Chapter 2

Fine Potential Theory

Since fine potential theory has proven to be a powerful tool in studying the pluri-
fine topology and questions related to pluripolar sets, we found it convenient to
write a chapter summarizing the basic properties of finely (sub) harmonic func-
tions and finely holomorphic ones. The theory of finely harmonic functions was
developed by Fuglede [56] in the general framework of abstract harmonic spaces.
However, we shall restrict our exposition to the particular case of a Green set
$\Omega \subset \mathbb{R}^n$, i.e, an open set having a Green function. This amounts, in view of
Merberg’s theorem, to $\Omega$ being an arbitrary open set if $n > 2$ or $\Omega$ has non-polar
complement if $n = 2$. Furthermore, since the case $n > 2$ is not relevant to the sub-
ject of the next chapters, and for the sake of simplicity, we will confine ourselves to
the case where $\Omega \subset \mathbb{C}$ is an open subset of the complex plane with non-polar com-
plement. Nevertheless, all the results presented in Sections 2.2 and 2.3.2, unless
otherwise indicated, are valid also in the general case, i.e, $n \geq 2$.

2.1 Introduction

Potential theory in the complex plane $\mathbb{C}$ can be roughly characterized as the theory
of the Laplace operator
\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\] (2.1.1)

A function $h$ is said to be harmonic on the open set $\Omega \subset \mathbb{C}$, if $h$ has continuous
second partial derivatives on $\Omega$ and $\Delta h(z) = 0$ for all $z \in \Omega$.

A considerable part of potential theory was developed in the course of studying
the Dirichlet problem and related questions. The problem may be stated, in its
simplest form, as follows: for a given continuous function $f : \partial \Omega \rightarrow \mathbb{R}$, determine,
if possible, a harmonic functions $h$ on $\Omega$ such that $h(z) \rightarrow f(\zeta)$ as $z \rightarrow \zeta$ for each
$\zeta \in \partial \Omega$. Such a function $h$ is called the solution of the Dirichlet problem on $\Omega$ with
boundary function $f$, and the maximum principle guarantees the uniqueness of the
solution if it exists. The set $\Omega$ is said to be regular, provided that the Dirichlet
problem has a solutions for all continuous function on $\partial \Omega$. Simple examples of regular sets are a disk or a half-plane. For these particular sets, the solution is even given by a nice integral representation involving the Poisson kernel, see [2, 76]. On the other hand, there are quite simple examples of non-regular sets. The first such a simple example goes back to S. Zaremba [138].

**Example 2.1.1.** If $\Omega = \mathbb{D}(0, \frac{1}{2}) \setminus \{0\}$ is the punctured unit disk in $\mathbb{C}$, and $f : \partial \Omega \rightarrow \mathbb{R}$ is defined by $f(0) = 1$ and $f(\zeta) = 0$ when $|\zeta| = 1$, then it is an immediate consequence of the removable singularity theorem and the maximum principle that there is no harmonic function on $\Omega$ with the preassigned boundary values.

Thus Zaremba discovered that domains with isolated boundary points are not regular. In other words, the Dirichlet problem does not always have a solution for such domains. A non trivial example (Lebesgue spine) of a non-regular domain in $\mathbb{R}^3$ was given by H. Lebesgue in (1913), cf. [76].

It was Lebesgue who explicitly proposed to "separate" the investigation of the Dirichlet problem into two parts: Firstly to produce a harmonic function depending in a way on the given boundary function and then investigate the boundary behavior of the resulting candidate for a solution. The classical Perron-Wiener-Brelot method (PWB-method for short) does associate to each boundary function $f$ such a candidate function (denoted by $H_f$) as the supremum of a certain family of subharmonic functions associated to $f$. See e.g. [2, 76]. This powerful method has brought in the concept of a regular boundary point. A boundary point $x \in \partial \Omega$ is called regular if for any continuous function $f$ on $\partial \Omega$, $\lim_{z \to x} H_f(z) = f(x)$. Otherwise $x$ is called an irregular boundary point. In other words, an irregular point of the boundary of a domain $\Omega$ is a point at which the continuity of the PWB solution may be violated. For instance, the origin in Example 2.1.1 is irregular and the PWB solution is the constant $H_f \equiv 0$. See [117, chapter 4].

In the first half of the 20th century quite an extensive study of the irregular boundary points was carried out. A major achievement was Wiener's characterization of these points in terms of capacity in 1924, cf. Theorem 2.2.16. This famous criterion of irregularity turned out to be useful and has certainly contributed to further development of potential theory. However, in general it is hard to compute or estimate the capacity of a given set, and therefore Wiener’s criterion is not easily applicable in some practical situations. The need to find more manageable criteria for irregular points perhaps motivated Marcel Brelot to introduce the notion of thinness of a set $E \subset \mathbb{C}$ at a point $x \in \mathbb{C}$. In 1939 he defined $E$ to be thin at $a$ if either $a$ is not a limit point of $E$, or else if there exists a subharmonic function $\varphi$ in a neighborhood of $a$ such that

$$\limsup_{x \to a, \ x \in E \setminus \{a\}} \varphi(x) < \varphi(a).$$ (2.1.2)

Observe that since $\varphi$ is upper-semicontinuous, we always have $\leq$ in (2.1.2).

Brelot showed that for a closed set $E$ which contains $x$, this definition amounts to $x$ being an irregular boundary point for the Dirichlet problem in the complement of $E$. 
Immediately after, H. Cartan observed, in a letter to Brelot, that an equivalent definition of thinness would be to say that $E$ is thin at $x$ if and only if the complement of $E\setminus \{x\}$ is a neighborhood of $x$ in the weakest topology on $\mathbb{C}$ that makes all subharmonic functions continuous. Since this topology is strictly finer (in case $n > 1$) than the Euclidean one, Cartan called it just the fine topology.

It is fairly easy to see that the fine topology is Hausdorff, completely regular and Baire. However, as we will see in Example 2.2.11, the open sets in this topology can be rather nasty and awkward. In particular they need not have any Euclidean interior points. Moreover the fine topology has no countable base (hence not metrizable) and it has no infinite compact sets.

These latter “shortcomings” have discouraged (for a while) any further interest in this topology. And for quite some time, the fine topology was only regarded as a mean of expressing results more elegantly. For instance, an irregular boundary point of a domain $\Omega$ is now simply a finitely isolated point of the complement $E = \mathbb{C} \setminus \Omega$. However, in 1954 J.L. Doob [34] discovered a spectacular link between the fine topology and probability theory. He proved that a Borel set $\Omega$ is open in the fine topology if and only if a Brownian particle, starting at a point of $\Omega$, remains in $\Omega$ with probability 1 through some positive interval of time. About a decade later, Doob [35] showed that the fine topology is quasi-lindelöf in a sense made precise in Theorem 2.2.20.

In a question raised by Ch. Berg, B. Fuglede [55] proved in 1969 that the fine topology is locally connected and enjoys other interesting connectedness properties, cf. Section 2.2.3. This was the starting point for Fuglede for developing fine potential theory which is exposed in his book from 1972, cf. [56]. Thus despite the absence of non-trivial finely compact sets and the failure of the countability axioms, it turned out that a major part of classical potential theory can be extended to fine domains. We will provide an amount of this theory in Section 2.3.

Shortly after that fine potential theory was established, many authors struggled to develop a kind of complex analysis on fine domains in $\mathbb{C}$. The first definition of a “holomorphic” function on a fine domain $U \subseteq \mathbb{C}$ was given by Fuglede [60] in 1974, cf. Definition 2.4.1. A second definition was proposed in the same year by Debiard and Gaveau [29]; they called a function finely holomorphic on a finely open set $U$ if it is finely harmonic and satisfies the Cauchy-Riemann equation $\partial f = 0$ a.e in $U$ in the sense of stochastic differentiation along Brownian paths. Fuglede’s definition seemed to be less manageable and had the drawback that it did not allow for a simple proof that the algebra of these functions is closed under uniform limits. Thus, Lyons [97, 98] and Nguyen-Xuan-Loc [106, 107] adopted the second definition. Using probabilistic methods combined with the theory of uniform algebra, Lyons obtained several interesting results. And, few years later, together with A. G. O’Farrell, Lyons discovered the missing link with Fuglede’s approach by proving that the two definitions are equivalent. Fuglede resumed the study of fine holomorphy in [62, 61] using analytic methods instead of probability. His theory is based on the theory of quasi-continuous Beppo-Levi functions [32], and the Cauchy-Pompeiu transform of square integrable functions, relying ultimately on a key lemma of T. Lyons, cf. [61, Lemma 6]. We will summarize the properties of
finely holomorphic functions in Section 2.4 and discuss their connection to Borel’s monogenic functions.

