices to
integrate the following inequality with respect to $d\theta/2\pi$ to get the submean value (1.2.1).

$$\varphi(a) \leq \frac{\varphi(a + e^{i\theta}b) + \varphi(a - e^{i\theta}b)}{2}. $$

More interesting examples of psh functions include $|f|$ and $c \log |f|$ for a constant $c > 0$ and a holomorphic function $f$ in $\Omega$. In fact, this class of functions generates locally the set $\text{PSH}(\Omega)$ in the following way.

**Theorem 1.2.3 (Bremermann [18]).** Every function $\varphi \in \text{PSH}(\Omega)$ is given locally as

$$\varphi(z) = \limsup_{w \to z} \left(\limsup_{j \to \infty} \frac{1}{j} \log |f_j(w)|\right), $$

(1.2.4)

for some sequence $\{f_j\}$ of holomorphic functions.

It should be mentioned here that for a plane domain $\Omega$, this theorem was discovered by Lelong [83].

For an upper semi-continuous function the second property in Definition 1.2.1 says that $\varphi \circ f$ should be subharmonic wherever it is defined, for all affine mappings $f : \mathbb{C} \to \mathbb{C}^n$. It is a remarkable fact that this property implies that $\varphi \circ f$ is subharmonic also for every holomorphic mapping $f$. Conversely, we have the following

**Theorem 1.2.4.** A function $\varphi : \Omega \to [-\infty, +\infty]$ defined on an open set $\Omega \subset \mathbb{C}^n$ is plurisubharmonic in $\Omega$ if and only if $\varphi \circ F$ is subharmonic in $F^{-1}(\Omega)$ for every $\mathbb{C}$-linear isomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$.

Theorem 1.2.3, and 1.2.4 explain already why psh functions are the right class of subharmonic functions to be studied in complex analysis.

## 1.3 Pluripolar Sets

A set $E \subset \mathbb{C}^n$ is called pluripolar if for every point $z \in E$ there is an open subset $\Omega \subset \mathbb{C}^n$ containing $z$ and a function $\varphi \in \text{PSH}(\Omega)$ ($\varphi \neq -\infty$) such that $E \cap \Omega \subset \{\varphi = -\infty\}$. These sets were first introduced and studied by Lelong [85, 86] who asked whether this local definition is equivalent to the global one; that is, given a pluripolar set $E \subset \mathbb{C}^n$, does there exist a globally defined plurisubharmonic function $\varphi \in \text{PSH}(\mathbb{C}^n)$ such that $E \subset \{\varphi = -\infty\}$? This problem has been called the first Lelong problem and was around for more than two decades. An affirmative answer was given by Josefson [78] with the use of delicate polynomial estimates. See also [77, page 285-290] for a more transparent version of Josefson’s proof.

Trivial examples of pluripolar sets are the zero sets of holomorphic functions, the intersection of such sets, and real-analytic curves, see [122, page 68-72]. However, despite the close connection (see, Theorem 1.2.3) between plurisubharmonic functions and holomorphic ones, pluripolar sets can be very nasty and far more...
complicated in structure than complex analytic varieties. In fact, there are examples of pluripolar sets that contain no non-trivial analytic variety and even no finely analytic curves. See [91] and Chapter 6.

In view of the important role in classical potential theory of the Cartan’s result, cf. Theorem 1.1.5, it is natural to ask whether the corresponding result for plurisubharmonic functions holds; that is, if \( \{ u_\alpha, \ \alpha \in A \} \subset \text{PSH}(\Omega) \) is a family of plurisubharmonic functions which are locally uniformly bounded above on \( \Omega \), and if \( U(z) = \sup_{\alpha \in A} u_\alpha(z) \), is then the set \( \{ U < U^* \} \) pluripolar? This problem (called the second Lelong problem) appeared first in the work of Lelong [87, 88] who called these kind of sets négligeables and studied them thoroughly. However, although Lelong succeeded to settle the problem in the particular case where \( U^*(z) \) is pluriharmonic, cf. [88, proposition 7], the problem turned out to be very hard in the general case, and had to await for further development of pluripotential theory.