2.2 The Fine Topology

As mentioned in the introduction the fine topology was introduced by H. Cartan in 1940 as the coarsest topology on \( \mathbb{C} \) which makes every subharmonic function on \( \mathbb{C} \) continuous. In this section we sum up some basic properties of this topology describing the ground on which fine potential theory is built. [2, 36, 76, 55, 58, 67]

Throughout this chapter (and Chapter 6), all topological notions referring to the fine topology will be qualified by the term "fine(ly)" to distinguish them from those pertaining to the usual (Euclidean) topology on \( \mathbb{C} \). For example, finely open means open in the fine topology. Note however that for the pluri-fine topology in \( \mathbb{C}^n \) in chapters 3, 4 and 5 the prefix "\( \mathcal{F}-\)" is used instead of "fine(ly)".

For any set \( A \subseteq \mathbb{C} \), let us use \( \overline{A} \) to designate the complement of \( A \). We denote by \( \overline{A}^\circ, \overline{A} \) and \( \partial_f A \) the fine interior, the fine closure and the fine boundary of \( A \) respectively. The set of finely isolated points of \( A \) is denoted by \( i(A) \), and

\[
\overline{b}(A) = \overline{A} \setminus i(A)
\]

designates the finely derived set of \( A \). Following Brelot’s terminology, we call \( b(A) \) the base of \( A \). Of course, this concept of a base of a set should not be confused with the usual topological notion of a base.

2.2.1 Base Elements and Properties

When studying topological properties of a given topological space, a knowledge of a base is usually necessary. However, a base is not unique. In fact an infinite number of bases, even of different "sizes", may generate the same topology. Thus, when studying some specific topological problem one base might be more suitable than another one. In this subsection we discuss different bases and derive some useful properties of the fine topology.

It is immediate that the fine topology on \( \mathbb{C} \) has a subbase consisting of all finely open sets of the form \( \{ z \in \mathbb{C} : \varphi(z) > 0 \} \) or \( \{ z \in \mathbb{C} : \varphi(z) < 0 \} \), where \( \varphi \in \text{SH}(\mathbb{C}) \). Sets of the last kind, however, are open in the Euclidian topology, in view of the upper semi-continuity of \( \varphi \). By considering the harmonic functions \( z \mapsto \pm \Re z \) en \( z \mapsto \pm \Im z \) we see that the open cubes (hence also usual open sets) are generated by finite intersection of sets of the form \( \{ z \in \mathbb{C} : \varphi(z) > c \} \), \( c \in \mathbb{R} \). This proves the following.

**Proposition 2.2.1.** Finite intersections of sets of the form

\[
B_c^\varphi = \{ z \in \mathbb{C} : \varphi(z) > 0 \},
\]

where \( \varphi \in \text{SH}(\mathbb{C}) \), constitute a base of the fine topology on \( \mathbb{C} \).
Since for any $a \in \mathbb{C}$ the function $z \mapsto |z - a|$ is subharmonic on $\mathbb{C}$, the open balls $B(a, r)$ are finely open for any $r > 0$. In particular, as may be seen from the above discussion, the fine topology is finer than the Euclidean one. In fact, we shall later give an example which shows that it is strictly finer.

If $\Omega$ is an open subset of $\mathbb{C}$, then the class $\text{SH}(\Omega)$ is a priori larger than the class consisting of those subharmonic functions on $\Omega$ that are restrictions to $\Omega$ of functions from $\text{SH}(\mathbb{C})$. Nevertheless, if $\varphi \in \text{SH}(\Omega)$, then $\varphi$ is finely continuous in $\Omega$. This follows from the fact that for any disk $D \subset \subset \Omega$ the function $\varphi|_D$ can be extended so as to be subharmonic in $\mathbb{C}$, see [76, Lemma 7.13]. We have therefore

**Proposition 2.2.2.** Let $\Omega \subset \mathbb{C}$. Then the restriction to $\Omega$ of the fine topology coincides with the coarsest topology on $\Omega$ in which every subharmonic function in $\Omega$ is continuous.

Using the above mentioned lemma from [76] one can describe a base of the fine topology as follows.

**Theorem 2.2.3.** Finite intersections of sets of the form

$$B_{B(a, r)}^\varphi = \{z \in B(a, r) : \varphi(z) > 0\},$$

where $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$, $\varphi \in \text{SH}(B(a, r))$, constitute a base of the fine topology on $\mathbb{C}$.

A detailed proof of this theorem is given in [80, page 178]. However, we shall give in the next chapter (Lemma 3.3.1) an easy proof to the fact that sets of the form $B_{B(a, r)}^\varphi$ constitute even a base of the fine topology.

As a corollary to the above theorem we have the following useful description of a local base.

**Corollary 2.2.4.** Let $\zeta \in \mathbb{C}$. Then a fine neighborhood base of $\zeta$ is given by sets of the form

$$\bigcap_{j=1}^n \{z \in \overline{B} : \varphi_j(z) \geq -1\},$$

where $B$ is a ball containing $\zeta$ and $\varphi_j \in \text{SH}(\overline{B})$ with $\varphi_j(\zeta) = 0$.

Notice that the sets in (2.2.1) are compact in the usual topology. In other words, the fine topology has a neighborhood base consisting of Euclidean compact set. This base will turn out to be extremely useful in many situations. For instance, relying on this it is an easy exercise to prove the following.

**Theorem 2.2.5.** The fine topology in $\mathbb{C}$ is Baire; that is, if $\{U_n\}$, $n \in \mathbb{N}$ is a countable collection of finely open finely dense sets in $\mathbb{C}$, then the set $\bigcap_n U_n$ is dense in $\mathbb{C}$.

Another interesting base of the fine topology which is less encountered in the literature is given by the following theorem, cf. [96, Theorem 2.3].
Theorem 2.2.6. The following collection is a base of the fine topology
\[ B = \{ \{ \varphi > \psi \} : \varphi, \psi \in SH(\mathbb{C}), \text{ with } \varphi \geq \psi \}. \quad (2.2.2) \]

Proof. By Proposition 2.2.1, the collection of sets \( \{ \varphi > 0 \}, \varphi \in SH(\mathbb{C}) \) is a subbase of the fine topology. Clearly, the following (larger) family \( \{ \varphi > \psi \} : \varphi, \psi \in SH(\mathbb{C}) \) is also a subbase. As \( \{ \varphi > \psi \} = \{ \max(\varphi, \psi) > \psi \} \), the collection \( B \) is a subbase of the fine topology. To see that \( B \) is a base it suffices to check that it is stable under taking finite intersection. This is settled by the following observation
\[ \{ \varphi_1 > \psi_1 \} \cap \{ \varphi_2 > \psi_2 \} = \{ \varphi_1 + \varphi_2 > \max(\varphi_1 + \psi_2, \varphi_2 + \psi_1) \}, \quad (2.2.3) \]
where \( \varphi_j \) and \( \psi_j, j = 1, 2, \) are subharmonic on \( \mathbb{C} \).

Remark 2.2.7. It follows from the above theorem that a non-empty finely open set must contain infinitely many points. Indeed, let \( \varphi, \psi \in SH(\mathbb{C}) \), and let \( a \in \{ \varphi > \psi \} \). Then there is an \( l \in \mathbb{R} \) such that \( a \in \{ \varphi > l \} \cap \{ l > \psi \} \subset \{ \varphi > \psi \} \). Since the set \( \{ l > \psi \} \) is open in the usual topology, it suffices to check that \( \{ \varphi > l \} \) contains infinitely many points near \( a \). This follows from the mean-value inequality for subharmonic functions, cf. Inequality (1.1.1). See also Remark 2.2.19 below.

Since subharmonic functions separate points, the fine topology is Hausdorff. The following corollary gives a stronger separation property.

Corollary 2.2.8. The fine topology is completely regular; that is, if \( A \subset C \) is a finely closed set and \( a \in \{ A \} \), then there exists a finely continuous function \( f \) from \( C \) to the real line \( \mathbb{R} \) such that \( f|_A = 0 \) and \( f(a) > 0 \).

Proof. By Theorem 2.2.6, there exists \( \varphi, \psi \in SH(\mathbb{C}) \) such that \( \varphi \geq \psi \) and \( a \in \{ \varphi > \psi \} \subset \{ A \} \).

The function \( f(z) := \varphi(z) - \psi(z) \) satisfies the assertion of the corollary.

A number of properties of the fine topology follow from the following well known result, cf. [36, page 58].

Theorem 2.2.9. Let \( E \) be a polar subset of \( \mathbb{C} \) and \( a \in \mathbb{C} \). Then there exists a subharmonic function \( \varphi \in SH(\mathbb{C}) \) such that \( \varphi = -\infty \) on \( E \setminus \{ a \} \) while \( \varphi(a) \neq -\infty \).

We should mention at this point that, although still valid for the potential theory of \( \mathbb{R}^n \), Theorem 2.2.9 has no analog in pluripotential theory. This is because pluripolar sets in \( \mathbb{C}^n \), \( n > 1 \), often propagate. In fact, understanding this phenomenon of propagation constitutes the central problem in this thesis.

Corollary 2.2.10. 1) If \( E \subset \mathbb{C} \) is polar, then \( E \) is finely closed. Moreover, \( E \) has no fine limit points in \( \mathbb{C} \), i.e. \( E \) is finely discrete.
2) A set is finely compact if and only if it is finite.
3) The fine topology is not separable.
4) The fine topology is not first countable; that is, no point of \( \mathbb{C} \) has a countable fundamental system of fine neighborhoods.
5) The fine topology is not metrizable.
Proof. 1) Let \( a \in \mathbb{C} \setminus E \) and let \( \varphi \in \text{SH}(\mathbb{C}) \) be the function provided by Theorem 2.2.9. Then the set \( \{ \varphi > \varphi(a) - 1 \} \) is a fine neighborhood of \( a \) that does not intersect \( E \). Hence \( \mathbb{C} \setminus E \) is finely open.

By the first assertion in 1), the polar set \( E \setminus \{ z \}, z \in \mathbb{C} \), is finely closed. Hence no point \( z \) in \( \mathbb{C} \) can be a fine accumulation point of \( E \).

2) Any finite set is finely compact. Conversely, suppose that \( K \) is an infinite finely compact subset of \( \mathbb{C} \). Then there is an infinite countable sequence \( (a_n)_{n \geq 1} \) in \( K \). The set \( V_k = \mathbb{C} \setminus \bigcup_{n \neq k} \{a_n\} \) is finely open, because countable sets are polar and hence finely closed in view of 1). Observe now that \( \{ V_k : k \in \mathbb{N} \} \) forms a cover of \( K \) which has no finite subcover.

3) Countable sets are polar, hence finely closed by 1).

4) Suppose that \( U_n, n \geq 1 \), is a countable fundamental system of fine neighborhoods of a point \( z \). In view of Remark 2.2.7, for each \( n \), there exists \( z_n \in U_n \setminus \{ z \} \). Since countable sets are polar and hence finely closed by 1), the set \( \mathbb{C} \{ z_n : n \geq 1 \} \) is a fine neighborhood of \( z \) which can not contain any \( U_n \).

5) follows from 4).

Example 2.2.11. 1) Let \( (a_n)_{n \geq 1} \) be a sequence of points in the punctured unit disk \( \mathbb{D}(0,1) \setminus \{0\} \) that converges to 0, and form the series

\[
\varphi(z) = \sum_{k=1}^{\infty} \frac{\log |z - a_k|}{2^k \log |a_k|}, \quad z \in \mathbb{C}. \quad (2.2.4)
\]

Then \( \varphi \in \text{SH}(\mathbb{C}) \), \( \varphi(a_k) = -\infty \) for every \( k \geq 1 \) and \( \varphi(0) = 1 \). This function is discontinuous at 0. Accordingly, the fine topology is effectively strictly finer than the Euclidean one. Observe also that the sequence \( (a_n)_{n \geq 1} \) has no fine limit points, in view of Corollary 2.2.10.

2) Let \( (w_n)_{n \geq 1} \) be a countable dense sequence in the closed unit disk \( \overline{\mathbb{D}}(0,1) \), and let \( (a_n)_{n \geq 1} \) be strictly positive numbers such that \( \sum_{n} a_n < \infty \). Define \( \varphi : \mathbb{C} \rightarrow [-\infty, +\infty] \) by

\[
\varphi(z) = \sum_{n \geq 1} a_n \log |z - w_n|, \quad z \in \mathbb{C}. \quad (2.2.5)
\]

Then:

a) \( \varphi \) is subharmonic in \( \mathbb{C} \) and \( \varphi \not\equiv -\infty \).

b) \( \varphi = -\infty \) on an uncountable dense subset of \( \overline{\mathbb{D}}(0,1) \).

c) \( \varphi \) is discontinuous almost everywhere on \( \overline{\mathbb{D}}(0,1) \).

If \( c \in \mathbb{R} \), then \( \{ z \in \mathbb{D}(0,1) : \varphi > c \} \) is, in view of a) and b), a finely open which has no usual interior points. See [117, page 41-42] for the proof of a), b) and c).

3) Let \( E \) be the sequence \( \mathbb{Q} \times \mathbb{Q} \), where \( \mathbb{Q} \) is the set of rational numbers. Then \( E \) is a polar subset of \( \mathbb{C} \) and there exists a subharmonic function \( \varphi \in \text{SH}(\mathbb{C}) \) (\( \varphi \not\equiv -\infty \)) such that \( E \subset \{ \varphi = -\infty \} \). Observe that if \( c \in \mathbb{R} \), then the set \( \{ \varphi > c \} \) is a finely open and again without usual interior points.
2.2.2 Thinness

**Definition 2.2.12.** A subset $E \subset \mathbb{C}$ is said to be thin at a point $a$ if either $a$ is not a limit point of $E$, or else if there exists a subharmonic function $\varphi$ in a neighborhood of $a$ such that

$$\limsup_{z \to a, \; z \in E \setminus \{a\}} \varphi(z) < \varphi(a). \quad (2.2.6)$$

It was proved by Brelot [17] that in the above definition one may choose the function $\varphi$ in such a way that the lim sup in the left hand side of (2.2.6) is $-\infty$ while $\varphi(a) = 0$.

**Example 2.2.13.** If $E \subset \mathbb{C}$ is a polar set, then $E$ is thin at every point $z \in \mathbb{C}$. This follows by application of Theorem 2.2.9. A less obvious and a typical example of a thin set, say at the origin, is given after Theorem 2.2.16.

The following properties follow immediately from the definition of thinness.

**Proposition 2.2.14.** a) The union of a finite number of sets that are thin at a given point is thin at this point.

b) A set $E \subset \mathbb{C}$ is thin at a point $a \in \overline{E}$ if and only if there exists an open set $\Omega \supset E \setminus \{a\}$ which is thin at $a$.

A less obvious and very useful result is the next theorem. Its proof is based on the fact that thinness is invariant under contractions, cf. [76, Theorem 10.14]. Another proof of it is implicit in our proof of Lemma 3.4.4.

**Theorem 2.2.15.** If $E \subset \mathbb{C}$ is thin at $a$, then there are arbitrarily small circle $\partial D(a;r)$ such that $\partial D(a;r) \cap E = \emptyset$.

An immediate consequence of this theorem is that a connected set containing more than one point in $\mathbb{C}$ is not thin at every point of its closure. It should be mentioned here that, as provided by the Lebesgue’s example, cf. [76, page 175], this conclusion and Theorem 2.2.15 fail for the case $\mathbb{R}^n, n \geq 3$. See also Example 3.4.13 in the next chapter.

**Theorem 2.2.16.** (Wiener’s criterion) Let $E \subset \mathbb{C}$, and let $\zeta_0 \in \mathbb{C}$. define

$$E_n = \{z \in E : 2^{-(n+1)} \leq |z - \zeta_0| \leq 2^{-n}\}. \quad (2.2.7)$$

Then $E$ is thin at $\zeta_0$ if and only if

$$\sum_{n \geq 1} \frac{n}{\log(1/\text{Cap}^*(E_n))} < \infty. \quad (2.2.8)$$

Here $\text{Cap}^*(E_n)$ denotes the outer Logarithmic capacity of the set $E_n$. If $\text{Cap}^*(E_n) = 0$, then the corresponding term in the sum (2.2.8) is considered to be equal to zero. See [2, page 218].
Example 2.2.17. Let $z_n = \frac{1}{2} (2^{-n} + 2^{-(n+1)})$, $n = 1, 2, \ldots$, and $r_n = 2^{-n^2}$. Put $E = \bigcup_n D(z_n, r_n)$, where $D(z_n, r_n)$ denotes the disk with center $z_n$ and radius $r_n$. Recall that the Logarithmic capacity of $D(z_n, r_n)$ is equal to $r_n$. Since the following series

$$\sum_{n=1}^{\infty} \frac{n}{\log(1/r_n)}$$

(2.2.9)

is convergent, the Wiener’s criterion shows that $\bigcup_n D(z_n, r_n)$ is thin at 0.

The following theorem of H. Cartan gives a characterization of thiness in terms of the fine topology.

Theorem 2.2.18. A set $E \subset \mathbb{C}$ is thin at a point $a \in \mathbb{C}$ if and only if $a$ is not a fine limit point of $E$. Equivalently, $C \setminus (E \setminus \{a\})$ is a fine neighborhood of $a$.

Notice that the complement $CE$ of the set $E$ in example 2.2.17 is, in view of Theorem 2.2.18, a fine neighborhood of 0 which is closed in the usual topology. See also the observation after Corollary 2.2.4.