In their pioneering paper [4], Bedford and Taylor introduced and studied the Monge-Ampère capacity. And they showed in particular that the zero sets of this capacity are precisely the pluripolar sets. This provided the missing instrument for the study of pluripolar sets, a new proof of Josefson’s theorem, and a solution to the second Lelong problem.

Since \( \text{PSH}(\Omega) \subset \text{SH}(\Omega) \), pluripolar sets are polar, and hence have Lebesgue measure zero in view of Theorem 1.1.3. Moreover, pluripolar sets should be conceived as even "smaller" since they have zero Monge-Ampère capacity. On the other hand, in an answer to a question by Bedford, Diederich and Fornaess [33] constructed a \( C^\infty \)-smooth, real, closed curve \( \gamma \) in \( \mathbb{C}^2 \) which is not pluripolar. In particular, the Hausdorff dimension of \( \gamma \) is equal to 1. This reveals a substantial difference between the notion of pluripolarity in \( \mathbb{C}^n \) and that of polarity in \( \mathbb{R}^{2n} \). Indeed, it is a well known result, cf. [2, page 156-159], that the Hausdorff dimension of a polar set \( P \subset \mathbb{R}^k \), \( k \geq 3 \) is smaller or equal to \( k - 2 \). In the opposite direction, every set \( P \subset \mathbb{R}^k \), \( k \geq 3 \) of Hausdorff dimension strictly smaller than \( k - 2 \) is necessarily a polar set. It is noteworthy that Carleman [20, Theorem 1] has constructed a non-polar set \( E \subset \mathbb{C} \) which has Hausdorff dimension zero. The product set \( E \times E \) is evidently non-pluripolar sets in \( \mathbb{C}^2 \), and again with Hausdorff dimension zero, see also [6, 81].

Pluripolar sets are an essential ingredient in the study of the growth of entire functions in \( \mathbb{C}^n \), \( n > 1 \); a well known result asserts that the exceptional set of the partial order of growth of an entire function is pluripolar, see [90]. In the opposite direction, Zeriahi [139] showed that every complete pluripolar set (see Section 1.4) \( E \subset \mathbb{C}^n \) of type \( F_\nu \) is the exceptional set of the partial order of the growth of some entire function in \( \mathbb{C}^n \). In approximation theory, Sadullaev [123] proved that every holomorphic function on a domain of holomorphy \( D \subset \mathbb{C}^n \) is rapidly approximable by rational functions if and only if the complement of \( D \) is a pluripolar set. Furthermore, pluripolar sets are usually removable for plurisubharmonic functions, holomorphic functions and closed positive currents. However, for a pluripolar set to be removable, it is sometimes necessary to require that it is complete. See [51, 126] and the next section.
1.4 Complete Pluripolar Sets and Thinness

A set $E \subset \mathbb{C}^n$ is *pluri-thin* at a point $a \in \mathbb{C}^n$ if and only if either $a$ is not a limit point of $E$ or there exist $r > 0$ and a plurisubharmonic function $\varphi$ on $B(0, r)$ such that

$$\limsup_{z \to a, z \in E \setminus \{a\}} \varphi(z) < \varphi(a).$$

We say that $E$ is pluri-thin if it is pluri-thin at all of its points. In such a case $E$ is negligible, cf. [4, Theorem 7.1], hence pluripolar, cf. [22, Theorem 2.1]. In contrast to the situation in classical potential theory (see Example 2.2.13), pluripolar sets are not necessarily pluri-thin as is shown by the following simple example.

Example 1.4.1. If $E = \{(z, w) \in \mathbb{C}^2 : |z| < 1, w = 0\}$, then $E$ is pluripolar but $E$ is not pluri-thin at any of its points.

Observe that the set $E$ in this example is contained in the complex line $\{w = 0\}$. Thus, one just uses property b) in Definition 1.2.1 together with the fact that a subharmonic function on $\{w = 0\}$ can not "jump" at points of $E$, see Subsection 2.2.2. Notice that this argument rests on the set $E$ being big in some complex analytic variety. There is a criterion of pluri-thinness involving pluriharmonic measure, cf. [5, page 228], which has limited applications, since the pluriharmonic measure is not easily computed. Thus usually, it is hard to determine whether a set $E$ is a pluri-thin at a given point $z$. See Theorem 1.4.6 and Theorem 1.4.7.