Remark 2.2.19. Another interesting topology that should be compared with the fine topology is the so called ordinary density topology (in e.g. $\mathbb{R}^2$). The neighborhoods of a point $a \in \mathbb{C}$ in this topology are the sets $CE (a \notin E)$ such that $a$ is a dispersion point of $E$ with respect to the Lebesgue measure $\lambda$ of $\mathbb{R}^2$. That is,

$$\lim_{r \to 0} \frac{\lambda^*(E \cap B(a, r))}{\lambda(B(a, r))} = 0,$$

(2.2.10)

where $\lambda^*$ is the outer Lebesgue measure. Relying on Wiener’s criterion, Fuglede proved in [55] that if a set $E$ is thin at $a$ ($a \notin E$), then $a$ is a dispersion point of $E$. In other words, the ordinary density topology is finer than the fine topology. In particular, every point of a finely open set has Lebesgue density 1.

The continuous functions with respect to the ordinary density topology, labeled approximately continuous, were introduced and studied by Denjoy [31] in his famous paper from (1915). Denjoy proved, in particular, that these functions are of Baire class 1. However, although the approximately continuous functions were extensively studied immediately after Denjoy’s paper, a systematic study of the density topology has started only in the fifties. The interested reader may find a detailed study of these concepts in [96, Chapter 6]. See also [73].

Theorem 2.2.20. (quasi-Lindelöf property). An arbitrary union of finely open subsets of $\mathbb{C}$ differs from a countable subunion by at most a polar set.

The following corollary will be useful in the next section

Corollary 2.2.21. Any finely open set can be written as the union of a (Euclidean) $F_\sigma$ set and a polar set.

Proof. Let $U$ be a finely open subset of $\mathbb{C}$. By Corollary 2.2.4, $U = \bigcup_{z \in U} K_z^*$, where $K_z^*$ is a compact fine neighborhood of $z$. According to Theorem 2.2.20, there is a countable sequence $z_n$ and a polar set $P$ such that $U = \bigcup_n K_{z_n} \cup P = \bigcup_n K_{z_n} \bigcup P$. \qed
Remark 2.2.22. The statement of the above corollary holds for finely closed sets as well. In fact, a fundamental theorem of Choquet asserts that a finely open (finely closed) set differ from some usual open (closed) by a set of arbitrary small capacity. We shall omit any further discussion in this direction and we refer to [2].

In view of the above remark one can expect that finely continuous functions should not differ much from usual continuous ones. We will discuss this in Section 2.3.2.

2.2.3 Connectedness in the Fine Topology

Recall once more that, except when otherwise indicated, all the results in this chapter are valid in $\mathbb{R}^n, n \geq 2$.

The first serious study of connectedness in the fine topology was carried out by Fuglede [55] around 1969. Answering a question by Ch. Berg, he proved the following.

Theorem 2.2.23. The fine topology on $\mathbb{C}$ is locally connected. Moreover, any usual domain is also a fine domain; that is, finely open and finely connected.

The proof of this theorem is based on deep results from the theory of balayage of measures (to be defined in the next section). An alternative proof which uses the theory of finely harmonic functions may be found in [56, page 87-92]. In the next chapter we give yet another proof of Theorem 2.2.23, cf. Corollary 3.4.7, which is completely elementary.

In view of the quasi-Lindelof property, Theorem 2.2.20, the following corollary is obvious. See also [55].

Corollary 2.2.24. Every finely connected component of a finely open set $U \subset \mathbb{C}$ is finely open. Moreover, the set of these components is at most countable.

Perhaps inspired by the result of J.L. Doob [34] mentioned in the introduction, Nguyen-Xuan-Loc and T. Watanabe [105, Theorem 2.4] characterized in 1972 the fine domains as follows.

Theorem 2.2.25. A necessary and sufficient condition for a finely open set $U \subset \mathbb{C}$ to be finely connected is that, given any point $z \in U$ and any non-empty, finely open set $V \subset U$, there is a positive probability that a Brownian particle starting at $z$ should reach the set $V$ before it (possibly) leaves $U$.

Since Brownian trajectories are continuous (and with probability 1 even Lipschitzian of order $< \frac{1}{2}$), it is quite easy, as observed by Fuglede [58, page 263], to prove the following.

Corollary 2.2.26. Every fine domain is arcwise connected in the usual topology.

Remark 2.2.27. In the same paper, Fuglede gave an alternative proof of Corollary 2.2.26 without recourse to probability. He made use of Theorem 2.2.15 in an essential way. Even stronger, he proved that any two points in a fine domain
2.3 Finely Subharmonic Functions

$U \subset \mathbb{C}$ can be joined by a finite polygonal path in $U$. Since Theorem 2.2.15 breaks down in $\mathbb{R}^n$, $n \geq 3$, the polygonal connectedness for a fine domain $U \subset \mathbb{R}^n$, $n \geq 3$ remained open.

Using probabilistic methods, Lyons [97] proved polygonal connectedness and brought the following precision to Corollary 2.2.26 (also valid for $U \subset \mathbb{R}^n$, $n \geq 3$). See also [67] for a non-probabilistic proof.

**Theorem 2.2.28.** Every point of a fine domain $U \subset \mathbb{C}$ has a fine neighborhood $V \subset U$ such that any two point $z, w \in V$ can be joined by a polygonal path in $U$ consisting of just two straight segments $[z, \zeta]$ and $[\zeta, w]$ of equal length, and such that the angle between either of them and the segment $[z, w]$ is $\theta$ ($\theta > 0$ being prescribed).

We end this subsection with the following result which will be reconsidered in the next chapter (Theorem 4.2.5). See [56, Theorem 12.2] and [55, Theorem 6].

**Theorem 2.2.29.** Let $U \subset \mathbb{C}$ be a finely open set and $E$ a polar set. Then $U \setminus E$ is finely connected if and only if $U$ is finely connected.

2.3 Finely Subharmonic Functions

Throughout this section $\Omega$ denotes an open subset of the complex plane with non-polar complement. A finely subharmonic function in a finely open set $U \subset \Omega$ is loosely speaking a finely upper semi-continuous function which satisfies the submean value with respect to certain measure called swept-out measure. To give a precise definition we must start with the concept of reduced and swept-out functions leading to this measure. Later on we shall give a characterization of finely subharmonic functions which does not involve the sweeping-out process (cf. Theorem 2.3.20).

2.3.1 Swept-out Measure

Let $\text{SH}^-(\Omega)$ denote the collection of all non-positive subharmonic functions on $\Omega$.

**Definition 2.3.1.** If $u \in \text{SH}^-(\Omega)$ and $E \subseteq \Omega$, then the reduced function (or “réduite”) of $u$ relative to $E$ is defined by

$$R_E^u(z) = \sup\{v(z) : v \in \text{SH}^-(\Omega) \text{ and } v \leq u \text{ on } E\}, \quad z \in \Omega.$$  \hspace{1cm} (2.3.1)

It follows from Theorem 1.1.5 that the upper semicontinuous regularization $\hat{R}_E^u$ is subharmonic on $\Omega$. We call $\hat{R}_E^u$ the swept-out or balayage of $u$ relative to $E$ in $\Omega$. It is the biggest non-positive subharmonic function in $\Omega$ which minorizes $u$ on $E$. When $u$ is the Green potential $G_\mu$ of a measure $\mu$ on $\Omega$, the swept-out function $\hat{R}_E^u$ is the Green potential of a certain measure on $\Omega$ denoted by $\mu^E$ and called the swept-out of $\mu$ on $E$ (relative to $\Omega$). A detailed study of reduced functions was
Chapter 2. Fine Potential Theory

originated by Brelot in [15]. We refer the reader to [2, 76, 36] for a good exposition of the theory of balayage.

Thus this operation of sweeping out of subharmonic functions induces in a natural way a sweeping of measures. In the special case when the measure \( \mu \) is the unit mass placed at some point \( z \), the following result, fundamental for the development of fine potential theory, goes back to Brelot [15], see also [2, page 275].

**Theorem 2.3.2.** If \( E \subseteq \Omega \), then there exists a unique Radon measure \( \varepsilon_z^E \) such that

\[
\hat{R}_u^E(z) = \int u(y) d\varepsilon_z^E(y), \quad (z \in \Omega).
\]  

(2.3.2)

For all \( u \in SH^{-}(\Omega) \).

The measure \( \varepsilon_z^E \) is called the swept-out measure of the Dirac mass \( \varepsilon_z \) onto \( E \).

The next theorem, due to Brelot [13], summarizes some fundamental properties of the swept-out measure \( \varepsilon_z^E \).

**Theorem 2.3.3.** Let \( E \subseteq \Omega \subset \mathbb{C} \). Then

a) \( \varepsilon_z^E \) is carried by \( b(E) \).

b) \( \varepsilon_z^E = \varepsilon_z \) if and only if \( z \in b(E) \).

c) \( \varepsilon_z^E(P) = 0 \) for any polar subset \( P \) with \( z \not\in P \).

These two theorems have led to an extensive theory of balayage of measures, generally carried out in the framework of harmonic spaces by R. M. Hervé [75] and Boboc-Constantinesco-Cornea [9]. See also [27]. The reader not familiar with axiomatic potential theory may find an amount of this theory in Doob [36, Chapter X].

The next proposition gives other interesting properties of the finely harmonic measure. Assertions 1) and 2) are due to Fuglede. 3) is a reformulation of b) in the above theorem.

**Proposition 2.3.4.** Let \( U \subseteq \Omega \) be finely open.

1) If \( U \) is a fine domain, then the measures \( \varepsilon_z^{\Omega \setminus U} \), \( z \in U \), have all the same null sets.

2) \( \varepsilon_z^{\Omega \setminus U} \) is carried by the fine boundary of the finely connected component of \( U \) that contains \( z \).

3) \( \varepsilon_z^{\Omega \setminus U} \neq \varepsilon_z \) if and only if \( \Omega \setminus U \) is thin at \( z \).

It should be mention at this point that the measure \( \varepsilon_z^E \) boils down to the usual harmonic measure of \( \Omega \setminus E \) when \( E \) is closed and \( z \not\in E \), cf, e.g [36, page 157]. Thus, when \( E \) is finely closed \( \varepsilon_z^E \) may be called the finely harmonic measure for the finely open set \( \Omega \setminus E \) relative to \( \Omega \). Incidentally, a recent result of S. Roy [128] asserts that these measures are exactly the extreme Jensen measures for subharmonic functions.
2.3.2 Definition and Basic properties

The submean value property with respect to the measure \( d_{E_z} \), \( z \notin E \) involves integrals such as \( \int_E \varphi d_{E_z} \) where \( E \) is finely closed. The measure \( d_{E_z} \) is a Borel probability measure, but not all finely closed sets are Borel sets. However, since \( d_{E_z} \) puts zero mass on each polar set, cf. Theorem 2.3.3, it is clear that \( d_{E_z} \) has a natural extension to the following \( \sigma \)-algebra

\[
PB = \sigma - \text{algebra generated by the Borel sets and the polar sets.} \tag{2.3.3}
\]

It follows from Corollary 2.2.21 that \( PB \) contains the fine Borel sets; that is, the \( \sigma \)-algebra generated by the finely open sets. In other words, fine Borel sets are measurable with respect to \( E_z \). In particular, every finely semicontinuous function is \( E_z \)-measurable for any \( E \in PB \) and \( z \notin E \).

**Definition 2.3.5.** A function \( \varphi : U \to [-\infty, +\infty] \) defined on a finely open set \( U \) is said to be finely hypoharmonic if

1) \( \varphi \) is finely upper semicontinuous; that is, \( \{ \varphi < c \} \) is finely open for every \( c \in \mathbb{R} \).

2) Those finely open sets \( V \) with closure \( \overline{V} \subset U \) for which

\[
\varphi(z) \leq \int_{\partial V} \varphi d_{E_z}, \quad \forall z \in V. \tag{2.3.4}
\]

form a base for the fine topology in \( U \).

3) If moreover \( \varphi \neq -\infty \) on every finely connected component of \( U \), then we call \( \varphi \) finely subharmonic.

We shall let \( \text{FSH}(U) \) designate the set of finely subharmonic functions on \( U \). It is illuminating to notice that the base in question depends a priori on the function \( \varphi \). Therefore, it is, for instance, by no means obvious that the sum of two finely subharmonic functions is finely subharmonic. However, it is a deep result that for a finely subharmonic function, the submean property 2.3.4 holds for a larger collection of finely open sets than required. Namely we have the following

**Theorem 2.3.6.** [56, Lemma 9.5] Let \( \varphi : U \to [-\infty, +\infty] \) be a finely subharmonic function on a finely open set \( U \). Then

\[
\varphi(z) \leq \int_{\partial V} \varphi d_{E_z}, \quad \forall z \in U, \tag{2.3.5}
\]

for every finely open set \( V \) with \( \overline{V} \subset U \) and such that \( \varphi \) is bounded above in \( \overline{V} \).

About two decades later, Fuglede extended the above theorem (in the planar case) to include every finely open set \( V \) (independent of \( \varphi \)) with fine closure \( \overline{V} \) contained in \( U \). See [68] for a precise formulation.

As an answer to a problem left open by Fuglede in [56], T. Lyons [99] showed by use of Choquet theory that an equivalent definition of fine subharmonicity would
be to replace 2) in Definition (2.3.5) by the following apparently weaker property:

2') For each point \( z \) in \( U \) there is a base \( \mathcal{B}(z) \) of fine neighborhoods of \( z \) in \( U \) such that

\[
\varphi(z) \leq \int_{\partial V} \varphi d\mathcal{E}^z, \quad \text{for all } V \in \mathcal{B}(z). \tag{2.3.6}
\]

The interested reader may also refer to [95] for the equivalence between the various definitions of fine subharmonicity.

For any family \( \{\varphi_n\} \) of finely subharmonic functions on a finely open set \( U \) which is finely locally bounded from above one can select a base \( \mathcal{B}(U) \) of the fine topology on \( U \) such that the inequality (2.3.5) holds for all \( n \) and for every \( V \in \mathcal{B}(U) \). Hence, Theorem 2.3.6 has the following easy consequences:

**Theorem 2.3.7.** [56] 1) \( \text{FSH}(U) \) is a convex cone, i.e \( c_1 u_1 + c_2 u_2 \in \text{FSH}(U) \) for any \( c_1, c_2 \geq 0 \) and any \( u_1, u_2 \in \text{FSH}(U) \).

2) If \( u_1, ..., u_2 \in \text{FSH}(U) \), then the function \( \max(u_1(x), ..., u_2(x)) \) belongs also to \( \text{FSH}(U) \).

3) The limit of a monotonically decreasing sequence of finely subharmonic functions on a fine domain \( U \) is finely subharmonic or identically \(-\infty\).

4) If \( U \) is a fine domain, then \( \text{FSH}(U) \cup \{-\infty\} \) is closed under finely locally uniform convergence.

**Definition 2.3.8.** A function \( \varphi : U \rightarrow \mathbb{R} \) defined on a finely open set \( U \) is called finely harmonic if \( \varphi \) and \( -\varphi \) are both finely subharmonic, or equivalently if \( \varphi \) is finely continuous and those finely open sets \( V \) with closure \( \overline{V} \subset U \) for which

\[
\varphi(z) = \int_{\partial V} \varphi d\mathcal{E}^z, \quad \forall z \in V.
\]

form a base for the fine topology in \( U \).

Using Theorem 2.3.2 one can easily prove that in the case of a usual open set \( U \) any sub(harmonic) function is finely sub(harmonic). The next theorem with the remark that follows establishes the converse.

**Theorem 2.3.9.** [56, Theorem 9.8] Let \( U \) be an open subset of \( \mathbb{C} \). Then a function \( \varphi : U \rightarrow \mathbb{R} \) is subharmonic if and only if \( \varphi \) is finely subharmonic and, moreover, locally bounded from above in the Euclidean topology.

**Remark 2.3.10.** It was proved by Fuglede [57] that one may drop the local boundedness in the above theorem. He also gave examples which prove that local boundedness can not be removed in higher dimensions, i.e., open subset of \( \mathbb{R}^n \), \( n > 2 \).

**Theorem 2.3.11.** [57, Theorem 2.3] Let \( \varphi \) be a finely subharmonic function on a bounded finely open set \( U \subset \mathbb{C} \). If

\[
\ell \limsup_{z \to x, z \in U} \varphi(z) \leq 0, \tag{2.3.7}
\]

for every \( x \in \partial_f U \), then \( \varphi \leq 0 \).
Here and elsewhere $f$-lim sup denotes the lim sup with respect to the fine topology. Theorem 2.3.11 does not hold in $\mathbb{R}^n$, $n > 2$, in this form.

A deep result in fine potential theory, cf. [56, Theorem 9.9] asserts that a finely locally bounded finely subharmonic function can be represented (finely locally) as the difference of two usual subharmonic functions. As a consequence of this we have

**Theorem 2.3.12.** Every finely subharmonic function is finely continuous.

The following results shows that finely subharmonic functions glue together, cf. [56, Lemma 10.1]. This will be one of the decisive tools in the first part of Chapter 3.

**Lemma 2.3.13.** Suppose that $V \subset U \subset \mathbb{C}$ are finely open sets, and let $\psi$ (resp $\varphi$) be a finely subharmonic function on $U$ (resp $V$). Assume that:

$$f \text{- lim sup}_{z \to x, z \in V} \varphi(z) \leq \psi(x) \text{ for all } x \in U \cap \partial_f V.$$ 

Then the following function $\Psi$ is finely subharmonic in $U$:

$$\Psi(z) = \begin{cases} 
\max\{\varphi(z), \psi(z)\} & \text{if } z \in V, \\
\psi(z) & \text{if } z \in U \setminus V.
\end{cases}$$

Recall once more that $f$-lim sup denotes the lim sup with respect to the fine topology, i.e.

$$\inf_{O} \sup_{z \in O} \psi(z),$$

where $O$ ranges over the set of all fine open sets in $V$ which contain $z$.