We shall now confine the discussion to pluripolar sets. The question: "When is a pluripolar set $E$ pluri-thin at a given point $z$?" hides a whole series of problems. We need some definitions.

Definition 1.4.2. Let $\Omega$ be an open subset of $\mathbb{C}^n$. A pluripolar set $E \subset \Omega$ is called *complete pluripolar* in $\Omega$ if there exists $\varphi \in \text{PSH}(\Omega)$ such that

$$E = \{z \in \Omega : \varphi(z) = -\infty\}.$$ 

In view of the upper semi-continuity of psh functions, a complete pluripolar set must be a $G_\delta$ set. In the classical potential theory of $\mathbb{R}^k$, a well known result of Choquet [24], see also [82, Theorem 3.1], asserts that a polar set $E \subset \mathbb{R}^k$ is complete polar in $\mathbb{R}^k$ if and only if $E$ is polar and a $G_\delta$ set. The proof of this is heavily based on the Riesz decomposition theorem, which ensures that every polar set $E \subset \mathbb{R}^k$ is contained in the infinity set of some potential. The situation for pluripolar sets is far more complicated. There are easy examples of $G_\delta$ pluripolar sets that are not complete in $\mathbb{C}^n$. Moreover, since plurisubharmonic functions are assumed to be subharmonic when restricted to lower dimensional complex manifolds, the pluripolar sets will obviously exhibit a propagation behavior as the following simple example illustrates. See [135].

Example 1.4.3. Let $E = \{(z, w) \in \mathbb{C}^2 : |z| = 1, w = 0\}$. The function $\log |w| \in \text{PSH}(\mathbb{C}^2)$ equals $-\infty$ on $E$, so $E$ is pluripolar. Moreover $E$ is a $G_\delta$. 
However, every function $\varphi \in \text{PSH}(\mathbb{C}^2)$ which is equal to $-\infty$ on $E$ will be equal to $-\infty$ on the larger set $\mathbb{C} \times \{0\}$. Indeed, the function $\psi(z) = \varphi(z, 0)$ is subharmonic in $\mathbb{C}$ and is equal to $-\infty$ on the circle $|z| = 1$. By the maximum principle $\psi(z) = -\infty$ for every $|z| \leq 1$. Theorem 1.1.3 implies now that $\psi(z) = -\infty$.

Trivial examples of complete pluripolar sets in $\mathbb{C}^n$ are graphs of entire functions. Indeed, if $f(z)$ is a holomorphic function in $\mathbb{C}^n \setminus \{0\}$, then the function $(z, w) \mapsto \log|w - f(z)|$ is plurisubharmonic in $\mathbb{C}^n$ and assumes the value $-\infty$ on the graph

$$
\Gamma_f(\mathbb{C}^{n-1}) = \{(z, f(z)) : z \in \mathbb{C}^{n-1}\},
$$

but nowhere else. This means that $\Gamma_f(\mathbb{C}^{n-1})$ is a complete pluripolar set in $\mathbb{C}^n$. Less obvious examples of complete pluripolar sets will be given below.

The study of completeness of pluripolar sets has naturally led to the following concept.

**Definition 1.4.4.** Let $E$ be a pluripolar set in $\mathbb{C}^n$. The **pluripolar hull** of $E$ relative to an open subset $\Omega$ of $\mathbb{C}^n$ is the set

$$
E^*_\Omega = \{z \in \Omega : \text{for all } \varphi \in \text{PSH}(\Omega) : \varphi|_E = -\infty \implies \varphi(z) = -\infty\}.
$$

For example, it is clear that the pluripolar hull of the compact set $E$ in example 1.4.3 is the set $\mathbb{C} \times \{0\}$. An example of a compact pluripolar set $K \subset \mathbb{C}^2$ such that $K^* \subset_\Omega K$ is dense in $\mathbb{C}^2$ may be found in [6]. Let us remark that $(E \cup F)^*_\Omega = E^*_\Omega \cup F^*_\Omega$. Moreover the "star" operation is idempotent; that is, $(E^*_\Omega)^*_\Omega = E^*_\Omega$.