The following important theorem is a consequence of fundamental properties of the swept-out measure and the quasi-Lindelöf property, cf. Theorem 2.2.20. It will be very useful in the study of pluripolar hulls in Chapter 5.

**Theorem 2.3.14.** [56, page 158]. Let $h : U \mapsto [-\infty, +\infty]$ be a finely subharmonic function on a finely open set $U \subset \mathbb{C}$. Then the set $\{z \in U : h(z) = -\infty\}$ is a polar subset of $U$.

We end this section by the following removable singularity theorem for finely harmonic functions.

**Theorem 2.3.15.** Let $\varphi$ be finely harmonic in $U \setminus E$, where $E$ denotes a polar subset of the finely open set $U$. Then $\varphi$ has a finely harmonic extension to $U$ if (and only if) $\varphi$ is bounded in some fine neighborhood of each point of $E$. The extension is then unique and given by

$$\varphi(a) = f \text{- lim sup}_{z \to a, z \in U \setminus E} \varphi(z), \text{ } a \in E$$

(2.3.8)
2.3.3 Continuity and Fine Differentiability Properties

In view of the striking result (Theorem 2.3.12) that finely subharmonic functions are finely continuous, it is tempting to surmise that finely harmonic functions would at least be finely differentiable. But this is not the case. Thus, while the usual harmonic functions are infinitely differentiable and even real analytic, the finely harmonic ones are not.

As we mentioned in Remark 2.2.19, if \( \varphi \) is finely continuous, then \( \varphi \) is approximately continuous and hence of Baire class 1 in view of Denjoy’s result. In particular, every finely subharmonic function is, by Theorem 2.3.12, of Baire class 1. The next fundamental result establishes that fine continuity is not very far from usual continuity, cf. [60, Lemma 1].

**Theorem 2.3.16.** (The Brelot property) Consider a countable family of finely continuous functions \( f_n : U \rightarrow \mathbb{C} \) (\( U \) finely open in \( \mathbb{C} \)). Every point of \( U \) has a fine neighborhood \( K \subset U \) (\( K \) a standard compact set, if we like) such that the restriction of each \( f_n \) to \( K \) is continuous in the standard topology.

The following result is proved in [30].

**Theorem 2.3.17.** Let \( K \subset \mathbb{C} \) be a compact set, and \( h \) be a continuous function on \( K \). Then the following are equivalent

1) \( f \) is finely harmonic in the fine interior \( K' \) of \( K \).
2) \( f \) can be uniformly approximated on \( K \) by a sequence of usual harmonic functions \( h_n \) defined in Euclidean neighborhoods \( W_n \) of \( K \).

Using a kind of localization result for harmonic approximation, Gauthier and Ladouceur proved in [72] that Theorem 2.3.17 is still valid if one assumes that \( K \) is merely a closed subset of \( \mathbb{R}^n \), \( n \geq 2 \). The following fine local version of Theorem 2.3.17 has been given in [57, Theorem 4.1]

**Theorem 2.3.18.** A function \( h \) defined in a finely open set \( U \subset \mathbb{C} \) is finely harmonic if and only if every point of \( U \) has a compact fine neighborhood \( K \subset U \) such that \( h|_K \) is the uniform limit of usual harmonic functions \( h_n \) defined in Euclidean neighborhoods \( W_n \) of \( K \).

Theorem 2.3.17 was proved by Debiard and Gaëtan using probabilistic methods. In [63], Fuglede devised an alternative proof which also works for the finely subharmonic case. His proof uses the theory of balayage and the Hahn-Banach theorem relying, ultimately, on a deep study of what he calls a "localization operator".

**Theorem 2.3.19.** Let \( K \subset \mathbb{C} \) be a compact set, and \( h \) be a continuous function on \( K \). Then the following are equivalent

1) \( f \) is finely subharmonic in the fine interior \( K' \) of \( K \).
2) \( f \) can be uniformly approximated on \( K \) by a sequence of usual subharmonic functions \( h_n \) defined in Euclidean neighborhoods \( W_n \) of \( K \).
In view of the the Brelot property (Theorem 2.3.16), the following corollary to Theorem 2.3.19 is immediate. As we promised at the beginning of Section 2.3, this may serve as an equivalent definition for fine subharmonicity.

**Theorem 2.3.20.** A function \( h \) defined in a finely open set \( U \subseteq \mathbb{C} \) is finely subharmonic if and only if every point of \( U \) has a compact fine neighborhood \( K \subset U \) such that \( h|_K \) is the uniform limit of usual subharmonic functions \( h_n \) defined in Euclidean neighborhoods \( W_n \) of \( K \).

Let \( f \) be a real valued function on a finely open set \( U \subseteq \mathbb{C} \). We say that \( f \) is finely differentiable at a point \( a \in U \) if there exists a vector \( \nabla f(a) \in \mathbb{C} \) (called the fine gradient of \( f \) at \( a \)) such that

\[
\frac{f(z) - f(a) - \langle z - a, \nabla f(a) \rangle}{|z - a|} \quad (2.3.9)
\]

converges to 0 as \( z \) converges finely to \( a \). (If \( z = x + iy, \ w = u + iv \), then \( \langle z, w \rangle = xu + yv \)). In view of Cartan’s theorem (cf. e.g [76, Theorem 10.15]), this amounts to saying that \( a \) has a fine neighborhood \( V \) such that the above expression converges to 0 as \( z \) converges (in the usual sense) to \( a \) in \( V \).

The following theorem was proved in [64].

**Theorem 2.3.21.** Every finely subharmonic function in a finely open set \( U \) is almost everywhere (w.r.t Lebesgue measure) finely differentiable in \( U \).

A. M. Davie and B. Øksendal [28] constructed an example of a finely harmonic function which is not everywhere finely differentiable.

## 2.4 Finely Holomorphic Functions

As pointed out in the introduction the study of finely holomorphic functions was carried out along two different lines. The methods used by Debiard and Gaveau [29], and likewise by Lyons [97, 98] involved uniform algebras combined with probabilistic tools like stochastic integration and the famous Ito’s formula. We shall avoid this probabilistic approach here and follow the alternative presentation of Fuglede [60, 61, 66]. The later involves analytical methods, notably the Cauchy-Pompeiu transform. It is much more convenient and natural for a complex analyst. Thus our main reference here will be [61]. In fact almost all the results presented below are included in [61].

**In the sequel** \( U \) will always denote a finely open subset of \( \mathbb{C} \).

The characterization of fine harmonicity in Theorem 2.3.18 by finely local uniform approximation by harmonic function suggested in [60] the following definition.

**Definition 2.4.1.** A function \( f : U \to \mathbb{C} \) is called finely holomorphic if every point of \( U \) has a compact (in the usual topology) fine neighborhood \( K \subset U \) such that the restriction \( f|_K \) belongs to \( R(K) \).
Here $R(K)$ denotes the uniform closure of the algebra of all restrictions to $K$ of rational functions on $\mathbb{C}$ with poles off $K$, or equivalently, in view of Runge’s theorem, of holomorphic functions in open neighborhoods of $K$ in $\mathbb{C}$.

Notice that by Definition 2.4.1 a holomorphic function $f$ in a usual domain $U$ is of course finely holomorphic. See also Corollary 2.4.6 below for the converse statement.

To give an idea of how a finely holomorphic function might look like, it is illuminating to start at least with the following simple example which is a slight modification of an example in [61, page 74].

**Example 2.4.2.** Let $a_n > 0$, $n = 1, 2, \ldots$ be a sequence of positive numbers such that $\frac{\sum_{n=1}^{\infty} a_n}{\sum_{n=1}^{\infty} n} \log(1/r_n) < +\infty$. Let $z_n = \frac{1}{2}(2^{-n} + 2^{-n+1})$, $n = 1, 2, \ldots$, and $r_n = 2^{-n^2}$. Put $K = \mathbb{D}(0,1) \setminus \bigcup_n D(z_n, r_n)$, where $D(z_n, r_n)$ denotes the disk about $z_n$ with radius $r_n$. Recall that the Logarithmic Capacity of $D(z_n, r_n)$ is equal to $r_n$. Since the following series

$$\sum_{n=1}^{\infty} \frac{n}{\log(1/r_n)}$$

is convergent, Wiener’s criterion shows that $\bigcup_n D(z_n, r_n)$ is thin at $0 \in K$. Thus $K$ is a (compact) fine neighborhood of $0$, cf. Theorem 2.2.18.

Define

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z - z_n}. \quad (2.4.2)$$

Clearly, this series converges absolutely, and locally uniformly, in the open set $\Omega = \mathbb{C} \setminus \{z_n\}_{n\in\mathbb{N}} \cup \{0\}$. Accordingly, $f$ is holomorphic in $\Omega$ and meromorphic in $\mathbb{C} \setminus \{0\}$. Suppose that $a_n$ has been chosen in such a way that

$$\sum_{n=1}^{\infty} \frac{a_n}{r_n} < +\infty. \quad (2.4.3)$$

Then the series $f_k(z) = \sum_{n=1}^{k} \frac{a_n}{z - z_n}$ of rational functions converges uniformly on $K$ to $f$. This shows that $f$ is finely holomorphic in the fine interior $K'$ of $K$. Thus altogether $f$ is finely holomorphic in $\Omega \cup \{0\}(= \mathbb{C} \setminus \{z_n\}_{n})$.

The next theorem shows that finely holomorphic function are closely related to finely harmonic function.

**Theorem 2.4.3.** A function $f : U \to \mathbb{C}$ is finely holomorphic if and only if $f$ and $z \mapsto zf(z)$ are both finely harmonic in $U$.

The theory of finely harmonic functions from the previous section combined with Theorem 2.4.3 allows us therefore to derive a number of important properties for finely holomorphic functions.

**Corollary 2.4.4.** The uniform limit of finely holomorphic functions in a finely open set $U \subseteq \mathbb{C}$ is finely holomorphic.
2.4. Finely Holomorphic Functions

Proof. Follows from Theorem 2.3.7 4).

Corollary 2.4.5. Let $E$ be a polar set contained in $U$. If $f : U \rightarrow \mathbb{C}$ is finely locally bounded in $U$ and finely holomorphic in $U \setminus E$, then $f$ extends to a unique finely holomorphic functions in $U$.

Proof. Follows from Theorem 2.3.15.

Corollary 2.4.6. Let $U \subseteq \mathbb{C}$ be a usual open set. Then a function $f : U \rightarrow \mathbb{C}$ is finely holomorphic in $U$ if and only if it is holomorphic.

Proof. Follows from the corresponding version of Theorem 2.3.9 for finely harmonic functions.

The following theorem is the fine analog of the classical open mapping theorem for holomorphic functions. It was first proved by Debiard and Gaveau [29]. See also [60].

Theorem 2.4.7. Let $f : U \rightarrow \mathbb{C}$ be a non-constant finely holomorphic function. Then the image $f(V)$ of every finely open set $V \subset U$ is finely open. Moreover $f^{-1}(E)$ is a polar subset of $U$ for every polar set $E \subset \mathbb{C}$.

Theorem 2.4.8. A function $f : U \rightarrow \mathbb{C}$ is finely holomorphic if and only if every point of $U$ has a fine neighborhood $V \subseteq U$ in which $f$ coincides with the Cauchy-Pompeiu transform of some function $\varphi \in L^2(\mathbb{C})$ with $\varphi = 0$ a.e. on $V$

$$f(z) = \int_{\mathbb{C}} \frac{1}{z - \zeta} \varphi(\zeta) d\lambda(\zeta), \quad z \in V.$$  

Theorem 2.4.8 is the heart of Fuglede's theory. It allowed him, in particular, to recover the result of Lyons [98] that every finely holomorphic function $f$ in $U$ has a fine derivative $f'$ at every point of $U$ and $f'$ is finely holomorphic in $U$; that is, at every point $z_0 \in U$ the fine limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$  \hspace{1cm} (2.4.4)

exists and is finely holomorphic in $U$. This should be interpreted as follows: To any $\epsilon > 0$ there shall correspond a fine neighborhood $V \subset U$ of $z_0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \quad \text{for all } z \in V.$$  \hspace{1cm} (2.4.5)

This is further equivalent, in view of Cartan's theorem (cf. e.g. [76, Theorem 10.15]), to saying that $z_0$ has a compact fine neighborhood $K$ such that

$$f'(z_0) = \lim_{z \rightarrow z_0, z \in K} \frac{f(z) - f(z_0)}{z - z_0},$$  \hspace{1cm} (2.4.6)
where the limit here is taken in the usual sense. Thus a finely holomorphic function is infinitely finely differentiable, and all its fine derivatives \( f^{(n)} \) are finely holomorphic.

In fact finely holomorphic functions can be characterized as follows, cf. [62].

**Theorem 2.4.9.** A function \( f : U \to \mathbb{C} \) is finely holomorphic if and only if \( f \) is a fine \( C^1 \)-function in the complex variable sense. Explicitly

\[
f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}
\]

exists for every \( z \in U \) and \( f' \) is finely continuous in \( U \).

Theorem 2.4.9 may thus serve as an alternative definition of fine holomorphy. Observe also that taking this as a primary definition is perhaps more natural since it is similar to the classical Cauchy’s definition of usual holomorphic functions.

The next corollary follows by application of the chain rule of fine differentiation. See also [57, page 126].

**Corollary 2.4.10.** Let \( f : U \to \mathbb{C} \) and \( f : V \to \mathbb{C} \) be finely holomorphic with \( f(U) \subset V \). Then \( g \circ f \) is finely holomorphic in \( U \), and \( (g \circ f)' = (g' \circ f)' \).

Another interesting consequence of Theorem 2.4.8 is the following precise fine local description of finely holomorphic functions.

**Theorem 2.4.11.** If \( f : U \to \mathbb{C} \) is finely holomorphic, then every point of \( U \) has a compact fine neighborhood \( V \subset U \) satisfying a), b), and c) below.

a) (Approximation by rational functions.) There exists a sequence of rational functions \( f_j \) with poles off \( V \) such that, for each integer \( n \geq 0 \), the \( n \)-th derivative \( f_j^{(n)} \) converges uniformly on \( V \) to the \( n \)-th fine derivative \( f^{(n)} \) as \( j \to \infty \).

b) (Asymptotic Taylor expansion.) For any \( m \geq 0 \) there is a constant \( A_m \) (depending also on \( f \) and \( V \)) such that the inequality

\[
f(w) - \sum_{k=1}^{m-1} \frac{1}{k!} (w - z)^k f^{(k)}(z) \leq A_m |w - z|^m
\]

holds for every \( z, w \in V \).

c) (\( C^\infty \)-extension.) There exists a usual \( C^\infty \)-function \( \tilde{f} : \mathbb{C} \to \mathbb{C} \) such that \( \tilde{f} = f \) on \( V \), and

\[
\partial^n \tilde{f} = f^{(n)}, \quad \overline{\partial} \tilde{f} = 0 \text{ on } V
\]

for every \( n \geq 0 \).

Here \( \partial \) and \( \overline{\partial} \) denote as usual

\[
\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \overline{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]
2.4.12. Property b) in the above theorem shows that the Taylor series of \( f \) at a given point \( z_0 \), although divergent in general, represents \( f \) asymptotically in a fine neighborhood of \( z_0 \). In fact it is a deep result of Fuglede that \( f \) is uniquely determined by the sequence of its fine derivatives \( f^{(n)}(z_0) \), \( n \geq 0 \), at any given point \( z_0 \in U \). Explicitly, we have the following uniqueness theorem.

**Theorem 2.4.13.** A finely holomorphic function \( f \) in a fine domain \( U \) is uniquely determined by the sequence of its fine derivatives \( f^{(n)}(z_0) \), \( n \geq 0 \), at any given point \( z_0 \in U \); that is, if \( f^{(n)}(z_0) = 0 \), for all \( n \geq 0 \), at any given point \( z_0 \in U \), then \( f \equiv 0 \).

The proof of Theorem 2.4.13 follows from property b) of theorem 2.4.11 and a certain properties of the fine Green function studied in [59]. We note in passing that the sequence \( f^{(n)}(z_0) \), \( n \geq 0 \) might exhibit a rather wild behavior; in an example (similar to Example 2.4.2) communicated by Fuglede to P. Pyrih [115] it is shown that for any prescribed sequence of real numbers \( c_n \) there exists a finely holomorphic functions \( f \) such that the sequence \( f^{(n)}(0) \) grows faster than \( c_n \).

Theorem 2.4.13 tells us in particular that the set of zeros of a finely holomorphic function \( f \) has empty fine interior (unless \( f \equiv 0 \)). In fact, it is shown in [61, Theorem 15] that this set is at most countable. See also [66, page 292].

The next theorem is the analog of the classical local inversion mapping theorem.

**Theorem 2.4.14.** Let \( f : U \rightarrow \mathbb{C} \) be finely holomorphic. If the fine derivative \( f'(z_0) \neq 0 \) for some \( z_0 \in U \), then \( f \) is injective in some fine neighborhood \( V \subset U \) of \( z_0 \). Moreover the inverse \( g := f^{-1} \) of \( f \) is finely holomorphic in \( f(V) \).

The "quasi-analytic" property stated in Theorem 2.4.13 suggest that a finely holomorphic function on a fine neighborhood of the real line might belong to some quasi-analytic class in the sense of Denjoy and Carleman, cf. [120]. However this is not always the case, see [115] for an interesting discussion on this topic.

An extension of the above theory of finely holomorphic functions to functions of several complex variables has been made in [65]. Explicitly, if \( U_1 \) and \( U_2 \) are two finely open subsets of \( \mathbb{C} \), then a rich theory of fine holomorphy in \( U_1 \times U_2 \), retaining almost all the above results, exists. However, this theory is not very satisfactory since the product fine topology in \( \mathbb{C}^2 \) is not biholomorphically invariant as explained in [65, page 144]. In contrast, the pluri-fine topology is biholomorphically invariant, and this was one of our motivation to study this topology in Chapter 3.