If a point $z \in \Omega$ does not belong to $E^*_\Omega$, then $E$ is evidently pluri-thin at $z$. Whether the converse holds is not known.

Clearly, if $E$ is complete pluripolar in $\Omega$, then $E$ is a $G_\delta$ and $E = E^*_\Omega$. It is not known whether the converse is true. However, in [139] Zeriahi proved the following.

**Theorem 1.4.5.** Let $E$ be a pluripolar subset of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and suppose that $E = E^*_\Omega$. If $E$ is a $G_\delta$ and an $F_\sigma$ set, then $E$ is complete pluripolar in $\Omega$.

The notion of the pluripolar hull was first introduced and studied by Zeriahi in [139]. The paper [93] of Levenberg and Poletsky contains a more detailed study of this concept, and a proof of the following theorem.

**Theorem 1.4.6.** Let $E = \{(t, t^\alpha) \in \mathbb{C}^2 : 0 < t \leq 1\}$ with $\alpha$ irrational. Then $E^*_\mathbb{C}^2 = \{(z, z^\alpha) \in \mathbb{C}^2 : z \neq 0\}$, where $z^\alpha$ runs over all possible values. In particular, $E$ is pluri-thin at the origin.

This solved an old problem of Sadullaev [122], who observed already that the set $E$ is not pluri-thin at the origin when $\alpha$ is rational. This is because the curve $E$ is real analytic at $t = 0$ in this case. See [122, Proposition 4.1]. The set $E^*_\mathbb{C}_z$ in Theorem 1.4.6 is not a $G_\delta$. Hence it is not complete.

In the same direction Wiegerinck [133] proved the following theorem which answers a second question of Sadullaev [122].
Theorem 1.4.7. Let $E = \{(t, e^{-1/t}) \in \mathbb{C}^2 : 0 < t \leq 1\}$. Then $E^*_{\mathbb{C}^2} = \{(z, e^{-1/z}) \in \mathbb{C}^2 : z \neq 0\}$. In particular, $E$ is pluri-thin at the origin.

These two theorems should be compared with Theorem 2.2.15. Also, observe that the graph $(E^*_{\mathbb{C}^2})$ of $e^{-1/z}$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$ in view of Zeriahi's theorem. In [133], it was actually proved that Theorem 1.4.7 remains valid if $e^{-1/z}$ is replaced by certain other holomorphic functions with an essential singularity at 0. However, soon after this, Wiegerinck [134] discovered the following general result. See also [38].

Theorem 1.4.8. Suppose that $D$ is a domain in $\mathbb{C}$ and $A$ a sequence of points in $D$ without density point in $D$. Let $f$ be holomorphic on $D \setminus A$, and not extendible over $A$. Let $E$ denote the graph of $f$ in $D \setminus A \times \mathbb{C}$. Then $E$ is complete pluripolar in $D \times \mathbb{C}$.

It should be mentioned at this point that the above results are related to questions of analytic continuation of holomorphic functions in $\mathbb{C}^n$. To illustrate this connection let us quote here the following result, known as the strong "disc" theorem, of Bremermann [19]. See also [54].

Theorem 1.4.9. Let $z(t) : [0,1] \rightarrow \mathbb{C}^n$ be a curve contained in a complex line $L \subset \mathbb{C}^n$, and let the function $f(z,w), z \in \mathbb{C}^n, w \in \mathbb{C}$, be holomorphic in a domain $\Omega \subset \mathbb{C}^n$, containing the set $D(t) = \{(z,w) : z = z(t), |w| < \rho\}$ for $0 < t < 1$ and $\rho > 0$. Suppose that the function $f(z(0),w)$ is holomorphic at at least one point of the set $D(0) = \{(z,w) : z = z(0), |w| < \rho\}$. Then $f(z(0),w)$ is holomorphic everywhere in $D(0)$.