## 2.5 Borel’s Monogenic Functions

The possibility of extending the classical concept of a holomorphic function to sets that are not necessarily open while retaining its distinctive property of uniqueness, cf. Theorem 2.4.13, was extensively investigated by Borel in the period 1892-1914. Borel’s initial idea, cf. [10], was to generalize the Weierstrass’s concept of holomorphic continuation by which can be defined single valued maximal holomorphic
function as the collection of "functional elements" created by power series developments. In other words, Borel believed that it must be possible to continue certain holomorphic functions beyond their maximal domain of existence. This idea was almost a heretic one to Weierstrass’s concept of maximal domain of existence.

In contrast to Borel’s deep intuition, Poincaré [112] had already constructed analytic expressions presenting certain singularities and believed that he had sufficient grounds for concluding to the impossibility of extending the theory of holomorphic functions beyond the bounds fixed by Weierstrass; that is, he believed in the impossibility of a process of holomorphic continuation by which one could extend a function beyond its maximal domain of existence. In this connection it is interesting to state the following paragraph from the preface of Borel’s book [12]:

"A la suite des travaux de Poincaré que j’ai rappelés il y a un instant, ce point de vue paraissait universellement admis; mais tandis que Poincaré accueillait avec bienveillance le premier essai dans lequel je montrais que les limites imposées par Weierstrass n’étaient pas aussi infranchissables qu’on l’avait cru, les disciples fidèles de Weierstrass ne consentaient même pas à discuter; je me rappellerai toujours l’étonnement avec lequel je vis M. Mittag-Leffler, auquel j’avais essayé d’exposer mes projets de recherches, ne faire aucun effort pour entrer dans ma pensée et se contenter de retirer de sa malle un Mémoire de Weierstrass pour me montrer une phrase qui devait clore définitivement toute discussion: Magister dixit."

We shall illustrate Borel’s idea to generalize holomorphic continuation by an example due to Borel himself and bring some precisions using the theory of finitely holomorphic functions.

**Example 2.5.1.** Let \( a_j, j = 1, 2, \ldots \) be a dense sequence on the unit circle \(|z| = 1\), and let \( c_j, j = 1, 2, \ldots \) be a sequence of positive numbers such that

\[
\sum_{j=1}^{\infty} c_j < \infty. \tag{2.5.1}
\]

Consider the following series

\[
f(z) = \sum_{j=1}^{\infty} \frac{c_j}{z - a_j}. \tag{2.5.2}
\]

Clearly, this series converges absolutely, and locally uniformly, in the open set \( \mathbb{C} \setminus \{|z| = 1\} \). Accordingly, the series \( f \) represents a function which is holomorphic both inside and outside the unit circle and we call these sums \( f_1(z) \) and \( f_2(z) \) respectively to distinguish the two. It follows by a result of Goursat, cf. [118, page 114], that each of the two function has radial limit \( \infty \) at each of the points \( a_n \). In particular the unit circle is a natural boundary for both \( f_1(z) \) and \( f_2(z) \) in the sense of Weierstrass; That is \( f_1(z), f_2(z) \) cannot be continued holomorphically one into the other. This is an apparent "paradox" since one can then argue that \( f_1(z) \) and \( f_2(z) \) do not represent the "same" function even though they are both representable by the series in (2.5.2).

Borel, in his thesis, was able to show that, under more stringent conditions on the sequence \((c_j)\), any two points \( z, w \) respectively inside and outside the
unit circle, can be joined by an arc of a circle $C$ which will necessarily intersect
the unit circle but on which the series (2.5.2) and the series obtained by term
by term differentiation any number of times will converge uniformly. Thus the
series represents a $C^\infty$ function at all points of $C$ but is not holomorphic on $C$,
in particular at the point of intersection of $C$ with the unit circle. Moreover, and
most importantly, the function $f(z)$ in (2.5.2) has the property that a knowledge
of $f(z)$ for $z$ on an arbitrarily small open subset of the complex plane determines
$f(z)$ uniquely on the part of the plane where it is defined. In this sense $f_1(z), f_2(z)$
can thus be considered "continuation" of each other.

We will now proceed as in example 7.2.2 from chapter 7. Define subharmonic
functions $g_j(z) = \log |z - a_j| - 2$ and $u_n$ by

$$ u_n(z) = \sum_{j=n}^{\infty} j^{-3} g_j(z). \quad (2.5.3) $$

The terms in the sum of (2.5.3) are subharmonic and they are negative for $|z| < k$
as soon as $j > k$. Hence $u_n$ represents a subharmonic functions in $\mathbb{C}$. Observe
that $u_n \not\equiv -\infty$ since $u_n(0)$ is finite. Let $D = (\cup_n \{u_n > -10\}) \setminus \{a_1, a_2, \ldots\}$. We
claim that $D = \{u_1 > -\infty\}$. Indeed, observe first that

$$ \{u_{k_1} = -\infty\} \setminus \{a_1, a_2, \ldots\} = \{u_{k_2} = -\infty\} \setminus \{a_1, a_2, \ldots\} \quad (2.5.4) $$

for any natural numbers $k_1$ and $k_2$. Next, if $z_0 \in \cup_n \{u_n > -\infty\} \setminus \{a_1, a_2, \ldots\}$. Then there exists, by (2.5.4), a natural number $k$ such that $|z_0| < k$ and $u_k > -\infty$.

Since, as mentioned before, all the terms of the series $u_k(z_0)$ are negative, a suitable
tail, say $u_N(z_0)$, will be very close to 0. In other words, $z_0 \in \{u_N > -10\}$. Hence
$z_0 \in D$ and consequently $D = \cup_n \{u_n > -\infty\} \setminus \{a_1, a_2, \ldots\}$. Therefore,

$$ \mathbb{C} \setminus D = \cap_{n=1}^{\infty} \{u_n = -\infty\} \cup \{a_1, a_2, \ldots\}. \quad (2.5.5) $$

Again, by (2.5.4), we conclude that

$$ \mathbb{C} \setminus D = \{u_1 = -\infty\} \cup \{a_1, a_2, \ldots\} = \{u_1 = -\infty\}. \quad (2.5.6) $$

This proves the claim. In particular, $D$ is, by Theorem 2.2.29, a fine domain.

For every $j > 3$ there exists $0 < \tilde{c}_j < 1$ such that if $|z - a_j| < \tilde{c}_j$, then

$$ u_n(z) < -11, \text{for every } n \leq j \quad (2.5.7) $$

Indeed, $\sum_{k>j} k^{-3} g_k(z) < 0$, while

$$ \sum_{k=n}^{j-1} k^{-3} g_k(z) < \log j \sum_{k=n}^{j-1} k^{-3} < 10 \log j. $$

So it suffices to take $\tilde{c}_j = j^{-21/3}$. 
Next put $c_j = 2^{-j} \tilde{c}_j$ and define a function on $D$ by

$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{z - a_j},$$

(2.5.8)

We claim that the function $f$ is finely holomorphic on $D$. Indeed, let $z_0 \in D$. Then $z_0$ belongs to the finely open set $\{u_m > -10\}$ for some $m$. By inequality (2.5.7), $\{u_m > -10\} \subset \mathbb{C} \setminus \bigcup_{j \geq m} B(a_j, c_j)$. Therefore, the compact set $K = \{|z| \leq 2|z_0|\} \setminus \bigcup_{j \geq m} B(a_j, c_j)$ is a fine neighborhood of $z_0$. Since the series of $f$ in (2.5.8) is uniformly convergent on the compact set $K$, the claim is proved.

Note that since $\mathcal{C}D$ is a polar set (containing the sequence $a_j$), the set $F = D \cap \{|z| = 1\}$ has length $2\pi$, and $f$ is finely holomorphic at each point of $F$. Moreover, and most importantly, if $a \in F$ then the series $f$ in (2.5.8) is uniquely determined by the sequence of its fine derivatives $f^{(n)}(a)$, $n \geq 0$, cf. Theorem 2.4.13. Furthermore, the above two functions $f_1(z)$ and $f_2(z)$ are finely holomorphic continuation of each other. And $f_1(z)$, $f_2(z)$ should thus be conceived as restrictions of one and the same function. So, Weierstrass theory of holomorphic continuation does not really give a complete picture.

During about two decades, Borel struggled to give his ideas a more general form and a solid foundation. This resulted in the creation of the theory of monogenic functions defined on a class of sets broader than open sets, and still possess a number of important properties usually associated with holomorphic functions. These sets where called Cauchy domains and they are a countable increasing unions of certain Swiss cheeses without interior points (w.r.t the usual topology). We shall not describe the details of Borel’s construction. The interested reader is referred to [11, 12, 137]. A discussion about the connection between Borel’s monogenic functions and finely holomorphic ones can be found in [66, 102].