The real analytic curves in Theorem 1.4.6 and Theorem 1.4.7 are not analytically continuable at the origin. This shows that non-analytic continuation might be the obstruction to the propagation of pluripolar sets. This point of view has been tested on graphs of holomorphic functions too. Indeed, inspired by a fundamental example of Sadullaev [122], Poletsky, Levenberg and Martin showed that the graphs of many lacunary series are complete pluripolar in $\mathbb{C}^2$. This has led them to conjecture that if $f$ is a holomorphic function, that is defined on its maximal domain of existence $D \subset \mathbb{C}$, then the graph $\Gamma_f$ of $f$ is complete pluripolar in $\mathbb{C}^2$.

A break through was achieved by Edigarian and Wiegerinck [39] by showing that the above conjecture fails. Their counterexample is given by a function

$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{z - a_j},$$

where $\{a_n\}_{n=1}^{\infty}$ is a countable and dense sequence, say in the boundary $\partial \mathbb{D}$ of the unit disk $\mathbb{D}$. Choosing $c_j$ to be very rapidly decreasing to 0, and using very precise harmonic measure estimates, they showed that there is a point $a \in \partial \mathbb{D}$ for which $(a, f(a)) \in (\Gamma_f(\mathbb{D}))^*_{\mathbb{C}^2}$. Here $\Gamma_f(\mathbb{D})$ is the graph of $f$ over $\mathbb{D}$.
This result gave a substantial impulse to the study of the pluripolar hulls of graphs of holomorphic functions. Siciak [125] observed that the function $f$ above has a pseudocontinuation $\tilde{f}$ across almost every point of the unit circle, and showed that the graph $\Gamma_f$ of $\tilde{f}$ is contained $(\Gamma_f(\mathbb{D}))_{C^2}$. In the opposite direction, he gave an example of a holomorphic function $g$ in the unit disk which has no pseudocontinuation, while $(\Gamma_g(\mathbb{D}))_{C^2} \neq \Gamma_g(\mathbb{D})$.

In [44], Edlund and Jöricke observed subsequently that the function $f$ of Edigarian and Wiegerinck is actually finely holomorphic (see, Section 2.4) in a set larger than the unit disk. In other words, the function $f$ has a finely holomorphic extension $\tilde{f}$ beyond the unit disk. Similarly to the above quoted result of Siciak, they showed that $(\Gamma_f(\mathbb{D}))_{C^2}$ contains a part of the graph of the finely analytic extension $\tilde{f}$, see Theorem 5.1.1. Thus, for a holomorphic function $g$, say in the unit disk $\mathbb{D}$, fine analytic continuation is a sufficient condition for $(\Gamma_g(\mathbb{D}))_{C^2}$ to be strictly larger than $(\Gamma_g(\mathbb{D}))$. Whether this is also a necessary condition remains unknown. It is however worth mentioning that a partial result in this direction was obtained by T. Edlund in his thesis [43]. See also [42] for a related result.

One of our goals in this thesis is to investigate to what extent the theory of finely holomorphic functions of one and several variables is related to pluripolar hulls. However, no theory of finely holomorphic functions of several variables exists. To develop it one must start with a study of the plurifine topology and finely plurisubharmonic functions. Thus we arrive at the plan of this thesis below.

1.5 Overview of the Thesis

Many of the results in this thesis were obtained in collaboration with Jan Wiegerinck. The contents is based on the following papers.


To get a quick idea about the contents of this thesis, it is illuminating to notice that all our results were obtained in the course of studying the following two problems.

**Problem 1.** Study the pluri-fine topology in $\mathbb{C}^n$ and develop the theory of finely plurisubharmonic and pluri-finely holomorphic functions.

Having some grip on Problem 1 allows us to attack Problem 2

**Problem 2.** Study the pluripolar hulls and understand their structure.
1.5. Overview of the Thesis

Problem 1 is also related to the study of the complex Monge-Ampère operator [5], and the more precise pointwise behavior of psh functions.

Let us state the following two problems which were presented at the Problem Session during a conference at Hanstholm, cf. [69, page 195-196]

**Question 1.** (Fuglede) Is the pluri-fine topology locally connected?

**Question 2.** (P. M. Gauthier) Let $K$ be a compact subset of $\mathbb{C}^n$ and let $PS(K)$ denote the set of functions on $K$ which can be uniformly approximated by functions which are continuous and plurisubharmonic on (neighborhoods of) $K$. Give a characterization of functions in $PS(K)$.

In the case $n = 1$, a rich theory of finely subharmonic (resp. finely holomorphic) functions was developed in the period 1969-1981, and Questions 1 and 2 were solved.

In Chapter 2 we will survey the development of fine potential theory, and finely holomorphic functions. This will serve as a source of inspiration and a model for our study of Problem 1. It may be convenient for the reader as well. Chapters 3-7 contain the original work of the thesis.

1.5.1 Fine Pluripotential Theory

The \textit{pluri-fine topology} on an open set $\Omega \subseteq \mathbb{C}^n$ is the coarsest topology on $\Omega$ that makes all plurisubharmonic functions on $\Omega$ continuous. We shall use the notations from Subsection 3.1.1.

In Chapter 3 we show that the pluri-fine topology has pleasant connectedness properties. It is locally connected (Theorem 3.3.4), and a usual open set $U \subseteq \mathbb{C}^n$ is $\mathcal{F}$-connected if and only if $U$ is connected in the usual topology. This gives an affirmative answer to Question 1 above. See also [6, 65].

As a consequence of the local connectedness and the so-called quasi-Lindelöf property, cf. Theorem 3.2.5, it follows quite easily that every $\mathcal{F}$-open set has a countable number of $\mathcal{F}$-connected components. Moreover, these components are $\mathcal{F}$-open. Our proof of the local connectedness relies ultimately on two properties of finely subharmonic functions in $\mathbb{C}$: Theorem 2.3.9, and Lemma 2.3.13 (the gluing lemma).

Next we introduce an analogue of plurisubharmonic functions on $\mathcal{F}$-domains.

A natural definition is as follows: a real function $f$ on an $\mathcal{F}$-open set $U \subseteq \mathbb{C}^n$ is said to be \textit{finely plurisubharmonic} in $U$ (or $\mathcal{F}$-plurisubharmonic) if it is upper semicontinuous in the pluri-fine topology and finely subharmonic on each complex line where it is defined. The key ingredients in the study of these functions are contained in the following result which is proved in Chapter 3.

**Theorem 1.5.1.** Let $U \subseteq \mathbb{C}^n$ be an $\mathcal{F}$-open subset and let $a \in U$. Then there exists a constant $\kappa = \kappa(U, a)$ and an $\mathcal{F}$-neighborhood $V \subseteq U$ of $a$ with the property that for any complex line $L$ through $z \in V$ the $\mathcal{F}$-component $C_{z,L}$ of the $\mathcal{F}$-open set $U \cap L$ that contains $z$, contains a circle about $z$ with radius at least $\kappa$.

The proof of Theorem 1.5.1 relies on a classical harmonic measure estimate of
A. Beurling and R. Nevanlinna, and leads to a second proof of the local connectedness of the pluri-fine topology. See Theorem 3.4.10 and Remark 3.3.7.

In Chapter 4 we study $\mathcal{F}$-plurisubharmonic functions. The first non-trivial result reads as follows.

**Theorem 1.5.2.** Let $f$ be an $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega$. If $f = -\infty$ on an $\mathcal{F}$-open subset $U$ of $\Omega$, then $f \equiv -\infty$.

Theorem 1.5.2 has interesting applications. Firstly, the $\mathcal{F}$-plurisubharmonic functions satisfy the maximum principle in a sense made precise in Theorem 4.2.6. Secondly, pluripolar sets don’t separate $\mathcal{F}$-domains, cf. Theorem 4.2.4. Finally, Theorem 1.5.2 can be applied to study pluripolar hulls, cf. [49].

Furthermore we prove in Section 4.3 that every bounded $\mathcal{F}$-plurisubharmonic functions can be $\mathcal{F}$-locally written as the difference of two usual plurisubharmonic functions, cf. Theorem 4.3.1. As a consequence of this, we show that $\mathcal{F}$-plurisubharmonic functions are pluri-finely continuous (not just pluri-finely upper semicontinuous, by definition). This means that there is no larger “plurifine-plurifine” topology. See Theorem 2.3.12 for the finely subharmonic case.

Another consequence of this $\mathcal{F}$-local decomposition of $\mathcal{F}$-plurisubharmonic functions and Theorem 1.5.2 is the following Theorem, which gives an affirmative answer to a question in [47].

**Theorem 1.5.3.** Let $f$ be an $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega$ such that $f \not\equiv -\infty$. Then the set $\{f = -\infty\}$ is pluripolar subset of $\Omega$.

Note that since $\mathcal{F}$-plurisubharmonic functions in a planar domain are finely subharmonic, our proof of the above theorem provides a second proof of Theorem 2.3.14.

Theorem 1.5.3 will be applied to questions about pluripolar hulls. This is discussed in the next subsection.

### 1.5.2 Applications to Pluripolar Hulls

Here we shall outline the progress we have made so far in studying Problem 2.

To formulate the main result of Chapter 5 we need the following definition.

**Definition 1.5.4.** Let $U \subseteq \mathbb{C}^n$ be $\mathcal{F}$-open. A function $f : U \rightarrow \mathbb{C}$ is said to be $\mathcal{F}$-holomorphic if every point of $U$ has a compact $\mathcal{F}$-neighborhood $K \subseteq U$ such that the restriction $f|_K$ belongs to $H(K)$.

As was pointed out at the end of the preceding section, Theorem 1.5.3 allows us to prove the following result, cf. Theorem 5.3.1, and Theorem 5.4.3.

**Theorem 1.5.5.** Let $U \subset \mathbb{C}^n$ be an $\mathcal{F}$-domain. Let $f$ be $\mathcal{F}$-holomorphic in $U$. Then the zero set of $f$ is pluripolar. In particular, the graph of $f$ is also pluripolar. Moreover, if $E$ is a non-pluripolar subset of $U$, then $\Gamma_f(U) \subset (\Gamma_f(E))_{\mathbb{C}^{n+1}}$.

The above theorem generalizes the following result, which was obtained in collaboration with Edigarian and Wiegerinck, cf. Theorem 5.4.1.
**Theorem 1.5.6.** Let \( f : U \rightarrow \mathbb{C}^n, \ f(z) = (f_1(z), \ldots, f_n(z)), \) be a finitely holomorphic map on an \( F \)-domain \( U \subseteq \mathbb{C}. \) Then the image \( f(U) \) of \( U \) is a pluripolar subset of \( \mathbb{C}^n. \) Moreover, if \( E \) is a non polar subset of \( U, \) then the pluripolar hull of \( f(E) \) contains \( f(U). \)

It is interesting to observe that the set \( U \) in the above theorems need not have any Euclidean interior points.

It turns out that the arguments in the proof of Theorem 1.5.6 can be equally well applied to arbitrary pluripolar hulls, rather than the particular case of graphs, see Propositions 5.5.1.

Chapter 6 is joint work with Tomas Edlund. Here we constructed an example, cf. Theorem 6.2.4, of a compact pluripolar set \( Y \) which hits every graph of a finitely holomorphic function in a polar set, and yet \( Y \neq (Y)^{\circ} \mathbb{C}^2. \) This initial example is further elaborated to construct a pluripolar set with a very large pluripolar hull without fine analytic structure. Explicitly, we have the following.

**Theorem 1.5.7.** For each proper non polar subset \( S \subset \mathbb{C} \) there exists a pluripolar set \( E \subset (S \times \mathbb{C}) \) with the property that \( E^{\circ} \mathbb{C}^2 \) contains no fine analytic structure and the projection of \( E^{\circ} \mathbb{C}^2 \) onto the first coordinate plane equals \( \mathbb{C}. \)

The set \( E \) in the above theorem is a subset of a complete pluripolar set \( X \subset \mathbb{C}^2, \) which is constructed in the same spirit as Wermer’s polynomially convex compact set without analytic structure. What is actually used is that \( X \) is the graph of an analytic multifunction.

Finally, in Chapter 7 we illustrate our results by examples and present some open problems.